2

STOCHASTIC INTEREST RATE MODELS

2.1. Continuous Time Finance Recap

Note: Please see Hull (2018), Chap.14.

- Stochastic process – any variable whose value changes over time in an uncertain way => different random trajectories for the variable.

- Discrete vs continuous time stochastic processes:
  - Discrete – the variable value can change only at certain fixed points in time
  - Continuous – changes can take place at any time

- Continuous vs discrete variables:
  - Discrete – only certain values are possible
  - Continuous – can take any value within a certain range

- Continuous-variable, continuous-time – variables can assume any value and changes can occur at any time.

STOCHASTIC PROCESSES

- **Continuous-variable, continuous-time stochastic processes are key to understanding the pricing of options and other derivatives.**
- However, **in practice, most asset prices do not follow continuous-variable, continuous-time stochastic processes.**
- For instance, stock prices are restricted to discrete values (e.g. multiples of a cent) and changes can be observed only when the markets are open.
- Nonetheless, continuous-variable, continuous-time stochastic processes are useful for many valuation purposes.
- **Markov Stochastic Process** – stochastic process where **only the current value of a variable is relevant for predicting the future** => **all past information is irrelevant**, as it is already incorporated into today's stock price (**market efficiency**).

- The probability distribution at any particular future time is independent from the path followed by the variable in the past.
- If market efficiency didn't hold, market participants could make above-average returns by interpreting the past behavior of asset prices.

STOCHASTIC PROCESSES

- Assuming a Markov process $X(t)$, the 1-year change ~ $\mathcal{N}(0,1)$.

- 2-year change = $\mathcal{N}(0,1)$ + $\mathcal{N}(0,1)$ = $\mathcal{N}(0,2)$, as both distributions are independent - given that this is a Markov process, the 2nd distribution does not depend on the 1st.

Δt (very small period of time) change ~ $\mathcal{N}(0, \Delta t)$

The expected value of any future outcome is equal to the current value (Martingale): z=25 => 1 year after,   z ~ N(25,1); 5 years after, z ~ N(25,5)

WIENER PROCESS

Wiener Process – a particular type of Markov process with:

- a mean change = 0
- a variance rate (per year) =1

A stochastic process z follows a **Wiener process (or the continuous random walk)** if it has the following properties:

Property 1. *The change Δz during a small period of time Δt is*

$$\boxed{\Delta z = \epsilon\sqrt{\Delta t}} \tag{14.1}$$

where ϵ has a standard normal distribution $\phi(0,1)$.

Property 2. *The values of Δz for any two different short intervals of time, Δt, are independent.*

It follows from the first property that Δz itself has a normal distribution with

$$\text{mean of } \Delta z = 0$$

$$\text{standard deviation of } \Delta z = \sqrt{\Delta t}$$

$$\text{variance of } \Delta z = \Delta t$$

The second property implies that z follows a Markov process

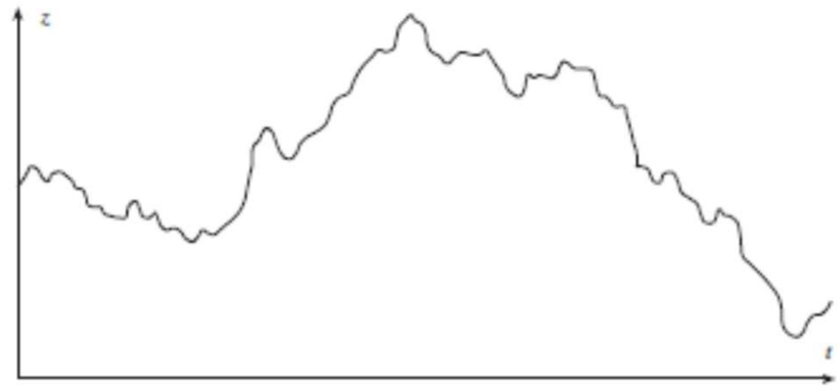
Hull, John (2018), "Options, Futures and Other Derivatives", Pearson Prenctice Hall, 10th Edition

WIENER PROCESS

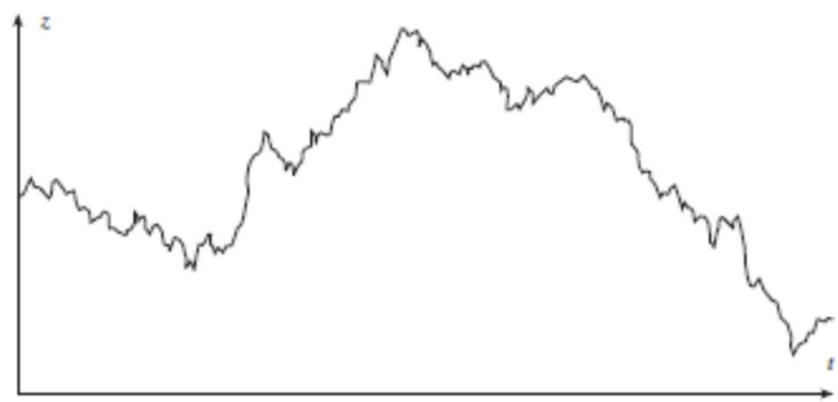
Wiener processes for different magnitudes of change in time:

When Δt -> 0, the path becomes much more irregular, as the size of the movement in the variable in time Δt is proportional to the $\sqrt{\Delta t}$. When Δt is small, $\sqrt{\Delta t}$ is much larger than Δt => the changes in z will be much larger than Δt, as $\Delta z = \epsilon\sqrt{\Delta t}$

Relatively large value of Δt



Smaller value of Δt



The true process obtained as $\Delta t \to 0$

Source: Hull, John (2018), “Options, Futures and Other Derivatives”, Pearson Prenctice Hall, 10th Edition

Generalized Wiener Process

- Instead of a drift = 0 and a variance rate = 1 as in the Wiener process (*dz*), we may have a stochastic process **where the drift can assume any value *a* and the variance rate can be b^2** => **Generalized Wiener Process**.

$$\Delta x = a\,\Delta t + b\epsilon\sqrt{\Delta t} \qquad \text{where } a \text{ and } b \text{ are constants.}$$

- In the Wiener Process, a=0 and b=1: $\Delta z = \epsilon\sqrt{\Delta t}$

- For very small time changes Δt: $\Delta x = a\,\Delta t + b\epsilon\sqrt{\Delta t}$

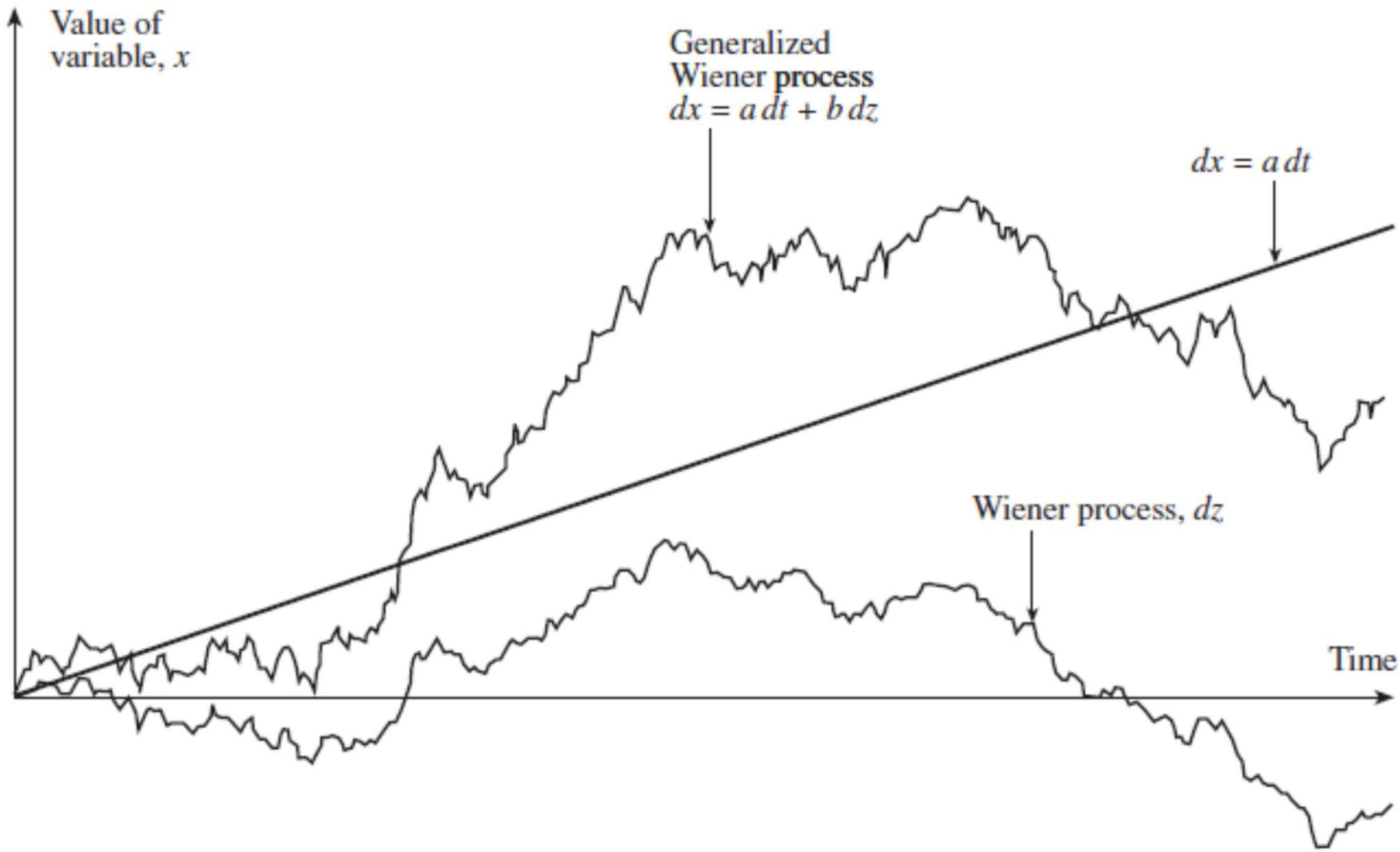
Δx ~ N, with

$$\text{mean of } \Delta x = a\,\Delta t$$
$$\text{standard deviation of } \Delta x = b\sqrt{\Delta t}$$
$$\text{variance of } \Delta x = b^2\,\Delta t$$

- The average increases in *x* are proportional to time (if there is no drift, i.e. a = 0, the mean of *x* doesn't change, i.e. Δx = 0).

GENERALIZED WIENER PROCESS

Figure 14.2 Generalized Wiener process with $a = 0.3$ and $b = 1.5$.



Source: Hull, John (2018), “Options, Futures and Other Derivatives”, Pearson Prenctice Hall, 10th Edition

Itô Process

- Definition: Generalized Wiener process with average and standard-deviation as a function of the underlying variable and time (instead of being only a function of time):

$$dx = a(x,t)\,dt + b(x,t)\,dz$$

$$\Delta x = a\,\Delta t + b\epsilon\sqrt{\Delta t}$$

- For small time intervals, we may assume that the average and the standard-deviation don't change (we're assuming that the drift and the variance rate don't change between *t* and *t+Δt*): $\Delta x = a(x,t)\Delta t + b(x,t)\epsilon\sqrt{\Delta t}$

- **This is still a Markov process, as *a* and *b* only depend on the current value of x, not on previous values.**
- It may be tempting to assume that a stock price follows a Generalized Wiener process (constant drift and variance).
- However, this assumption is not valid, having in mind that investors require or expect a given level of returns (as a % variation) regardless the price level, i.e. for higher prices, expected changes will also be higher.
- One can acknowledge this fact by replacing the assumption of constant expected drift by the assumption of constant expected returns (i.e. constant expected drift divided by the stock price ⇔ variable drift along time).

ITÔ PROCESS

- Expected drift rate in $S = a(x,t)) = \mu S$

  (being S the stock price at time t and μ the expected rate of return of the stock, expressed in decimal form, for a time unit = 1).

- For a short interval of time Δt, the expected increase in S is $\mu S \Delta t$, i.e. the expected rate of return on the stock, times the stock price, times the time interval:

$$\boxed{\Delta S = \mu S \Delta t}$$

$$\Delta x = a(x,t)\Delta t + b(x,t)\epsilon\sqrt{\Delta t}$$

- If Δ**t -> 0 =>** $$dS = \mu S dt \Leftrightarrow \frac{dS}{S} = \mu dt$$

- This corresponds to the price of an asset following a continuously compounding process (under no uncertainty, being μ = risk-free rate in a risk-neutral world)

$$S_T = S_0 e^{\mu T}$$

Geometric Brownian Motion

- Given that in practice there is uncertainty, a reasonable assumption is that the variability of the percentage return (σ) in a short period of time Δt is the same regardless the stock price.

- An investor is as uncertain about his return when the stock price is high or low.

- Accordingly, the standard deviation of the change in a short period of time must also be proportional to the stock price, as the standard deviation for the percentage change is constant – **Geometric Brownian Motion**:

$$dS = \mu S dt + \sigma S dz \Leftrightarrow$$

$$\frac{dS}{S} = \mu dt + \sigma dz$$

Geometric Brownian Motion

- Example:

Consider a stock that pays no dividends, has a volatility of 30% per annum, and provides an expected return of 15% per annum with continuous compounding. In this case, $\mu = 0.15$ and $\sigma = 0.30$. The process for the stock price is

$$\frac{dS}{S} = 0.15\,dt + 0.30\,dz$$

If S is the stock price at a particular time and ΔS is the increase in the stock price in the next small interval of time, the discrete approximation to the process is

$$\frac{\Delta S}{S} = 0.15\Delta t + 0.30\epsilon\sqrt{\Delta t}$$

where ϵ has a standard normal distribution. Consider a time interval of 1 week, or 0.0192 year, so that $\Delta t = 0.0192$. Then the approximation gives

$$\frac{\Delta S}{S} = 0.15 \times 0.0192 + 0.30 \times \sqrt{0.0192}\,\epsilon$$

or

$$\boxed{\Delta S = 0.00288S + 0.0416S\epsilon}$$

Source: Hull, John (2018), "Options, Futures and Other Derivatives", Pearson Prenctice Hall, 10th Edition

GEOMETRIC BROWNIAN MOTION

- Monte Carlo simulation:

A path for the stock price over 10 weeks can be simulated by sampling repeatedly for ϵ from $\phi(0,1)$ and substituting into equation (14.10). The expression =RAND() in Excel produces a random sample between 0 and 1. The inverse cumulative normal distribution is NORMSINV. The instruction to produce a random sample from a standard normal distribution in Excel is therefore =NORMSINV(RAND()). Table 14.1 shows one path for a stock price that was sampled in this way. The initial stock price is assumed to be \$100. For the first period, ϵ is sampled as 0.52. From equation (14.10), the change during the first time period is

$$\frac{\Delta S}{S} = 0.15 \times 0.0192 + 0.30 \times \sqrt{0.0192}\,\epsilon$$

$$\Delta S = 0.00288S + 0.0416S\epsilon$$

Table 14.1 Simulation of stock price when $\mu = 0.15$ and $\sigma = 0.30$ during 1-week periods.

Stock price at start of period	*Random sample for ϵ*	*Change in stock price during period*
100.00	0.52	2.45
102.45	1.44	6.43
108.88	−0.86	−3.58
105.30	1.46	6.70
112.00	−0.69	−2.89
109.11	−0.74	−3.04
106.06	0.21	1.23
107.30	−1.10	−4.60
102.69	0.73	3.41
106.11	1.16	5.43
111.54	2.56	12.20

Source: Hull, John (2018), "Options, Futures and Other Derivatives", Pearson Prenctice Hall, 10th Edition

Itô's Lemma

- An option price (G) is a function of the underlying asset's price and time.

- It is key to understand the **behavior of functions of stochastic variables**.
- An important result was discovered by K. Itô in 1951 and is known as **Itô's lemma**.

[6] See K. Itô, "On Stochastic Differential Equations," *Memoirs of the American Mathematical Society*, 4 (1951): 1–51.

- Assuming that a variable x follows an Itô process:

$$dx = a(x,t)\,dt + b(x,t)\,dz$$

where dz is a Wiener process and a and b are functions of x and t. The variable x has a drift rate of a and a variance rate of b^2. Itô's lemma shows that a function G of x and t follows the process

$$dG = \left(\frac{\partial G}{\partial x}a + \frac{\partial G}{\partial t} + \frac{1}{2}\frac{\partial^2 G}{\partial x^2}b^2\right)dt + \frac{\partial G}{\partial x}b\,dz$$

ITÔ'S LEMMA

- Thus, G also follows an Itô process with a drift rate = $\frac{\partial G}{\partial x}a+\frac{\partial G}{\partial t}+\frac{1}{2}\frac{\partial^2 G}{\partial x^2}b^2$ and a variance rate of $\left(\frac{\partial G}{\partial x}\right)^2 b^2$

$$dx = a(x,t)\,dt + b(x,t)\,dz$$

- Assuming that the stock price follows a Geometric Brownian Motion, with constant μ and σ: $dS = \mu S dt + \sigma S dz$

- Applying the Ito's Lemma to the previous equation:

$$dG = \left(\frac{\partial G}{\partial x}a+\frac{\partial G}{\partial t}+\frac{1}{2}\frac{\partial^2 G}{\partial x^2}b^2\right)dt+\frac{\partial G}{\partial x}b\,dz$$

$$dG = \left(\frac{\partial G}{\partial S}\mu S+\frac{\partial G}{\partial t}+\frac{1}{2}\frac{\partial^2 G}{\partial S^2}\sigma^2 S^2\right)dt+\frac{\partial G}{\partial S}\sigma S\,dz$$

$$\Downarrow$$

- Therefore, both *S* and *G* are affected by the same volatility source – *dz*.
- In the Black-Scholes option pricing formula, G (the option price) is determined by the instantaneous volatility of the returns of the underlying asset price.

APPLICATION TO FORWARD CONTRACTS

- Forward: $F_0 = S_0 e^{rT}$

- Forward at *t*: $F = Se^{r(T-t)}$

- Ito's Lemma => $dG = \left(\frac{\partial G}{\partial S} \mu S + \frac{\partial G}{\partial t} + \frac{1}{2} \frac{\partial^2 G}{\partial S^2} \sigma^2 S^2 \right) dt + \frac{\partial G}{\partial S} \sigma S \, dz$

- The stochastic process of *F* can be defined calculating the derivatives of *F* in order to *S* and *t* (i.e. *F* now corresponds to *G*):

$$\frac{\partial F}{\partial S} = e^{r(T-t)}, \qquad \frac{\partial^2 F}{\partial S^2} = 0, \qquad \frac{\partial F}{\partial t} = -rSe^{r(T-t)}$$

Substituting F for $Se^{r(T-t)}$

$$dF = [e^{r(T-t)} \mu S - rSe^{r(T-t)}] \, dt + e^{r(T-t)} \sigma S \, dz$$ => $$dF = (\mu - r) F \, dt + \sigma F \, dz$$

- Like *S*, *F* follows a GMB, with the same volatility and a trend of μ-*r* (instead of μ).

PROBABILITY DISTRIBUTION

- From the stochastic process of the rate of returns,

$$\frac{dS}{S} = \mu dt + \sigma dz$$

- Its distribution gets

$$\frac{\Delta S}{S} \sim \phi(\mu\, \Delta t,\ \sigma^2 \Delta t)$$

- Assuming $G = \ln S$, since $\frac{\partial G}{\partial S} = \frac{1}{S}$, $\frac{\partial^2 G}{\partial S^2} = -\frac{1}{S^2}$, $\frac{\partial G}{\partial t} = 0$, it follows from the Itô's lemma that

$$dG = \left(\mu - \frac{\sigma^2}{2}\right) dt + \sigma\, dz$$

$$dG = \left(\frac{\partial G}{\partial S}\mu S + \frac{\partial G}{\partial t} + \tfrac{1}{2}\frac{\partial^2 G}{\partial S^2}\sigma^2 S^2\right) dt + \frac{\partial G}{\partial S}\sigma S\, dz$$

PROBABILITY DISTRIBUTION

Since μ and σ are constant, this equation indicates that $G = \ln S$ follows a generalized Wiener process. It has constant drift rate $\mu - \sigma^2/2$ and constant variance rate σ^2. The change in $\ln S$ between time 0 and some future time T is therefore normally distributed, with mean $(\mu - \sigma^2/2)T$ and variance $\sigma^2 T$. This means that

$$\ln S_T - \ln S_0 \sim \phi\left[\left(\mu - \frac{\sigma^2}{2}\right)T,\ \sigma^2 T\right]$$

or

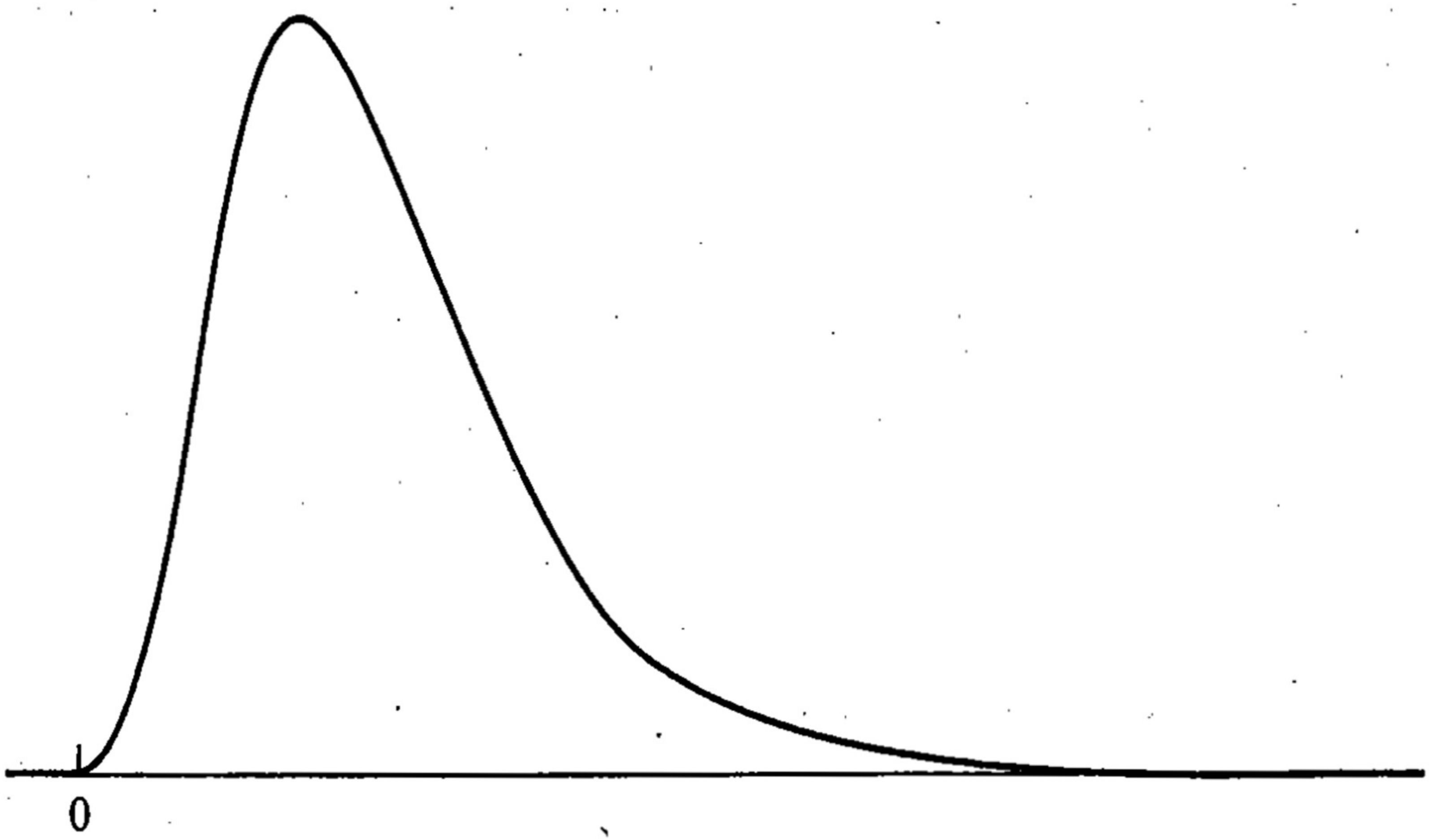
$$\ln S_T \sim \phi\left[\ln S_0 + \left(\mu - \frac{\sigma^2}{2}\right)T,\ \sigma^2 T\right]$$

- **ln S_T is normally distributed (and S_T has a log normal distribution)**, with a standard deviation $\sigma\sqrt{T}$ (proportional to the square root of time).

- **The growth rate of the asset price is normally distributed => the asset price is lognormally distributed.**

PROBABILITY DISTRIBUTION

Figure 13.1 Lognormal distribution.



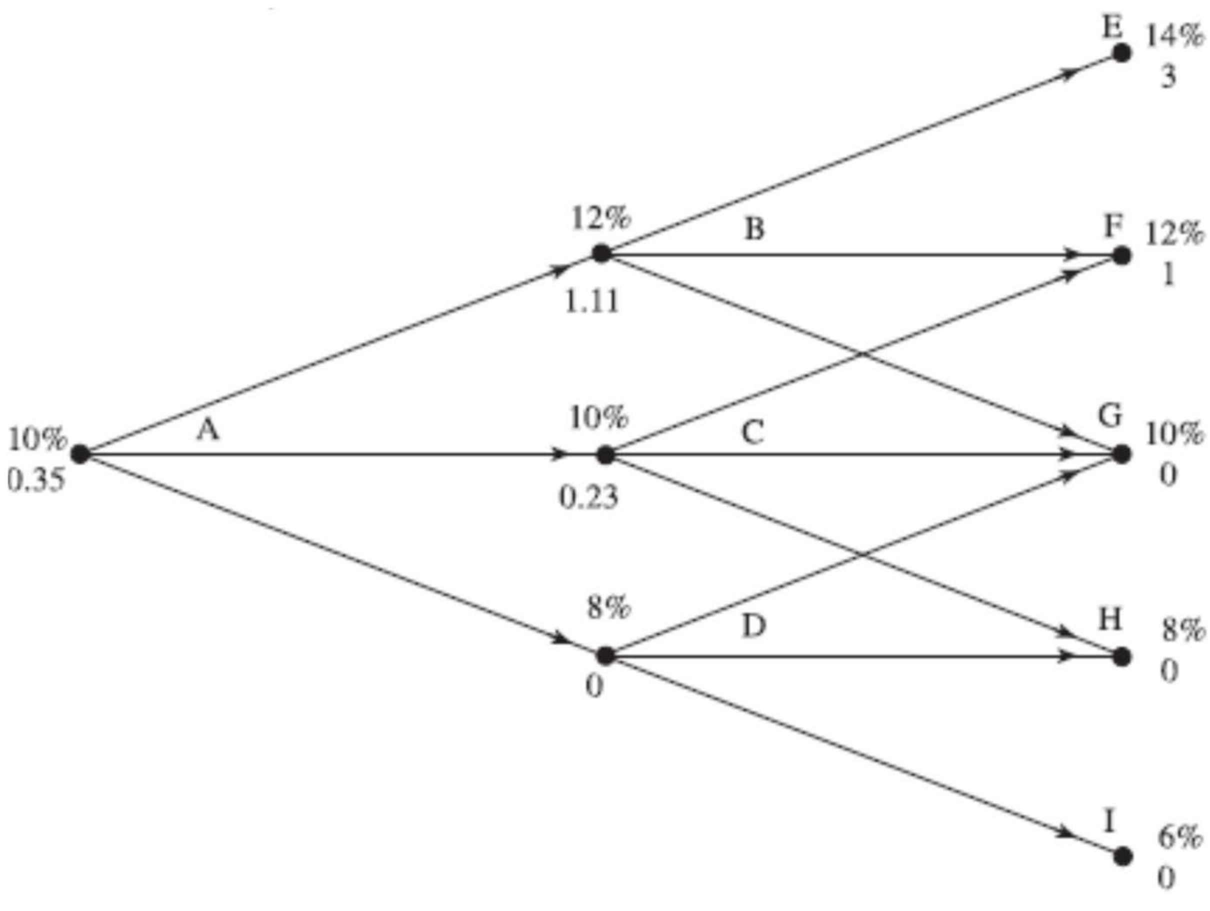
Source: Hull, John (2009), "Options, Futures and Other Derivatives", Pearson Prenctice Hall, 7th Edition

2.2. Short Rate Models

2.2.1. Interest Rate Trees

- Focus: How to model the TSIR by specifying the behavior of the short-term interest rate?
- Bond and interest rate derivative prices depend on the behavior of the risk-free short-term interest rate (or instantaneous short rate).
- The variable to be modeled by trees will be the instantaneous short rate.
- Why are trees used? a tree is a discrete-time representation of the stochastic process.
- Binomial trees are often used, even though trinomial trees are recommended to value interest rate derivatives.
- At the final nodes, the value of the derivative equals its pay-off.
- At previous nodes, the value of the derivative is calculated through a rollback procedure, calculating the expected value of the derivative according to the probabilities attached to the different scenarios and discounting this expected value using the interest rate at that node.

Figure 32.4 Example of the use of trinomial interest rate trees. Upper number at each node is rate; lower number is value of instrument.



The tree is used to value a derivative that provides a payoff at the end of the second time step of

$$\max[100(R - 0.11), 0]$$

Assumptions:

Probabilities of up, middle and down are 0.25, 0.5 and 0.25, respectively.

Derivative value at Node B:

$$[0.25 \times 3 + 0.5 \times 1 + 0.25 \times 0]e^{-0.12 \times 1} = 1.11$$

Derivative value at Node C:

$$(0.25 \times 1 + 0.5 \times 0 + 0.25 \times 0)e^{-0.1 \times 1} = 0.23$$

Derivative value at Node D = 0

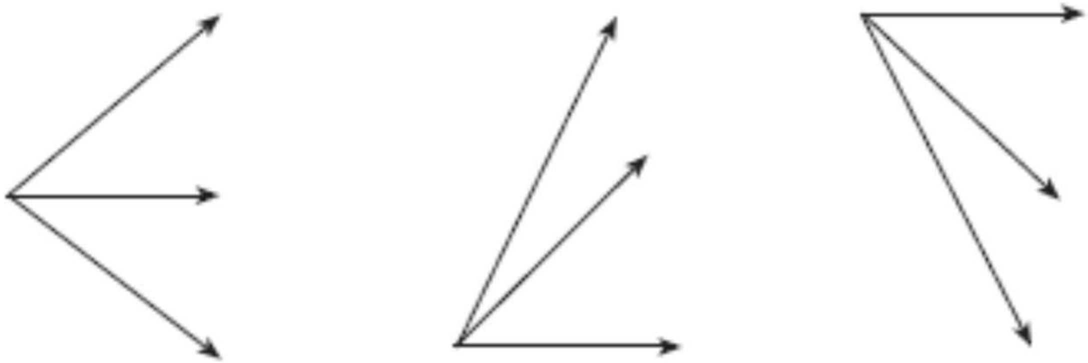
Derivative value at Node A:

$$(0.25 \times 1.11 + 0.5 \times 0.23 + 0.25 \times 0)e^{-0.1 \times 1} = 0.35$$

Source: Hull, John (2018), “Options, Futures and Other Derivatives”, Pearson Prenctice Hall, 10th Edition

Non-standard branching:

Figure 32.5 Alternative branching methods in a trinomial tree.



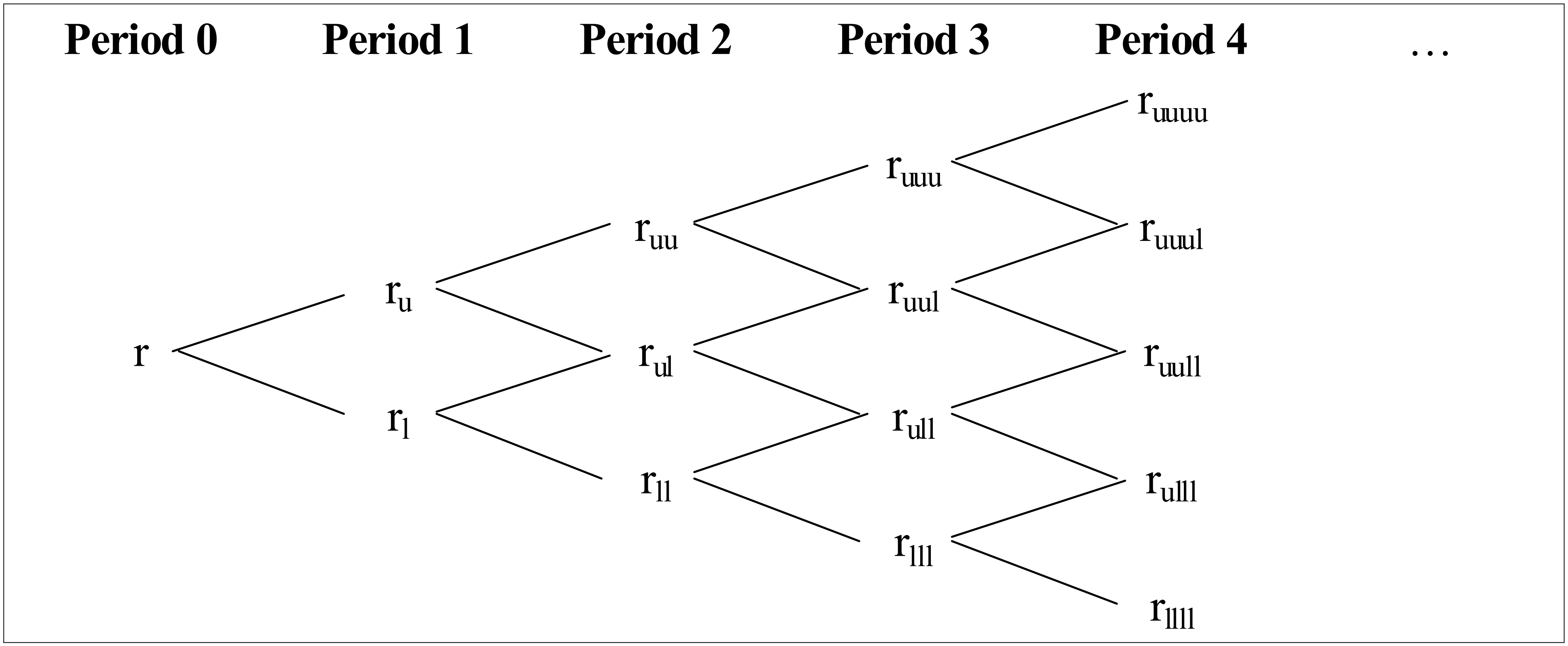
(a) (b) (c)

Source: Hull, John (2018), "Options, Futures and Other Derivatives", Pearson Prenctice Hall, 10th Edition

- (b) and (c) are useful to represent <u>mean-reverting interest rates</u> when interest rates are either very low (and are not supposed to move even lower) or very high (and are not supposed to move even higher), respectively.

- **General binomial model**
  - Given the current level of short-term rate *r*, the next-period short rate can take only two possible values: an upper value r_u and a lower value r_l, with equal probability 0.5.
  - In the following period, the short-term interest rate can take four possible values: r_{uu}, r_{ul}, r_{lu}, r_{ll} .
  - More generally, in period *n*, the short-term interest rate can take on 2^n values => very time-consuming and computationally inefficient.
- Recombining trees
  - Means that an upward-downward sequence leads to the same result as a downward-upward sequence (regardless being binomial or trinomial trees)
  - For example, $r_{ul} = r_{lu}$ => only (n+1) different values at period *n*.

INTEREST RATE TREE - RECOMBINING



INTEREST RATE TREE – ANALYTICAL

- We may write down the binomial process as:

$$\Delta r_t \equiv r_{t+1} - r_t = \sigma \varepsilon_t$$

Itô process

$$\boxed{dx = a(x,t)\,dt + b(x,t)\,dz}$$

$$\Delta r_t \equiv r_{t+\Delta t} - r_t = \mu\left(t, \Delta t, r_t\right) + \sigma\left(t, \Delta t, r_t\right)\varepsilon_t$$

- Specific case – assuming that the drift and the variance are proportional to the time increment:

$$\Delta r_t \equiv r_{t+\Delta t} - r_t = \mu \Delta t + \sigma \sqrt{\Delta t}\, \varepsilon_t$$

- Continuous-time limit (Merton (1973)):

$$dr_t \equiv r_{t+dt} - r_t = \mu dt + \sigma dW_t$$

2.2.2. CT SINGLE FACTOR MODELS

- In the initial term structure models, the short rate is the only driver of the yield curve, being assumed as a continuous and stochastic or random variable.
- Therefore, a single-factor continuous-time model specifies the dynamics of the short-term rate: $dr_t = \mu(t, r_t)dt + \sigma(t, r_t)dW_t$
- The term *W* denotes a Brownian motion - process with independent normally distributed increments:

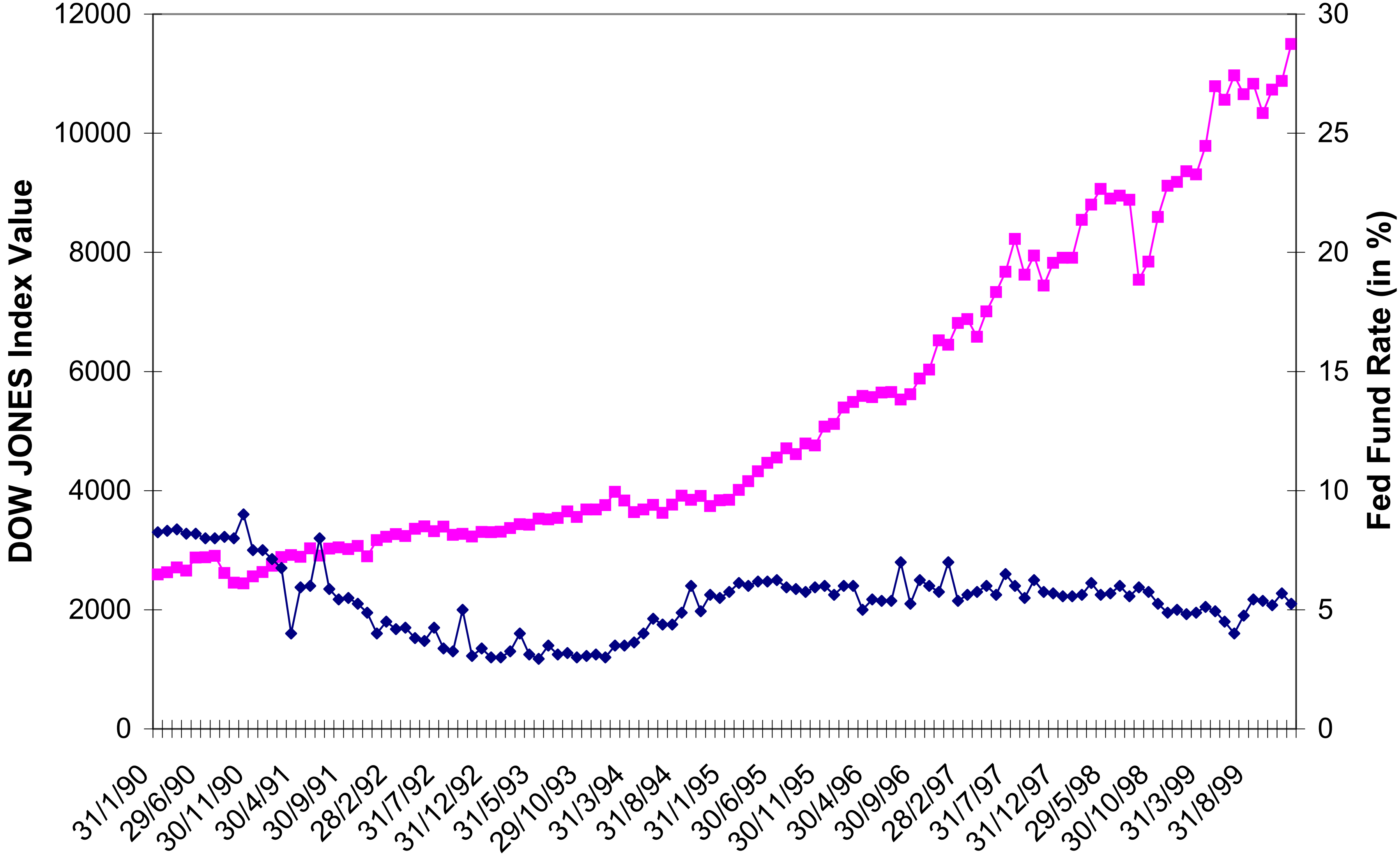
$$dW_t = \varepsilon_t \sqrt{dt}$$

  - *dW* represents the instantaneous change;
  - It is stochastic (uncertain);
  - It is a stochastic variable with a normal distribution with zero mean and variance *dt*;

- A good model is a model that is consistent with reality =>
  - Tractable
  - Parsimonious

- Stylized facts about the dynamics of the term structure:
  - Fact 1: (nominal) interest rates are (usually) positive
  - Fact 2: interest rates are mean-reverting
  - Fact 3: interest rates with different maturities are imperfectly correlated
  - Fact 4: the volatility of interest rates evolves (randomly) in time

Empirical Facts 1, 2 and 4



Note: Interest Rate in blue

EMPIRICAL FACT 3

	1M	3M	6M	1Y	2Y	3Y	4Y	5Y	7Y	10Y
1M	1									
3M	0.999	1								
6M	0.908	0.914	1							
1Y	0.546	0.539	0.672	1						
2Y	0.235	0.224	0.31	0.88	1					
3Y	0.246	0.239	0.384	0.808	0.929	1				
4Y	0.209	0.202	0.337	0.742	0.881	0.981	1			
5Y	0.163	0.154	0.255	0.7	0.859	0.936	0.981	1		
7Y	0.107	0.097	0.182	0.617	0.792	0.867	0.927	0.97	1	
10Y	0.073	0.063	0.134	0.549	0.735	0.811	0.871	0.917	0.966	1

EQUILIBRIUM VS NO-ARBITRAGE MODELS OF THE SHORT RATE

Equilibrium models:

(i) Start with assumptions about economic variables and derive a process for the short rate (*r*).

(ii) Accordingly, the initial yield curve is given by an analytical formula as a function of the short-term rate and the model parameters, assuming that the economy is in equilibrium.

(iii) The process for *r* in a one-factor equilibrium model involves only one source of uncertainty -– the (instantaneous) short-term rate itself => endogenous models.

(iv) A one-factor model implies that all rates move in the same direction over any short time interval, but not that they all move by the same amount => the shape of the zero curve can change with the passage of time.

(v) The process for the short rate is usually assumed to be stationary, in the sense that the parameters of the process are not functions of time.

(vi) If the instantaneous short rate follows a Markov process, all rates can be calculated at all times as a function of the short rate.

(vii) A shortcoming of these models is that they do not automatically fit today's term structure of interest rates.

EQUILIBRIUM VS NO-ARBITRAGE MODELS OF THE SHORT RATE

No-arbitrage models:

(i) A no-arbitrage model is designed to be exactly consistent with today's TSIR.

(ii) Essential difference between an equilibrium and a no-arbitrage model – in an equilibrium model, today's TSIR is an output, while in a no-arbitrage model it is an input.

(iii) the drift is, in general, dependent on time, as the shape of the initial spot curve governs the average path taken by the short rate in the future – positively sloped zero curve => positive drift for the short rate.

(iv) Some equilibrium models can be transformed into no-arbitrage models by including a function of time in the drift of the short rate.

EQUILIBRIUM ONE-FACTOR MODELS OF THE SHORT RATE

- Interest rate dynamics:

$$dr = m(r)\,dt + s(r)\,dz \longrightarrow$$

The drift is not a function of time, but of the interest rate itself.

- Main type of models:

$$m(r) = \mu r;\ s(r) = \sigma r \qquad \text{(Rendleman and Bartter model)}$$

$$m(r) = a(b - r);\ s(r) = \sigma \qquad \text{(Vasicek model)}$$

$$m(r) = a(b - r);\ s(r) = \sigma\sqrt{r} \qquad \text{(Cox, Ingersoll, and Ross model)}$$

EQUILIBRIUM ONE-FACTOR MODELS OF THE SHORT RATE

1. Rendleman and Bartter

- The short-term interest rate follows a GMB: $dr = \mu r\,dt + \sigma r\,dz$

Rendleman, R. and B. Bartter (1980). "The Pricing of Options on Debt Securities". *Journal of Financial and Quantitative Analysis*. **15**: 11–24).

Pros:

- More tractable model, as it follows a GMB.

Cons:

- Assumes that interest rates follow a stochastic process similar to stocks, while they usually exhibit a mean-reversion behavior.

EQUILIBRIUM ONE-FACTOR CT MODELS OF THE SHORT RATE

2. Vasicek (1977) $$dr = a(b - r)\,dt + \sigma\,dz$$

Vasicek, O., 1977, "An Equilibrium Characterization of the Term Structure," *Journal of Financial Economics*, 5, 177–188.

Also known as Hull and White (1990) model or an Ornstein–Uhlenbeck process.

Hull, J., and White, A., "Pricing Interest Rate Derivative Securities", *Review of Financial Studies*, 1990, pp. 573–592.

Pros:

- More tractable model, due to constant volatility.
- Interest rates are mean-reverting (to *b*), at a reversion rate (pace) *a*.

Cons:

- The model assumes a constant volatility, while interest rate volatility is often variable, namely during periods of higher uncertainty, when the estimation of interest rates becomes more complex but also more useful.

EQUILIBRIUM ONE-FACTOR CT MODELS OF THE SHORT RATE

3. Cox, Ingersoll and Ross (CIR)

$$dr = a(b - r)\,dt + \sigma\sqrt{r}\,dz$$

Stochastic volatility model => higher volatility with higher interest rates.

Cox, Ingersoll, and Ross. 1985, "A Theory of the Term Structure of Interest Rates", *Econometrica*, Vol 53, March.

Pros:

- Model closer to reality, as interest rates have stochastic volatilities (higher volatilities with higher interest rates).

Cons:

- Model becomes less tractable, as it requires the single factor to be >0, which is not a problem when the single factor is the short-term interest rate and this is positive, but becomes a problem if the short-term rate turns negative or the factor is a different variable.

NO-ARBITRAGE SHORT RATE CT MODELS

1. Ho-Lee (1986)

Ho, T.S.Y., and S.-B. Lee, "Term Structure Movements and Pricing Interest Rate Contingent Claims," *Journal of Finance*, 41 (December 1986): 1011–29.

$$dr = \theta(t)\,dt + \sigma\,dz$$

$\theta(t)$ defines the average direction that r moves at time t:

2. Hull-White One-Factor Model (1990)

Extended version of Vasicek, to provide an exact fit to the initial TSIR:

$$dr = [\theta(t) - ar]\,dt + \sigma\,dz \quad \text{or} \quad dr = a\left[\frac{\theta(t)}{a} - r\right]dt + \sigma\,dz$$

Corresponds to the Ho-Lee model, with mean reversion at rate a.

Hull, J. C., and A. White, "Pricing Interest Rate Derivative Securities," *The Review of Financial Studies*, 3, 4 (1990): 573–92.

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3. Black-Derman-Toy (1990)

Black, F., E. Derman, and W. Toy, "A One-Factor Model of Interest Rates and Its Application to Treasury Bond Prices," *Financial Analysts Journal*, January/February 1990: 33–39.

$$d \ln r = [\theta(t) - a(t) \ln r]\, dt + \sigma(t)\, dz$$

with $a(t) = -\frac{\sigma'(t)}{\sigma(t)}$ and $\sigma'(t)$ is the derivative of σ with respect to *t*.

- It is similar to Hull-White One-Factor Model, but in logs and with mean reversion rate *a* being time-dependent.
- It doesn't allow negative interest rates.

Constant volatility => $\sigma'(t)$ = 0 => a(t)=0 => BDT model: $d \ln r = \theta(t)\, dt + \sigma\, dz$

Log-normal version of Ho-Lee model

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4. Black-Karasinski (1991)

Black, F., and P. Karasinski, "Bond and Option Pricing When Short Rates Are Lognormal," *Financial Analysts Journal*, July/August (1991): 52–59.

- Extended version of BDT (1990) model, where the reversion rate and volatility are determined independently of each other:

$$d\ln r = [\theta(t) - a(t)\ln r]\,dt + \sigma(t)\,dz$$, with $a(t) = -\dfrac{\sigma'(t)}{\sigma(t)}$ - BDT

- The model is the same as BDT (1990), but with no relation between *a(t)* and σ*(t)*.
- As in practice *a(t)* and σ*(t)* are often assumed to be constant, the model becomes:

$$d\ln r = [\theta(t) - a\ln r]\,dt + \sigma\,dz$$

2.2.3. CT Multi Factor Models

1. Fong and Vasicek (1991) model - short rate and its volatility (v) as two-state variables

H. G. Fong and O. A. Vasicek: Fixed-income volatility management. Journal of Portfolio Management, 41-56, 1991.

$$dr = \alpha(\bar{r} - r)dt + \sqrt{v}dz_1$$

$$dv = \gamma(\bar{v} - v)dt + \xi\sqrt{v}dz_2$$

2. Longstaff and Schwartz (1992) model

Longstaff, F. A. and E. S. Schwartz, "Interest Rate Volatility and the Term Structure: A Two Factor General Equilibrium Model," *Journal of Finance*, 47, 4 (September 1992): 1259–82.

- Longstaff and Schwartz (1992) uses the same two-state variables (the short rate and its volatility), but with a different specification, as the drift is governed by two factors or state variables, while the variance is a function of only one of them:

$$\frac{dQ}{Q} = (\mu X + \theta Y)\, dt + \sigma\sqrt{Y}\, dZ_1$$

- With this specification, it is ensured that the drift and the variance are not perfectly correlated.

- The dynamics of the state variables are as follows:

$$dX = (a - bX)\, dt + c\sqrt{X}\, dZ_2$$

$$dY = (d - eY)\, dt + f\sqrt{Y}\, dZ_3 :$$

3. Balduzzi et al. (1996) models

Balduzzi, P., S. R. Das, S. Foresi, and R. Sundaran, 1996, "A Simple Approach to Three-Factor Affine Term Structure Models," *The Journal of Fixed Income*, 6, 14–31.

- Balduzzi *et al.* (1996) suggest the use of a 3-factor model by adding the mean of the short-term rate (θ) to a 2-factor model.

$$dr = \mu_r(r, \theta, t)dt + \sigma_r(r, V, t)dz$$

$$d\theta = \mu_\theta(\theta, t)dt + \sigma_\theta(\theta, t)dw$$

$$dV = \mu_V(V, t)dt + \sigma_V(V, t)dy$$

$$dr = \kappa(\theta - r)dt + \sqrt{V}dz$$

$$d\theta = \alpha(\beta - \theta)dt + \eta dw$$

$$dV = a(b - V)dt + \phi\sqrt{V}dy$$