2 Duality and Sensitivity Analysis

1. A cautious investor, after a difficult experience with real estate-based funds, has recovered 25000 monetary units (m.u.) and now wants to invest this amount over a certain period. This time, the investor's main goal is to **minimize risk**, while still achieving a minimum return of 2000 m.u. by the end of the investment period. After analyzing two available financial products, the investor builds the following linear programming model:

$$\begin{array}{rll} \min & z = x_1 + 2x_2 & \text{(overall risk index)} \\ \text{s. to} & x_1 + x_2 \leq 25 & \text{(budget constraint)} \\ & 0.5x_1 + 0.8x_2 \geq 2 & \text{(minimum return constraint)} \\ & x_1 \geq 0, \quad x_2 \geq 0 & \text{(non-negativity)} \end{array}$$

where x_1 is the amount (in thousands of m.u.) to invest in Product 1, and x_2 is the amount (in thousands of m.u.) to invest in Product 2.

   (a) Graphically solve the linear programming problem. Present and interpret the optimal values of the decision variables and the slack/surplus variables.

   (b) Formulate the dual of the problem and determine its optimal solution (only the decision variables). You may use the primal solution and resolution to support your answer.

   (c) Determine the range of values for the right-hand side coefficients (budget and return) for which the current optimal basis remains unchanged.

   (d) Determine the allowable range for the objective function coefficients such that the current solution remains optimal.

2. In factory Choco, three new types of chocolate bars are going to be made for the food industry. Each bar is made of sugar and chocolate only. The table below shows the quantity of sugar and chocolate (in kg) and the profit (in monetary units) of each chocolate bar.

Bar	Sugar (kg/bar)	Chocolate (kg/bar)	Profit (m.u./bar)
Type 1	1	2	3
Type 2	1	3	7
Type 3	1	1	5

The factory has 50 kg of sugar and 100 kg of chocolate available. To formulate the problem, we define variables x_j, representing the number of chocolate bars Type j to make, where $j = 1, 2, 3$. Answer to the following questions using, when needed, the Solver/Excel to find the solution of the LP problems.

   (a) For which unit profit values of chocolate bars of Type 2 does the current solution remains optimal? Which will be the optimal solution in case the unit profit is 13 m.u.?

   (b) Is it worth considering an increase in the availability of sugar?

(c) For which amount of sugar is the set of basic variables in the optimal solution the same?

(d) Is it worth considering an increase in the availability of chocolate?

(e) For which amount of chocolate is the set of basic variables in the optimal solution the same?

(f) If the amount of sugar available was of 60 kg, which would be the total profit of these products? Which should be the production plan that Choco should apply in these conditions?

(g) Repeat the previous question for an availability of sugar of 40 kg and 30 kg.

3. A humanitarian organization intends to plan a program for medicines distribution in two regions located in the Great Lakes area of Africa. For strategic and security purposes it is possible to use 3 airports from which, by land routes, the supply of the two regions will take place. Considering that the transportation cost of medicaments to the airports should be minimized, insuring, in each of the two regions, a minimum number of people is contemplated by the program, the following LP model has been formulated:

$$\begin{aligned} \min z = 40x_1 + 18x_2 + 30x_3 \quad & \text{(in m.u.)} \\ \text{s.t. } 4x_1 + x_2 + x_3 \geq 250 \quad & \text{(thousands of people)} \\ 4x_1 + 3x_2 + 6x_3 \geq 350 \quad & \text{(thousands of people)} \\ x_1, x_2, x_3 \geq 0 & \end{aligned}$$

where x_j represents the tons of medicaments to be shipped to airport $j = 1, 2, 3$.

(a) Obtain the optimal solution for the problem using the Solver/Excel.

(b) Write a short report presenting the problem's solution, referring the value of the dual decision variables as well as its meaning.

(c) Consider that the number of thousands of people to be contemplated in the first region is 350 instead of 250, which will be the new cost for the program?

(d) Determine the changes in the solution if airport 1 cannot receive more than 40 ton. of medicaments.

Some solutions

2.1 $x^* = (4, 0)$, $z^* = 4 = w^*, y^* = (0, 2)$.

2.2 $x^* = (0, 25, 25)$, $z^* = 300 = w^*, y^* = (4, 1)$.

2.3 $x^* = (57.5; 0; 20)$, $z^* = 2900 = w^*, y^* = (6, 4)$.