Advanced Microeconomics:

Game Theory (I) - Static games

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These class notes draw extensively from Jehle, G. A., & Reny, P. J. (2011). *Advanced microeconomic theory* (3rd ed.). Pearson. Unless otherwise noted, all figures and diagrams presented herein are adapted from or directly sourced from this text.

Roadmap of the lecture

- Static (simultaneous) games
  - Formal definitions
  - Nash Equilibrium
- Applications to producer theory
- Static games of incomplete information

What is game theory?

Context: In situations where the actions of an individual agent affect others, agents are incentivized to act strategically, considering how others will respond to their actions.

What is Game Theory?

- **Definition:** Game theory is the systematic study of how rational agents behave in such strategic contexts—referred to as *games*.
- In these games, each agent must account for the decisions of others before determining their optimal course of action.

Key Feature:

- The interdependence of agents' actions. Each agent's decisions influence and are influenced by others.

Example 1: The batter-pitcher game

Scenario:

- A pitcher can throw two types of pitches: a fastball and a curveball.
- The fastball is the pitcher's strongest pitch, while the curveball is average.
- If the pitcher were playing alone, his optimal decision would be to always throw the fastball.

What Changes?

- There **is** a batter in the game.
- The batter anticipates the pitcher's actions and adjusts his strategy accordingly.
- If the pitcher ignores the batter's response, his best pitch could become the worst decision.

Unpredictability as a Strategy

- In the batter-pitcher game, predictable behavior leads to a disadvantage.
- Rational and strategically behaving agents must sometimes act in an **unpredictable** manner to maintain an edge over their opponent.
- This unpredictability is formalized through the concept of **mixed strategies**, where players choose their actions according to specific probabilities.

The Prisoner's Dilemma

Scenario:

- Two members of a criminal gang are arrested and placed in solitary confinement.
- The police lack enough evidence to convict them on the main charge.
- Both are given the option to **testify** against the other or **remain silent**.
- If one testifies and the other remains silent, the testifier goes free, and the silent partner receives a three-year sentence.
- If both testify, they each receive a two-year sentence.
- If both remain silent, they each serve one year in jail on a lesser charge.

Normal form game

- We define the scenarios discussed as **games**.
- In game theory, a **game** is characterized by **strategic interaction** among participants.
- Strategic interaction implies that each participant's **payoff** depends not only on their own actions but also on the actions of other participants.
- The participants are referred to as **players**.
- The possible actions that each player can choose are called **strategies**.

Strategic Form Game

We will focus on static games: all players take actions simultaneously. A strategic form game is a tuple:

$$G = (S_i, u_i)_{i=1}^N,$$

where:

- For each player $i = 1, \ldots, N$, S_i is the set of strategies available to player i. This means that the representation will also specify the list of players.
- $u_i : \prod_{j=1}^N S_j \to \mathbb{R}$ describes player i's payoff as a function of the strategies chosen by all players.

Finite Strategic Form Game: A strategic form game is **finite** if each player's strategy set contains finitely many elements.

Dominant Strategies

For the following definitions, let $S = S_1 \times \cdots \times S_N$ represent the set of joint pure strategies. The symbol $-i$ refers to 'all players except player i'.

Strictly Dominant Strategies:

A strategy, $\hat{s}_i$ for player i is strictly dominant if $u_i(\hat{s}_i, s_{-i}) > u_i(s_i, s_{-i})$ for all $(s_i, s_{-i}) \in S$ with $s_i \neq \hat{s}_i$.

	Left	Right
Up	$\underline{3}, \underline{0}$	$\underline{0}, -4$
Down	$2, 4$	$-1, \underline{8}$

A rational player should play a dominant strategy, provided that there is one.

Dominant strategies

Strictly Dominated Strategies

A strategy $\hat{s}_i$ for player i strictly dominates another strategy $\bar{s}_i$ if $u_i(\hat{s}_i, s_{-i}) > u_i(\bar{s}_i, s_{-i})$ for all $s_{-i} \in S_{-i}$. In this case, $\bar{s}_i$ is said to be strictly dominated within S.

Consider the following game in matrix form:

	L	M	R
U	$\underline{3}, \underline{0}$	$0, -5$	$\underline{0}, -4$
C	$1, -1$	$3, 3$	$-2, \underline{4}$
D	$2, 4$	$\underline{4}, 1$	$-1, \underline{8}$

Dominant Strategies

Formal Definition of iterative elimination of strictly dominated strategies:

Let $S_i^0 = S_i$ denote the initial set of strategies for each player i. For $n \geq 1$, let S_i^n represent the set of strategies for player i that remain after the n-th round of elimination. Specifically, $s_i \in S_i^n$ if $s_i \in S_i^{n-1}$ and s_i is not strictly dominated in S_i^{n-1}.

Iteratively strictly Undominated Strategies

A strategy s_i for player i is said to be iteratively strictly undominated in S (or to survive the iterative elimination of strictly dominated strategies) if $s_i \in S_i^n$ for all $n \geq 1$.

Dominant strategies

Iterative elimination of strictly dominated strategies in practice:

1. **Step 1: Initial elimination.** Since rational players will never play a strictly dominated strategy, we can immediately remove these from their strategy sets.
2. **Step 2: Iteration through common knowledge.** With the assumption of common knowledge of rationality, all players know that strictly dominated strategies will be eliminated.
3. **Step 3: Stopping condition.** The process continues until no more strictly dominated strategies remain for any player. The remaining strategies form the set of surviving strategies.

Dominant Strategies

Two drawbacks to Iterative Elimination of Strictly dominated strategies:

1. 2 Assumptions needed: rationality + common konwledge: all players are rational, + all players know that all players are rational + all players know that all players know...
2. Imprecise prediction about the game: how would you play if there are no strictly dominated strategies?

	L	C	R
T	$0,4$	$4,0$	$5,3$
M	$4,0$	$0,4$	$5,3$
B	$3,5$	$3,5$	$6,6$

An example with a coordination game

"The battle of programming languages":

- **Context**: Alba and Joël, two Ph.D. friends, are working on a programming problem set with different language preferences:
  - Joël prefers **Julia**.
  - Alba prefers **Python**.
- **Challenge**:
  - Each would rather the other switch to their preferred language.
  - Failure to coordinate means they can't submit the problem set.
- They choose independently without prior discussion.

An example with a coordination game

	Joël (Py)	Joël (Ju)
Alba (Py)	$\underline{2}, \underline{1}$	$0, 0$
Alba (Ju)	$0, 0$	$\underline{1}, \underline{2}$

Dominant Strategies

Weakly Dominated Strategies

- A strategy $\hat{s}_i$ **weakly dominates** another strategy $\bar{s}_i$ for Player i if:

$$u_i(\hat{s}_i, s_{-i}) \geq u_i(\bar{s}_i, s_{-i}) \quad \text{for all } s_{-i} \in S_{-i},$$

  with at least one strict inequality (i.e., same payoff agains all the strategies except for one where $\hat{s}_i$ gives a strictly higher payoff).
- This implies that $\bar{s}_i$ is **weakly dominated** in S.
- In contrast to strict dominance, weakly dominated strategies can be equal to the dominating strategy in some cases.

	L	R
U	$\underline{1}, \underline{1}$	$\underline{0}, 0$
D	$0, \underline{0}$	$\underline{0}, \underline{0}$

Dominant Strategies

Iteratively Weakly Undominated Strategies

- Let $W_i^0 = S_i$ for each player i.
- For $n \geq 1$, let W_i^n be the set of strategies that remain after n-th round of elimination of weakly dominated strategies.
- A strategy $s_i \in W_i^n$ if s_i is not weakly dominated in W^{n-1}.
- A strategy s_i is called **iteratively weakly undominated** if it remains in W_i^n for all $n \geq 1$.

Thus, the final equilibrium outcomes from iterative deletion of weakly dominated strategies depend on the order of elimination.

Nash Equilibrium

In economics, **equilibrium** refers to a state where no agent has an incentive to change their behavior, as they are optimizing their situation (e.g., consumers maximize utility, firms maximize profits).

Game theory extends this concept by considering **strategic interactions** between players:

- Players act in their own self-interest while considering the actions of others.
- Equilibrium reflects mutual awareness of strategic decision-making.

Nash Equilibrium, introduced by John Nash in 1951:

- Defined as a set of strategies where no player has a profitable unilateral deviation.
- Each player's strategy is a best response to the other's.
- A player cannot improve their payoff by changing their strategy while others keep theirs unchanged.

Pure Strategy Nash Equilibrium

Given a strategic form game $G = (S_i, u_i)_{i=1}^{N}$, a joint strategy $\hat{s} \in S$ is a Nash equilibrium if:

$$u_i(\hat{s}) \geq u_i(s_i, \hat{s}_{-i}) \quad \text{for all} \quad s_i \in S_i.$$

Based on the following assumptions:

- Players are rational, aiming to maximize their utility.
- This rationality is common knowledge among players.
- Players form correct conjectures about the strategies of others.

  Throughout previous discussions, we have reasoned in terms of each player's best strategy. In some of the cases they were Nash Equilibria.

Nash Equilibrium

- **Proposition 1:** A strategy profile that survives the iterative elimination of strictly dominated strategies is a Nash equilibrium.

  That is, if the strategies $(s_1, \cdots, s_2)$ are a Nash equilibrium they will survive the iterated elimination of strictly dominated strategies, but there can be strategies that also survive and are not a Nash equilibrium

- **Proposition 2:** If the iterative elimination of strictly dominated strategies reduces the game to a single strategy profile, then this profile is the **unique** Nash equilibrium of the game.

Mixed strategies

A mixed strategy involves randomizing over possible actions with assigned probabilities to avoid predictability

- Motivation: Pure strategies might lead to predictable behavior. By randomizing, players can prevent opponents from exploiting their strategy.
- Key Idea: All players know each other's strategies and payoffs but may randomize to prevent being outmaneuvered.

Example: Batter-Pitcher Game

- If the pitcher always throws a fastball, the batter will always prepare for it.
- To avoid predictability, the pitcher might randomize between a fastball and a curveball.
- The batter, in turn, also randomizes her response.

Mixed Strategy Nash Equilibrium

- **Nash Equilibrium in Mixed Strategies**: A strategy profile where each player randomizes over actions such that no one has an incentive to unilaterally deviate.
- **Example**: In the batter-pitcher game, both players randomize with probability $\frac{1}{2}$ for each action.

Why is this an equilibrium?

- Neither player can improve their expected payoff by changing their strategy alone.
- The outcome is stable and self-enforcing: knowing the randomization, neither player has a reason to change their strategy.

Mixed Strategies in Finite Strategic Form Games

- A **mixed strategy** m_i for player i is a probability distribution over the player's pure strategies S_i.
- Formally, $m_i : S_i \to [0,1]$, where:

$$M_i = \left\{ m_i : S_i \to [0,1] \;\middle|\; \sum_{s_i \in S_i} m_i(s_i) = 1 \right\}.$$

- Each pure strategy can be viewed as a mixed strategy by assigning probability 1 to that strategy and 0 to others.
- Mixed strategies provide a broader set of choices, allowing for all possible probabilistic combinations of pure strategies.

Mixed strategies

- Players value each strategy by making expected-value calculations. That is, the uility function is von Neumann-Morgenstern utility function on S:

  $$v_i(m) \equiv \sum_{s \in S} m_1(s_1) \cdots m_N(s_N) u_i(s).$$

  Strategies are chosen independently: the probability that the pure strategy $s = (s_1, \ldots, s_N) \in S$ is chosen is the product of the probabilities that each separate component is chosen, namely

  $$\prod_{i=1}^{N} m_i(s_i).$$

In fact, this expected uility calculation allows us to define a new game which is an augmented version of our pure-strategy game:

$$\Delta G = (m_1, \cdots ,_N ; v_i, \cdots , v_N)$$

Mixed Strategies Nash Equilibrium

- In a finite strategic form game $G = (S_i, u_i)_{i=1}^N$, a joint strategy $\hat{m} \in M$ is a Nash equilibrium if:

$$v_i(\hat{m}) \geq v_i(m_i, \hat{m}_{-i}) \quad \forall m_i \in M_i.$$

- A mixed-strategy Nash equilibrium occurs when no player can increase their expected utility by unilaterally changing their probability distribution over actions.
- Each player's mixed strategy is a best response to the other players' mixed strategies.
- For a given game G, a mixed strategy profile $\hat{m}$ constitutes a mixed-strategy Nash equilibrium if $\hat{m}$ constitutes a Nash-equilibrium of the augmented game ΔG.
- **Deviation in Mixed Strategies**: Instead of switching to a single action, players alter their probability distribution over actions.

Simplified Nash Equilibrium Tests

The following statements are equivalent:

1. $\hat{m} \in M$ is a Nash equilibrium.
2. For every player i, $u_i(\hat{m}) = u_i(s_i, \hat{m}_{-i})$ for every $s_i \in S_i$ given positive weight by $\hat{m}_i$, and $u_i(\hat{m}) \geq u_i(s_i, \hat{m}_{-i})$ for every $s_i \in S_i$ given zero weight by $\hat{m}_i$.
3. For every player i, $u_i(\hat{m}) \geq u_i(s_i, \hat{m}_{-i})$ for every $s_i \in S_i$. That is, no pure strategy yields higher expected payoff than the mixed strategy.

Recommended strategy:

1. Iterated elimination of dominated strategies.
2. Look for pure NE.
3. Look for mixed-strategies using the tests above.

The Batter-pitcher game with mixed strategies

Recall our batter-pitcher game:

	B: F	B: C
P: F	$-1, \underline{1}$	$\underline{1}, -1$
P: C	$\underline{1}, -1$	$-1, \underline{1}$

Suppose the pitcher believes the batter will be ready for F with probability q and to be ready for C with probability $1 - q$.

- $EU_P(F, (q, 1 - q)) = q(-1) + (1 - q)1 = 1 - 2q$
- $EU_p(C, (q, 1 - q)) = q(1) + (1 - q)(-1) = 2q - 1$

The Batter-pitcher game with mixed strategies

When will always F (i.e., a pure-strategy) be a best-response to the mixed strategy of B?

- P will always do F iff
  $EU_P(F,(q,1-q) > EU_p(C,(q,1-q)) \quad \Rightarrow \quad q < 1/2$

So, the best-response of P in pure-strategies is:

$$BR_P(q(1-q)) = \begin{cases} F, & q < 1/2, \\ \textit{Indifferent} & q = 1/2 \\ C, & q > 1/2. \end{cases}$$

The batter-pitcher game with mixed strategies

Let r and $1-r$ be the mixed strategy in which P throws as F and C with probability r and $1-r$ respectively. Let's find $r^*(q)$ such that $(r, 1-r)$ is a best response of P to B doing $(q, 1-q)$:

Notice that the expected utility for P is:

$EU_P((r, 1-r), (q, 1-q)) = rq(-1) + r(1-q) \cdot 1 + (1-r)q \cdot 1 + (1-r)(1-q)(-1) = (2q-1) + r(2-4q)$

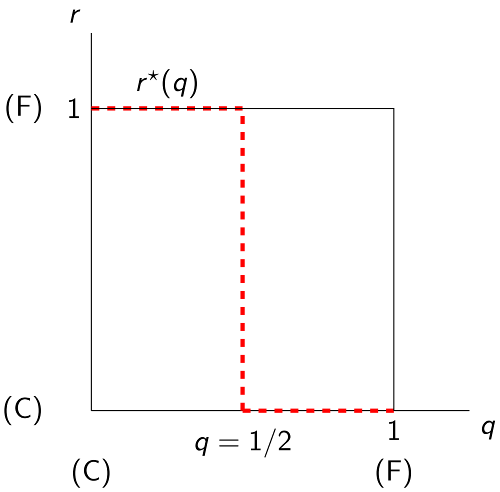
- If $(2-4q) > 0 \quad \Rightarrow \quad q < 1/2$, expected utility increases in r and decreases if $(2-4q) < 0$
- $r = 1$ if $q < 1/2$ (i.e., F), $r = 0$ if $q > 1/2$
- $q = 1/2$: P is indifferent between all pure strategies but also between all mixed strategies because the EU_F is independent of r.

The Batter-pitcher game with mixed strategies

Thus the best-response (correspondence):

$$r(q(1-q)) = r^*(q) = \begin{cases} r = 1 \quad \text{(i.e., } F\text{)}, & q < 1/2, \\ r \in [0,1] \quad \text{(i.e., indifferent)} & q = 1/2 \\ r = 0 \quad \text{(i.e., } C\text{)}, & q > 1/2. \end{cases}$$

The Batter-Pitcher game with mixed strategies



The Batter-Pitcher game with mixed strategies

We apply the same reasoning to the batter. We need to find $q^*(r)$ such that $(q, 1-q)$ is a best response for B to P doing $(r, 1-r)$ Notice that the expected utility for F is:

$EU_B((q, 1-q), (r, 1-r)) = rq(1) + r(1-q) \cdot (-1) + (1-r) \cdot q(-1) + (1-r)(1-q)(1) = (1-2r) + q(4r-2)$

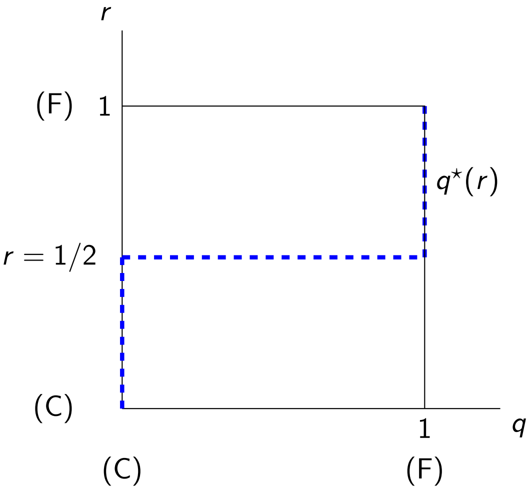
- If $(4r-2) > 0 \quad \Rightarrow \quad r > 1/2$, expected utility increases in q and decreases if $(4r-2) < 0$
- $q = 1$ if $r > 1/2$ (i.e., F), $q = 0$ if $r < 1/2$
- $r = 1/2$: B is indifferent between all pure strategies but also between all mixed strategies because the EU_B is independent of q.

The Batter-pitcher game with mixed strategies

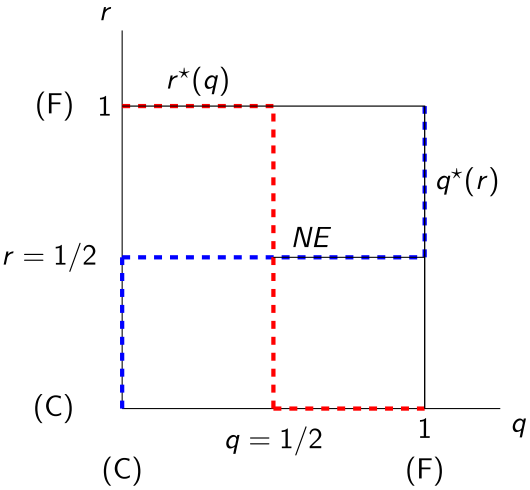
Thus the best-response (correspondence):

$$q((r,(1-r))) = q^*(r) = \begin{cases} q = 1 \quad (\text{i.e., } F), & r > 1/2, \\ q \in [0,1] \quad (\text{i.e., indifferent}) & r = 1/2 \\ q = 0 \quad (\text{i.e., } C), & r < 1/2. \end{cases}$$

The Batter-Pitcher game with mixed strategies



The Batter-Pitcher game with mixed strategies



The Batter Pitcher game with mixed strategies

The Nash Equilibrium, as we defined it, is the intersetion between the best-response correspondences $r^*(q)$ and $q^*(r)$:

- The Pitcher throws a fastball with probability $1/2$ and a curveball with probability $1/2$
- The batter throws a fastball with probability $1/2$ and a curveball with probability $1/2$

Existence of Nash Equilibrium

Proposition: Consider a game where:

1. $S_i \in \mathbb{R}^m$ is non-empty, convex and compact $\forall i$,
2. $u_i(s_1, \cdots, s_n)$ is continuous on all s_i and quasiconcave in s_i.

Then, there exists a pure Nash equilibrium of the game.

Corollary: Nash Theorem (1950):

- Every finite strategic form game possesses at least one Nash Equilibrium, potentially in mixed strategies.
- This holds true regardless of the number of players involved, as long as:
  - Each player has a finite number of pure strategies.
  - The game is not infinite.

Strategic interaction in product markets

- Many markets display a competitive structure combining elements of both monopoly and competition.
- Fewer firms in an industry increase interdependence among competitors.
- Firms adopt strategic decision-making due to the influence of each other's actions.

Cournot Duopoly

- Let q_1 and q_2 denote the quantities of a homogenous product produced by firm 1 and firm 2.
- Let $P(Q) = a - Q = a - q_1 - q_2$, $Q < a$ and $P(Q) = 0$ for $Q \geq a$.
- Total cost of producing for firm i: $C_i(q_i) = cq_i$.
- Assumce $c < a$.
- Firms choose the quantities simultaneously.

Cournot Duopoly

Normal form representation:

- 2 players
- Strategy space: $S_i = [0, \infty)$
- $s_i = q_i \in [0, a)$
  Notice that because $P(Q) = 0$ for $Q > a$, no firm produces $q_i \geq a$
- Assume payoff is equal to the profits:
  $u_i(s_i, s_j) = \pi(q_i, q_j) = q_i[a - q_i - q_j - c]$

Cournot-Nash equilibrium: output vector $\overline{\mathbf{q}}$ maximizes profit given competitors' actions (i.e., being a maximum it will imply that $u_i(s_i^*, s_j^*) \geq u_i(s_i, s_j^*)$)

Cournot Duopoly

Thus, the first-order condition of the maximization:

$$q_1^* = \frac{1}{2}(a - q_2^* - c),$$
$$q_2^* = \frac{1}{2}(a - q_1^* - c).$$

Solving the system of equations implies: $q_1 = q_2 = \frac{a-c}{3}$.

What's the intuition? Notice that every firm would like to be a monopolist:

- set q_i^* s.t. it maximizes $\pi_i(q_i, 0)$. That is: $q_m = \frac{a-c}{2}$.
- Profits would be: $\frac{(a-c)^2}{4}$

Cournot Oligopoly

- Suppose there are J identical firms in the market, and entry is blocked.
- All firms share identical costs:

$$C(q^j) = cq^j, \quad c \geq 0 \quad \text{for } j = 1, \ldots, J.$$

- Price is determined by total output, with linear inverse demand:

$$p = a - b\sum_{k=1}^{J} q^k, \quad a > 0, b > 0.$$

Bertrand Oligopoly

- Firms compete on setting prices rather than quantities.
- Consider a Bertrand duopoly: two firms producing a homogeneous good with marginal cost c.
- Linear market demand:

$$Q = a - \beta p.$$

- Firms decide prices simultaneously, supplying whatever is demanded.

Bertrand Oligopoly

- Profit function for Firm 1:

$$\pi_1(p_1, p_2) = \begin{cases} (p_1 - c)(a - \beta p_1), & c < p_1 < p_2, \\ \frac{1}{2}(p_1 - c)(a - \beta p_1), & c < p_1 = p_2, \\ 0, & \text{otherwise.} \end{cases}$$

- Firms must set prices $p \geq c$ to avoid negative profits.

Bertrand Oligopoly

- $p_1 = p_2 = c$ is a Nash equilibrium.
- No equilibrium exists with $p_i > c$:
  - If $p_1 > c$, Firm 2 can undercut and gain the market.
  - Both firms would have incentives to lower their prices.
- Bertrand competition leads to prices set equal to marginal cost, similar to perfect competition.

Perfect vs. Imperfect Information in Games

- **Assumption of Perfect Information**:
  - So far, we've solved games assuming players know all actions and payoffs of their rivals.
  - **Realistic?** Not always.

- **Introducing Imperfect Information**:
  - Players often lack complete knowledge of others' payoffs.
  - We introduce **beliefs** about other players' payoffs to account for uncertainty.
  - At least one player is uncertain about another's player's payoff function.
  - *Example:* Competing firms may not know each other's production costs but form beliefs about them.

Beliefs and Player Types

- **Beliefs as Part of the Game**:
  - Players hold beliefs about unknown factors (e.g., rivals' costs).
  - Beliefs can be probabilistic (e.g., firm 1 believes firm 2 has high or low costs with equal probability).
- **Types of Players**:
  - Each player has a finite set of **types** T_i (e.g., high cost, low cost).
  - Player's payoff depends on both their type and the strategies and types of others.
- **Strategy and Type Sets**:
  - S_i: Strategy set for player i
  - T_i: Type set for player i
  - A strategy, $s_i(t_i)$ is a function that specifies which action to take if your type is t_i.
- **Payoff Function**: t_i is privately known and determines the payoff function.
  - Payoff $u_i(s, t)$ depends on strategies s and types t.
  - $u_i : S \times T \to \mathbb{R}$: von Neumann-Morgenstern utility for each player.

Timing

Following Harsanyi (1967) we assume the following timing:

1. Nature draws a type vector $t = (t_1, \cdots, t_n)$ from T_i according to the prior probability distribution $p(t)$.
2. Nature reveals t_i only to player i
3. Players choose simultaneously their actions
4. Payoffs are realized.

Introducing nature transforms the game of incomplete information to imperfect information: you don't know the complete history so far.

Beliefs in Games

- **Beliefs about Others' Types**:
  - For each player i and type t_i, beliefs about others' types are denoted $p_i(t_{-i} \mid t_i)$.
  - These probabilities must sum to 1:

$$\sum_{t_{-i} \in T_{-i}} p_i(t_{-i} \mid t_i) = 1$$

- **Consistency of Beliefs**:
  - Beliefs should be consistent across players.
  - Achieved through a joint probability distribution p over the type space.

Common Prior Assumption

- **Common Prior**:
  - A single probability distribution p represents the joint distribution of types.
  - Each player's beliefs are derived from this common prior with Bayes' rule:

$$p_i(t_{-i}|t_i) = \frac{p(t_{-i}, t_i)}{p(t_i)} = \frac{p(t_{-i}, t_i)}{\sum_{t_{-i}} p(t_{-i}, t_i)}$$

- **Why Common Priors?**:
  - Empirical distribution from repeated observations.
  - Differences in beliefs arise from differences in information.

Game of Incomplete Information (Bayesian games)

Definition: A game of incomplete information is characterized by:

- A tuple $G = (p_i, T_i, S_i, u_i)_{i=1}^N$
- Each player i has:
  - A finite set of types T_i
  - A strategy set S_i
  - A utility function $u_i : S \times T \to \mathbb{R}$
- A probability distribution $p_i(\cdot \mid t_i)$ over other players' types T_{-i}.

Bayesian games

- To analyze Bayesian games, we relate the incomplete information game G to a strategic form game G^*:
  - Each type of player i in G is treated as a distinct player in G^*.
- This transformation allows us to use previously established techniques from strategic form games.
- Important: G^* must capture all relevant aspects of the original game G.

Associating Bayesian Games with Strategic Form Games

In general, we want to associate with each Bayesian game G a strategic form game G^* in which each type of each player is a *separate* player.

For each player $i \in \{1, \dots, N\}$ and each type $t_i \in T_i$, we consider t_i as a player in G^* whose finite set of pure strategies is S_i.
The set of players in G^* is given by

$$T_1 \cup \cdots \cup T_N$$

and the set of joint pure strategies is

$$S^* = S_{T_1} \times \cdots \times S_{T_N}.$$

Payoff Definition

To define the players' payoffs, let $s_i(t_i) \in S_i$ denote the pure strategy chosen by player $t_i \in T_i$. Given a joint pure strategy

$$s^* = (s_1(t_1), \ldots, s_N(t_N))_{t_1 \in T_1, \ldots, t_N \in T_N} \in S^*,$$

the payoff for player t_i is defined as:

$$v_{t_i}(s^*) = \sum_{t_{-i} \in T_{-i}} p_i(t_{-i} \mid t_i) u_i(s_1(t_1), \ldots, s_N(t_N), t_1, \ldots, t_N).$$

Interpretation of Payoffs

The payoff formula shows how G^* captures the essence of the incomplete information game. When players select pure strategies in G^*, the payoff for player i:

$$\sum_{t_{-i} \in T_{-i}} p_i(t_{-i} \mid t_i) u_i(s_1(t_1), \ldots, s_N(t_N), t_1, \ldots, t_N)$$

reflects the uncertainty player i has regarding the types of the other players, expressed through $p_i(t_{-i} \mid t_i)$.

By associating G with the strategically defined game G^*, we transform the analysis of incomplete information games into complete information games, specifically strategic form games.

This association allows us to apply the solution techniques available for G^*, particularly utilizing the Nash equilibria of G^*.

Bayesian Nash Equilibrium

- A Bayesian-Nash equilibrium is a Nash equilibrium in the strategic form game G^*. The intutition is that no player wants to change her strategy, even if the change involves only one action by one type.
- Every finite game of incomplete information has at least one Bayesian-Nash equilibrium.

Incomplete Information in Competition à la Bertrand

Consider two firms competing à la Bertrand. Firm 1 has a marginal cost of production equal to zero. Firm 2's marginal cost can be either 1 or 4, each occurring with a probability of $\frac{1}{2}$.

If the lowest price charged is p, market demand is $8 - p$. Each firm can choose from three possible prices: 1, 4, or 6.

The payoff structure assumes that if both firms' costs are strictly less than the common price, the market is evenly split between them. If Firm 1 lowers its price by a small amount ϵ, it can capture the entire market.

The strategic form game comprises three players: Firm 1, Firm 2 when of low cost type (2_l), and Firm 2 when of high cost type (2_h). The strategy set remains $\{1, 4, 6\}$.

Payoff structure

First number denotes Firm 1's payoff, the second indicates Firm 2's payoff depending on its type.

	$p_2 = 6$	$p_2 = 4$	$p_2 = 1$
$p_1 = 6$	$6, 5$	$0, 12$	$0, 0$
$p_1 = 4$	$16, 0$	$8, 6$	$0, 0$
$p_1 = 1$	$7, 0$	$7, 0$	$7, 0$

	$p_2 = 6$	$p_2 = 4$	$p_2 = 1$
$p_1 = 6$	$6, 2$	$0, 0$	$0, -21$
$p_1 = 4$	$16, 0$	$16, 0$	$0, -21$
$p_1 = 1$	$7, 0$	$7, 0$	$7, 0$

Payoff Structure

Firm 1's choice of price determines the matrix, and firms 2_l and 2_h's prices determine the row and column of the chosen matrix.

Firm 1 chooses $p_1 = 6$

	$p_h = 6$	$p_h = 4$	$p_h = 1$
$p_l = 6$	$6, 5, 2$	$3, 5, 0$	$3, 5, -21$
$p_l = 4$	$3, 12, 2$	$0, 12, 0$	$0, 12, -21$
$p_l = 1$	$3, 0, 2$	$0, 0, 0$	$0, 0, -21$

Firm 1 chooses $p_1 = 4$

	$p_h = 6$	$p_h = 4$	$p_h = 1$
$p_l = 6$	$16, 0, 0$	$16, 0, 0$	$8, 0, -21$
$p_l = 4$	$12, 6, 0$	$12, 6, 0$	$4, 6, -21$
$p_l = 1$	$8, 0, 0$	$8, 0, 0$	$0, 0, -21$

Firm 1 chooses $p_1 = 1$

	$p_h = 6$	$p_h = 4$	$p_h = 1$
$p_l = 6$	$7, 0, 0$	$7, 0, 0$	$7, 0, 0$
$p_l = 4$	$7, 0, 0$	$7, 0, 0$	$7, 0, 0$
$p_l = 1$	$7, 0, 0$	$7, 0, 0$	$7, 0, 0$

Finding the Bayesian-Nash Equilibrium

To identify the Bayesian-Nash equilibrium, we analyze the Nash equilibria of the strategic form game.

Notice that for firm $2l$ setting $p_l = 4$ is a weakly dominant strategy. So is setting $p_h = 6$ for firm $2h$.

That is the game would be reduced to:

$p_1 = 6$

	$p_h = 6$	$p_h = 4$
$p_l = 6$	$6, 5, \underline{2}$	$3, 5, 0$
$p_l = 4$	$3, \underline{12}, \underline{2}$	$0, \underline{12}, 0$

$p_1 = 4$

	$p_h = 6$	$p_h = 4$
$p_l = 6$	$\underline{16}, 0, \underline{0}$	$\underline{16}, 0, \underline{0}$
$p_l = 4$	$\underline{12}, \underline{6}, \underline{0}$	$\underline{12}, \underline{6}, \underline{0}$

$p_1 = 1$

	$p_h = 6$	$p_h = 4$
$p_l = 6$	$7, \underline{0}, \underline{0}$	$7, \underline{0}, \underline{0}$
$p_l = 4$	$7, \underline{0}, \underline{0}$	$7, \underline{0}, \underline{0}$

Finding the Bayesian- Nash Equilibrium

Firm 1 has a dominant strategy of setting $p_1 = 4$. If Firm 1 sets $p_1 = 4$, it secures a payoff of 16 when Firm 2 is of type (2_h) and chooses $p_h = 6$, and a payoff of 12 when Firm 2 is of type (2_l) and also sets $p_l = 4$.

Thus, the Bayesian Nash equilibrium occurs when Firm 1 sets a price of 4, Firm 2_h sets a price of 6, and Firm 2_l also sets a price of 4.

Revisiting Mixed Strategies

Recall that a mixed strategy for player j can be interpreted as representing player i's uncertainty about player j's choices. This uncertainty may arise either from player j randomizing their actions or from player i lacking knowledge about player j's type.

Thus, we can typically view a mixed-strategy Nash equilibrium as the Bayesian Nash equilibrium of an associated game with incomplete information that is structurally similar.

The battle of programming languages

Let's go back to our example of the battle of programming languages.

Of course, because Alba and Joël are Ph.D. friends -and that's a long journey- it was reasonable to believe that both knew each other preferences about programming languages.

Suppose instead is the very first day of the Ph.D., so they don't know the other's programming skills yet and they are working together on the Problem Set.

Assume the uncertainty is captured as follows:

- Payoff for Alba is both use Python is $2 + t_a$
- Payoff for Joël is both use Julia is $2 + t_j$
- t_a and t_j are independent draws from a uniform distribution on $[0, x]$ (i.e., small perturbations of the original game).
- The last part is common knowledge

The battle of programming languages

The normal form of the Bayesian game is now

- Strategies space $= S_A = S_j = \{Py, Ju\}$
- Type space $= T_a = T_j = [0, x]$
- Beliefs: $p_a(t_a) = p_j(t_j) = \frac{1}{x}$ (i.e. proba. density function of a uniform on $[0, x]$).

	Joël (Py)	Joël (Ju)
Alba (Py)	$2 + t_a, 1$	$0, 0$
Alba (Ju)	$0, 0$	$1, 2 + t_J$

The battle of programming languages

We will find a pure-strategy Bayesian Nash Equilibrium in which Alba chooses Python if t_a is greater than a value c, and chooses Julia otherwise. Similarly, Joël chooses Julia if t_j is higher than a value p.

- Probability that Alba chooses Python: $P(t_a \geq c) = 1 - \frac{c}{x} = \frac{x-c}{x}$
- Probability that Joël chooses Julia: $P(t_j \geq p) = 1 - \frac{p}{x} = \frac{x-p}{x}$

The idea is to find the values of c and x such that this is a Bayesian Nash Equilibrium.

The battle of programming languages

- Given the other strategy, the expected utility for Alba and Joël from each pure strategy is:

$$
\begin{aligned}
EU_A(Py, p) &= \frac{p}{x}(2 + t_a) + (1 - \frac{p}{x})0 = \frac{p}{x}(2 + t_a), \\
EU_A(Ju, p) &= \frac{p}{x}(0) + (1 - \frac{p}{x}) = 1 - \frac{p}{x}, \\
EU_J(c, Ju) &= (1 - \frac{c}{x})(0) + (\frac{c}{x})(2 + t_j) = \frac{c}{x}(2 + t_j), \\
EU_J(c, Py) &= (1 - \frac{c}{x})(1) = 1 - \frac{c}{x}.
\end{aligned}
$$

- This implies the following system of equations:

$$
\begin{aligned}
t_a &\geq \frac{x}{p} - 3 = c \\
t_j &\geq \frac{x}{c} - 3 = p.
\end{aligned}
$$

The battle of programming languages

Solving this system of equations implies:

$$\begin{aligned} x - 3p &= cp \\ x - 3c &= cp \qquad \Rightarrow -3p + 3c = 0 \Leftrightarrow c = p. \end{aligned}$$

Solving $p^2 + 3p - x = 0$ therefore and substituting in the probability that each chooses their preferred language (i.e., $\frac{x-c}{x}$ for Alba to choose Python and $\frac{x-p}{x}$ for Joël to choose Julia):

$$1 - \frac{-3 + \sqrt{9 + 4x}}{2x},$$

The battle of programming languages

Multiply both numerator and denominator by $3 + \sqrt{9 + 4x}$ to rewrite the expression as:

$$\frac{4}{3 + \sqrt{9 + 4x}}$$

Notice therefore than when x goes to zero (i.e., there's no private information), the behaviour of Alba and Joël is back to the Mixed-Strategy Nash equilibrium in the original game with complete information.