

FTF - Foundations of Financial Theory

MASTER IN FINANCE 2025 – 2026

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Course Details

Main Reference Book

Asset Pricing: John H. Cochrane: ISBN 9780691121376

Evaluation method 100% Exam

	11/05/2026		09h → 12h	
	02/06/2026		09h → 12h	

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Lectures

Fridays | 8h → 11h | F2 109

Why Should You Care About SDFs? 1/3

1. Foundational for Financial Economics

- Core to understanding asset pricing theory.
- Describes risk premiums in a comprehensive, rigorous way.
- Bridges the gap between macroeconomics and finance.
- Flexible framework that can be applied to many problems.

2. Multiple Practical Applications

- Values bonds, equities and currencies.
- Values financial derivatives, bonds, and complex structured products.
- Values insurance, pension funds, companies and their projects.
- Essential for UNBIASED Monte Carlo simulations and computational finance.

3. Career Relevance

- Investment banks, hedge funds, and central banks require knowledge of SDF models.
- Integral to fairly priced models used in risk management, and regulatory frameworks.

4. Research Opportunities

- Uncover historical patterns in risk premia.
- Exploring alternative models for SDFs can lead to innovations in financial modelling.
- Topics like mean-reverting processes and multi-currency models provide fertile ground for thesis work.

5. SDFs Change how you think about how You Price Assets

- Forces internal consistency across markets.
- Prevents arbitrage by construction.
- Clarifies which risks are priced vs. diversifiable.
- Provides a benchmark to test empirical anomalies.

Takeaway

If you understand the SDF, you understand *why* assets earn the returns they do.

Stochastic Discount Factors (SDF)

- The stochastic discount factor (SDF) is a stochastic process that discounts a projected future cashflow to give a present value.
- The SDF process takes a different value at each time point and depends on the state of the economy at that time.
- Only one SDF process is applied to all cashflows, pricing all assets simultaneously.

Terminology - The SDF is also referred to as:

- State price density
- State price kernel
- Marginal rate of substitution
- State price deflator^a (Popular in Actuarial literature)

^aNot to be confused with the economic, GDP deflator which measures historical changes in prices.

Asset Pricing Theory

Core Idea

Price = Expected Discounted Payoff $P = E[mx]$

Types of Pricing Approaches

- **Absolute Pricing:**
 - Price each asset by reference to exposure of fundamental factors.
 - Examples: Consumption-based models, general equilibrium models (most common).
- **Relative Pricing:**
 - Learn about asset values given the prices of other assets.
 - Example: Black-Scholes model.

Basic Pricing Equation

$$P_t = E [m_{t+1}x_{t+1}], \quad m_{t+1} = f(\text{data, parameters})$$

Discount Factor Concept

Definition

A discount factor is a random variable that generates prices from payoffs.

Law of One Price (LOOP)

If two portfolios have the same payoff in every state of nature, they must have the same price. **Key Point:** A discount factor exists if the law of one price holds.

Absence of Arbitrage

If payoff A is at least as good as B and sometimes better, then:

$$\text{Price of } A > \text{Price of } B$$

Key Point

A positive discount factor prices all payoffs if there are no arbitrage opportunities.

Introduction to State prices

Contingent Claims

What is a Contingent Claim?

A contingent claim is an asset that delivers a payoff only if certain conditions occur. It pays a predefined amount in the future if the world enters a specific state, where a state can represent any collection of events.

Example: Put Option

The payoff claimed by the holder of a put option on a share is contingent on the share price falling below a predefined value.

Everything can be made from Contingent Claims

Since contingent claims can cover any event, all assets can be redefined in terms of contingent claims.

What Are State Assets?

Introduced by [Arrow and Debreu, 1954] State assets are a special type of contingent claim asset that:

- Pay 1 unit of currency if a particular state ω occurs.
- Pay 0 if any other state occurs.

State Price Definition

Let $\Psi(\omega)$ be the current price of a state asset that pays 1 only if future state ω occurs.

- $\Psi(\omega)$ is called the state price of a ω state asset. Ψ is referred to as the state price.
- At time ω :
 - $\Psi(\omega) = 1$ if state ω occurs.
 - $\Psi(\omega) = 0$ otherwise.

State Assets Are Probabilistic LEGO

State assets can act as building blocks.

A portfolio of state assets can replicate the payoff of any contingent claim asset in all future states.

Recalling the probability triple $(\Omega, \mathcal{F}, \mathbb{P})$

Probability Spaces consist of three components:

- **Sample Space** (Ω):
 - The set of all possible outcomes of a random experiment.
 - Example: Rolling a die, $\Omega = \{1, 2, 3, 4, 5, 6\}$.
- **Sigma-algebra** (\mathcal{F}):
 - A collection of subsets of Ω (called events).
 - \mathcal{F} is closed under complementation and countable unions.
 - Example: $\mathcal{F} = \{\emptyset, \{1\}, \{2, 3\}, \Omega\}$.
- **Probability Measure** (\mathbb{P}):
 - A function assigning a number $\mathbb{P}(A)$ to each event $A \in \mathcal{F}$.
 - Properties:
 - $\mathbb{P}(\emptyset) = 0$ and $\mathbb{P}(\Omega) = 1$.
 - If A_1, A_2, \dots are disjoint, $\mathbb{P}(\bigcup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} \mathbb{P}(A_i)$.
 - Example: $\mathbb{P}(\{1\}) = \frac{1}{6}$, $\mathbb{P}(\{2, 3\}) = \frac{2}{6}$.

Together, $(\Omega, \mathcal{F}, \mathbb{P})$ provides a rigorous mathematical framework for studying randomness.

Further Reading: *Probability with Martingales*: David Williams.

State Price Equation

Defining the State Price Equation:

- Let $x(\omega)$ be the future value of a contingent claim asset x if the world enters state ω .
- The state price equation is given by:

$$price(x) = \sum_{\omega} \Psi(\omega)x(\omega) \quad (1)$$

- $price(x)$ is the market price now of the contingent claim asset x .
- $\Psi(\omega)$ represents the state price for state ω .

Probability Spaces are the foundation of state prices

- Each ω is in the **Sample Space** (Ω) and all are disjoint subsets.
- All contingent claims are part of the **Sigma-algebra** \mathcal{F} that derive from the event that occurred.
- **Probability Measures** \mathbb{P} (Real World) and \mathbb{Q} (Risk Neutral) will be defined later.

Arbitrage Opportunities

The current price of the portfolio of state assets should match the current price of the contingent claim asset.
If the prices of the portfolio and asset differ then an arbitrage opportunity exists.

Risk-Free Assets and State Prices

What is a Risk-Free Asset?

An asset is risk-free if its payoff is independent of the future state of the world.

$$x(\omega) = \text{Constant}, \quad \forall \omega$$

Pricing a Risk-Free Asset

Consider the price paid now for receiving a guaranteed amount of 1 in the future:

$$\text{price}(1) = \sum_{\omega} \Psi(\omega) \cdot 1 = \sum_{\omega} \Psi(\omega)$$

where $\Psi(\omega)$ is the state price for state ω .

Enforcing Non-Negative Interest Rates

The gross risk-free rate R^f satisfies:

$$R^f = \frac{1}{\text{price}(1)} = \frac{1}{\sum_{\omega} \Psi(\omega)} \geq 1 \quad \Rightarrow \quad \sum_{\omega} \Psi(\omega) \leq 1.$$

State Prices and Arbitrage

No Arbitrage Condition

To avoid arbitrage, all state prices must satisfy:

$$\Psi(\omega) \geq 0 \quad \forall \omega.$$

If $\Psi(\omega) < 0$, someone would pay you to hold an asset that either:

- Expires worthless, or
- Pays 1 if state ω occurs.

Showing How Negative State Prices Create Arbitrage

Suppose there exists a particular state ω^* with $\Psi(\omega^*) < 0$.

A portfolio holding these negatively priced state assets would:

- Have strictly non-negative payoffs of 1 in state ω^* .
- Have a negative current price.

This violates the no-arbitrage condition. You should not receive money when there is no risk of a downside.

Negative State Prices Imply Arbitrage

Scenario: Negative Prices for a particular State

Suppose there exists a state asset x^* with:

- Positive payoffs: $x^*(\omega^*) = 1$ for a specific state ω^* .
- Negative current price: $price(x^*) < 0$.

Portfolio Construction

Consider holding only these negatively priced state assets:

- Payoff is 1 in state ω^* .
- Payoff is 0 in all other states.

This portfolio has an arbitrage opportunity

- A strictly non-negative payoff in all possible states.
- A negative current price, creating an arbitrage opportunity.

This violates the no-arbitrage condition.

State Prices in Discrete Space (Single asset)

Defining Uncertainty

- In a discrete and finite setting, uncertainty is represented by a finite set of states:

$$\omega \in \{1, \dots, S\}, \quad S = \text{number of future states.}$$

- Only one state becomes true at a particular time in the future.

State Prices Vector Ψ

- A vector of state prices:

$$\Psi^T = [\Psi_1, \dots, \Psi_S].$$

- Each Ψ_ω represents the cost of obtaining an additional unit of currency in state ω .

Price Representation

The price of an asset i can be expressed as the sum of its payoffs over all states:

$$P_i = \sum_{\omega=1}^S x_{i\omega} \Psi_\omega, \quad \text{or in matrix form: } \mathbf{P} = \mathbf{X}\Psi.$$

State Prices in Discrete Space (Groups of Assets)

Asset Prices Vector \mathbf{P}

- Prices of the N assets can be represented as a vector:

$$\mathbf{P} = [P_1, \dots, P_N]^T.$$

Payoff Matrix \mathbf{X}

- An economy with N assets and S states has an $N \times S$ payoff matrix:

$$\mathbf{X} = [x_{i\omega}], \quad x_{i\omega} = x_i(\omega),$$

where $x_{i\omega}$ is the payoff of asset i in state ω .

Portfolio properties

- A portfolio θ of multiple assets has:
 - Market value: $\mathbf{P}^T \theta$, = The current price of the portfolio.
 - Payoff vector: $\mathbf{X}^T \theta$, = The matrix of possible future payoffs from the portfolio.

State Prices and Arbitrage-Free Conditions

Uncovering State Prices

When number of assets equals number of states:

- State price equation: $\mathbf{P} = \mathbf{X}\Psi$
- Matrix can be inverted: $\mathbf{X}^{-1}\mathbf{P} = \Psi$
- Requires: Linearly independent payoffs (i.e. an invertible matrix)

Arbitrage-Free Conditions

For payoff matrix \mathbf{X} , price vector \mathbf{P} , portfolio θ :

- System is weakly arbitrage-free if: $\mathbf{P}^T\theta \geq 0$ whenever $\mathbf{X}^T\theta \geq 0$

By Farkas's Lemma this is equivalent to

- The existence of $\Psi \geq 0$ such that $\mathbf{P} = \mathbf{X}\Psi$

Initial Setup: Market Prices and Payoffs

Prices and Payoffs

Price/Payoff matrix	Current Price	Boom state payoff	Bust state payoff
Asset A	1.65	3	1
Asset B	1.00	2	0.5

Written

Using the recently defined notation this is:

- Price vector: $\mathbf{P} = \begin{bmatrix} 1.65 \\ 1.00 \end{bmatrix}$
- Payoff matrix: $\mathbf{X} = \begin{bmatrix} 3 & 1 \\ 2 & 0.5 \end{bmatrix}$

Reminder: Inverse of a 2×2 Matrix

For $\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, the inverse is

$$\mathbf{A}^{-1} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix},$$

provided $ad - bc \neq 0$.

Finding State Prices

Required State Price Payoffs

State Asset	Price	Boom	Bust
Boom state	?	1	0
Bust state	?	0	1

Solution

- $\mathbf{X}^{-1} = \begin{bmatrix} -1 & 2 \\ 4 & -6 \end{bmatrix}$
- State prices: $\boldsymbol{\Psi} = \mathbf{X}^{-1}\mathbf{P} = \begin{bmatrix} 0.35 \\ 0.60 \end{bmatrix}$

Replicating Portfolios

Portfolio	Price	Units of A	Units of B
Boom state	0.35	-1	2
Bust state	0.60	4	-6

Verification and Implications

Verifying Asset A's Price

State	Price	Payoff	Calculation
Boom	0.35	3	$0.35 \times 3 = 1.05$
Bust	0.60	1	$0.60 \times 1 = 0.60$

Total price = $1.05 + 0.60 = 1.65$. Recovering the observable price

What about Asset B?

Key Insights

- Risk-free bond price = $0.35 + 0.60 = 0.95$. Buying 3 of A and selling 4 of B.
- Shows you can price any asset using state prices
- No probabilities are needed (since the market is complete)
- Only requirements: Known cashflows in all states. As many assets as states.

Numerical Example: State Prices and Portfolio Analysis

Input Parameters for 3 assets and 4 states

- **State Prices Vector (Ψ):**

$$\Psi = [0.2, 0.3, 0.2, 0.2]$$

- **Payoff Matrix (X):**

$$X = \begin{bmatrix} 10 & 15 & 20 & 25 \\ 5 & 10 & 15 & 20 \\ 8 & 12 & 18 & 22 \end{bmatrix}$$

- **Portfolio (θ):**

$$\theta = [2, 3, 1]$$

What is the:-

- Vector of Asset Prices?
- Portfolio Market Value?
- Portfolio Payoff Vector?

Numerical Example: Portfolio properties

Results:

- **Asset Prices Vector (P):**

$$\mathbf{P} = [15.5, 11.0, 13.2]$$

- **Portfolio Market Value:**

$$\mathbf{P}^T \theta = 77.2$$

- **Portfolio Payoff Vector:**

$$\mathbf{X}^T \theta = [43, 72, 103, 132]$$

Key Takeaways

- Asset prices are calculated as the weighted sum of state prices and payoffs.
- Portfolio value combines asset prices with the chosen portfolio weights.
- Portfolio payoffs represent possible future outcomes across states.

Bonus Question: What was the interest rate?

Market Completeness and State Prices

Properties of Complete Markets

- Number of linearly independent assets = Number of states
- All contingent claims can be synthesized
- Note that trading is not required for all claims, but spanning set s needed

Alternative Cases to Consider

Fewer States than Assets:

- Redundant assets exist
- Potential arbitrage opportunities
- Strategy: Calculate state prices using non-redundant assets to identify mispricing

More States than Assets:

- Market is incomplete
- Infinite possible state price combinations
- Requires different analytical approach
- (Similar issues arise with continuous state spaces)

Incomplete Market Example

Market Prices and Payoffs

	Market Price	Boom state	Bust state	Bad state
Asset A	1.65	3	1	0.5
Asset B	1.00	2	0.5	0.75

Table: Observable values

State Price Requirements

Two equations from market prices:

$$3\psi_{\text{Boom}} + \psi_{\text{Bust}} + 0.5\psi_{\text{Bad}} = 1.65$$

$$2\psi_{\text{Boom}} + 0.5\psi_{\text{Bust}} + 0.75\psi_{\text{Bad}} = 1.00$$

Multiple solution exist and additional constraints are required

Solving the State Prices in Terms of Bad State

$$\begin{aligned}\Psi_{\text{Boom}} &= 0.35 - \Psi_{\text{Bad}} \\ \Psi_{\text{Bust}} &= 0.60 + 2.5\Psi_{\text{Bad}}\end{aligned}$$

Constraints

1. No arbitrage requires positive state prices for all states:
 - $0 < \Psi_{\text{Bad}} < 0.35$
2. Positive interest rates adds an additional constraint:
 - $\Psi_{\text{Bad}} < 0.02$ (ensures sum of state prices < 1)

Key Implications

- Infinite possible state price combinations within bounds
- Cannot uniquely price non-traded assets
- Need to introduce probabilities for further analysis

Section Summary: State Prices

Key Concepts

- **Contingent Claims:** Assets delivering payoffs based on specific future states.
- **State Assets:** Simplest contingent claims, paying 1 in a specific state and 0 otherwise.
- **State Prices:** Current prices of state assets, forming the basis for pricing other assets.

Core Equations

- State price equation:

$$price(x) = \sum_{\omega} \Psi(\omega)x(\omega).$$

- Arbitrage-free condition: $\Psi(\omega) \geq 0 \forall \omega$ and $\sum_{\omega} \Psi(\omega) \leq 1$.

Why is this useful?

Implications of framework

- **Arbitrage-Free Pricing:** Ensures consistent prices across all assets.
- **Portfolio Construction:** Use state prices to replicate any asset, or hedge payoffs.
- **Market Completeness:** State prices enable the pricing of all assets with known cashflows.
- **Risk Free Rate:** The sum of all state prices is the price of a risk free bond paying 1 in all states.

Practical Applications

- **Pricing Financial Derivatives:**
 - Calculate prices for options, futures, and other contingent claims.
 - Example: Use state prices to replicate payoffs of complex derivatives.
- **Risk Management:**
 - Construct portfolios to hedge against specific state-dependent risks.
 - Minimize exposure to adverse economic scenarios.
 - Ensure models are coherent and without internal bias.

Papers

- [Varian, 1987] gives a good introduction to state prices with a Binomial option pricing application.
- [Jarvis et al., 2001] provide an excellent introduction to state prices (referred to in the paper as a deflator), considering complete markets in discrete time and space.
- [Etheridge, 2002] explains why weak arbitrage is impossible if and only if there exists non-negative state prices.

Exercise 1: Understanding Contingent Claims

1. Define a contingent claim in your own words. Provide two examples of financial instruments that are contingent claims and explain why they meet the definition.
2. The lecture mentions that "everything is made from contingent claims." Explain this statement in the context of asset pricing theory.

Exercise 2: State Assets and State Prices

1. A state asset pays 1 in state ω and 0 in all other states. Prove that the portfolio of all state assets can be used to replicate any contingent claim.
2. Suppose there are three states of the world $\omega_1, \omega_2, \omega_3$, and the state prices are:

$$\Psi(\omega_1) = 0.4,$$

$$\Psi(\omega_2) = 0.3,$$

$$\Psi(\omega_3) = 0.2.$$

Compute the price of a contingent claim that pays:

$$x(\omega_1) = 10,$$

$$x(\omega_2) = 5,$$

$$x(\omega_3) = 2.$$

Exercise 3: Arbitrage Opportunities

1. Explain why negative state prices create arbitrage opportunities. Illustrate your explanation with a numerical example.
2. Consider a market with two assets and two states:

$$\mathbf{P} = \begin{bmatrix} 2.5 \\ 1.5 \end{bmatrix}, \quad \mathbf{X} = \begin{bmatrix} 4 & 2 \\ 3 & 1 \end{bmatrix}.$$

Check if there is an arbitrage opportunity by solving for the state prices $\Psi(\omega_1)$ and $\Psi(\omega_2)$.

Exercise 4: Risk-Free Assets and Interest Rates

1. Derive the relationship between the gross risk-free rate R^f and the sum of state prices, showing that $R^f = \frac{1}{\sum_{\omega} \Psi(\omega)}$.
2. If the state prices in a two-state world are $\Psi(\omega_1) = 0.4$ and $\Psi(\omega_2) = 0.5$, calculate the gross risk-free rate. Verify that the condition $R^f \geq 1$ holds. What is the interest rate?

Exercise 5: Numerical Example (State Prices in Discrete Space)

1. Given the following price vector and payoff matrix:

$$\mathbf{P} = \begin{bmatrix} 3.5 \\ 2.5 \end{bmatrix}, \quad \mathbf{X} = \begin{bmatrix} 5 & 2 \\ 4 & 1 \end{bmatrix},$$

find the state prices Ψ .

2. Verify that the calculated state prices correctly reproduce the given asset prices.

Exercise 6: Portfolio Analysis

1. In a market with three assets and three states, the payoff matrix is given by:

$$\mathbf{X} = \begin{bmatrix} 2 & 4 & 1 \\ 1 & 3 & 2 \\ 3 & 6 & 4 \end{bmatrix}.$$

The current prices of the assets are $\mathbf{P} = \begin{bmatrix} 5 \\ 3 \\ 8 \end{bmatrix}$.

- 1.1 Solve for the state prices Ψ .
- 1.2 Confirm that these state prices result in no arbitrage opportunities.

Exercise 7: Advanced Problem – Replicating Portfolios

1. Consider the following market:

$$\mathbf{P} = \begin{bmatrix} 1.8 \\ 1.2 \end{bmatrix}, \quad \mathbf{X} = \begin{bmatrix} 3 & 0 \\ 2 & 1 \end{bmatrix}.$$

1.1 Find the state prices.

1.2 Create a portfolio that replicates the payoff of a contingent claim that pays $x(\omega_1) = 5$ and $x(\omega_2) = 3$.

Exercise 8: Conceptual Questions

1. Discuss the role of the probability measure \mathbb{P} (real-world) versus \mathbb{Q} (risk-neutral) in asset pricing.
2. Explain how the existence of a unique set of state prices is related to the completeness of a market. Provide examples of complete and incomplete markets.

Exercise 9: Numerical Simulation

1. Assume a market with four states and the following state prices:

$$\boldsymbol{\psi} = \begin{bmatrix} 0.25 \\ 0.20 \\ 0.30 \\ 0.25 \end{bmatrix}.$$

For an asset with payoffs $x(\omega) = [5, 3, 4, 2]$, calculate:

- 1.1 The current price of the asset.
- 1.2 The replication portfolio in terms of state assets.

Exercise 10: Inverse Payoff Matrix

1. Prove that the existence of an inverse payoff matrix \mathbf{X}^{-1} implies that the market is complete.
2. Provide an example of a payoff matrix that is not invertible, and explain the economic implications for state prices and arbitrage.

Introducing Stochastic Discount Factors

Stochastic Discount Factors

- The stochastic discount factor (SDF) is a stochastic process that discounts a projected future cashflow to give a present value.
- The SDF process takes a different value at each time point and depends on the state of the economy at that time.
- Only one SDF process is applied to all cashflows, pricing all assets simultaneously.

Terminology - The SDF is also referred to as:

- State price deflator^a
- State price density
- State price kernel
- Marginal rate of substitution

^aNot to be confused with the economic, GDP deflator which measures historical changes in prices.

Probabilities are introduced to complete an incomplete market

Definition: Stochastic Discount Factor (SDF)

$$m(\omega) = \frac{\text{State Price}}{\text{Probability}} = \frac{\Psi(\omega)}{\Pr(\omega)}.$$

- SDFs are well-defined only for states with positive probability.
- They simplify pricing equations by combining probabilities and state prices.

Definition: Basic Pricing Equation: $price(x) = \mathbb{E}[mx]$

$$price(x) = \sum_{\omega} \Psi(\omega)x(\omega) = \sum_{\omega} \Pr(\omega) \frac{\Psi(\omega)}{\Pr(\omega)} x(\omega) = \sum_{\omega} \Pr(\omega) m(\omega) x(\omega) = \mathbb{E}[mx].$$

Multiplying cashflows by a SDF and taking expectations cancels out the probability in the denominator.

The Role of Probabilities

- Introducing probabilities resolves ambiguities in state prices.
- Probabilities allow markets to be "completed" conceptually, even if not practically.

SDFs and Asset Pricing in Incomplete Markets

Remarks: State prices

- State prices are natural in discrete-state, complete markets.
- State prices become unwieldy in multi-period, multi-asset settings.

Remarks: Stochastic discount factors

- Summations over states and probabilities are replaced by easier-to-handle expectations.
- By focusing on expectations, the SDF approach offers flexibility for theoretical and empirical applications.
- Expected values generalize better to continuous time and space.
- The SDF framework generalizes well to multi-asset settings

Constructing the SDF in a Finite State Space

Overview

- The following method calculates SDFs for both complete and incomplete markets.
- Highlights the link between SDFs and the basic pricing equation (eq: 2).

Cashflow Space \underline{C}

- Defined as the set of all cashflows investors can purchase.
- In specific studies, \underline{C} is restricted to tradeable cashflows.
- Each cashflow is a portfolio combination:

$$\{\theta^T \mathbf{X}\} \in \underline{C},$$

where θ is a vector of portfolio weights, and \mathbf{X} is the cashflow matrix.

Remarks:

- The construction reveals connections between cashflows, state prices, and SDFs.
- Provides a foundation for analyzing pricing in incomplete markets.

Complete Markets: Defining the SDF

Cashflow Space and Asset Representation

- Assume the cashflow space \underline{C} is generated by S cashflows, matching the number of states ($n = S$).
- Asset i 's cashflows are organized as a vector:

$$\mathbf{x}_i = [x_{i1}, x_{i2}, \dots, x_{iS}]^T \in \underline{C}.$$

- Current asset prices form a vector:

$$\mathbf{P} = [\text{price}(x_1), \text{price}(x_2), \dots, \text{price}(x_n)]^T.$$

Defining the SDF Vector

- The SDF vector \mathbf{M} is defined as:

$$\mathbf{M} = \theta^T \mathbf{X},$$

where θ is a vector of portfolio weightings.

- To ensure $\mathbf{P} = \mathbf{X}\mathbf{M}^T$, the weights are: $\theta = (\mathbf{X}\mathbf{X}^T)^{-1} \mathbf{P}$.

Remarks:

- In complete markets, the SDF is unique and equals state asset prices.

Incomplete Markets: Introducing Expectations

Market Characteristics

- Incomplete markets occur when $n < S$, meaning not all states have contingent claim assets.
- Expectations are introduced to define state prices and complete the market.

Defining the SDF in Incomplete Markets

- The pricing equation becomes:

$$\mathbf{P} = E[\mathbf{X}\mathbf{M}^T] = E[\mathbf{X}\mathbf{X}^T\theta].$$

- Rearranging gives:

$$\theta = \left(E[\mathbf{X}\mathbf{X}^T]\right)^{-1} \mathbf{P}.$$

- The SDF vector is then:

$$\mathbf{M} = \mathbf{P}^T \left(E[\mathbf{X}\mathbf{X}^T]\right)^{-1} \mathbf{X}.$$

Remarks:

- The SDF found prices all assets but may include non-positive values, allowing arbitrage.
- Probabilities must be chosen to ensure positivity and prevent arbitrage.

Complete vs. Incomplete Markets

Key Differences

- In complete markets:
 - $n = S$: Each state has a corresponding asset.
 - The SDF is unique and equals state asset prices.
- In incomplete markets:
 - $n < S$: Not all states have associated assets.
 - Infinite possible SDFs exist, determined by probability weightings.

Completing the Market

- Selecting a set of probability weightings defines a specific SDF.
- This choice assigns prices to state assets and ensures no arbitrage.

Numerical Example

We consider an incomplete market example and assign probabilities to each state:

Price/Payoff matrix	Market Price	Boom state payoff	Bust state payoff	Bad state payoff
Real World Probability		10%	50%	40%
Asset A	1.65	3	1	0.5
Asset B	1.00	2	0.5	0.75

Next, we take expectations of the cashflows:

$$\mathbf{X} \Pr \mathbf{I} = E[\mathbf{X}] = \begin{bmatrix} 0.3 & 0.5 & 0.2 \\ 0.2 & 0.25 & 0.3 \end{bmatrix}$$

$$E[\mathbf{X} \mathbf{X}^T] = E[\mathbf{X} \mathbf{X}^T] = \begin{bmatrix} 1.5 & 1 \\ 1 & 0.75 \end{bmatrix}$$

Inverting gives:

$$E[\mathbf{X} \mathbf{X}^T]^{-1} = \begin{bmatrix} 6 & -8 \\ -8 & 12 \end{bmatrix}$$

State Price Vector

Multiplying by prices:

$$\mathbf{P} = \begin{bmatrix} 1.65 \\ 1.00 \end{bmatrix}$$

and cashflows leaves the Stochastic Discount Factor (SDF) vector:

$$\mathbf{M} = \mathbf{P}^T E[\mathbf{X}\mathbf{X}^T]^{-1} \mathbf{X} = [3.3 \quad 1.3 \quad 0.05]$$

and a corresponding state price vector of:

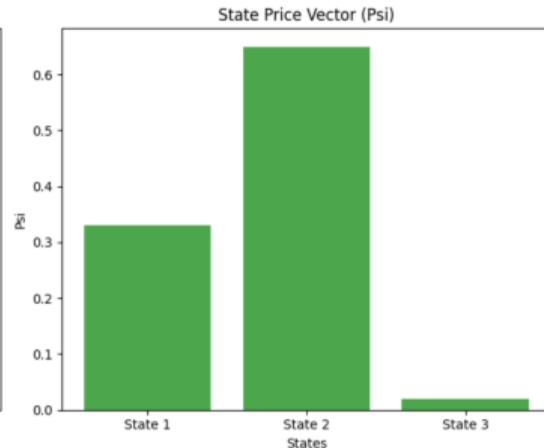
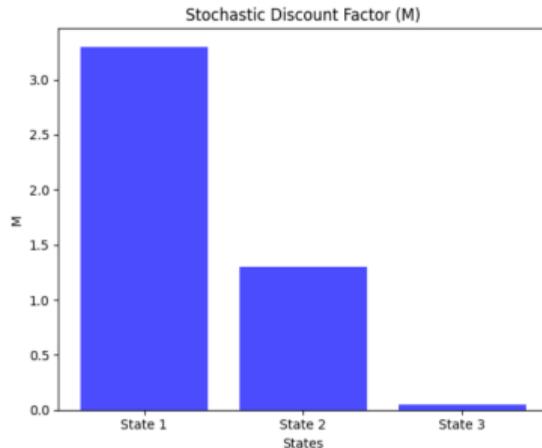
$$\boldsymbol{\Psi} = [0.33 \quad 0.65 \quad 0.02]$$

Key Insights:

- Different probabilities will produce different state prices
- The more likely a state is, the higher the price of the corresponding state asset.
- Different state prices lead to different SDFs.
- Different SDFs give different prices for non-traded cashflows.

Completing the market: Interactive example

```
Updated Results:  
Probabilities: [0.1 0.5 0.4]  
E[X]:  
[1. 0.75]  
E[XX^T]:  
[[1.5 1. ]  
 [1. 0.75]]  
Stochastic Discount Factor (M):  
[3.3 1.3 0.05]  
State Price Vector (Psi):  
[0.33 0.65 0.02]
```



[Click to enter the Google colab](#)

Contingent Claims

Reminder

- S = states of nature

$$\{s : s = 2 \cup (s = \text{rain} \cap s = \text{shine})\}$$

- A contingent claim pays 1 dollar (1 unit of consumption) tomorrow if in state s .
- Price today of contingent claim: $pc(s)$.

Complete Markets

- Complete market: Can buy contingent claims or synthesize them using enough securities.
- If complete contingent claims exist, a discount factor $m(s)$ exists:

$$m(s) = \frac{pc(s)}{\pi(s)}$$

Pricing and the Happy Meal Theorem

Pricing Contingent Claims

- Contingent claim to state $s = 1$: $x(1)$.
- Price of contingent claims:

$$p(x) = \sum_s p c(s) x(s)$$

Happy Meal Theorem

- "Happy Meal Theorem": Price of a happy meal equals burger + fries + drink + toy.
- Easier to take expectations than sum over states:

$$p(x) = \sum_s \pi(s) \left(\frac{p c(s)}{\pi(s)} \right) x(s) = E(mx)$$

- Define $m(s) = \frac{p c(s)}{\pi(s)}$ as state price density.

Contingent Claims Pricing

Contingent Claims Price Vector

- Price vector $pc = [pc(1), pc(2), \dots, pc(s)]^T$
- Payoffs vector $x = [x(1), x(2), \dots, x(s)]^T$

Inner Product Notation

$$p(x) = \sum_s pc(s)x(s) = pc \cdot x$$

This lies on a hyperplane perpendicular to the contingent claim price vector.

Key Insight

Inner product pricing implies a linear function mapping state space $R^s \rightarrow R$.

Arbitrage: Restrictions Inherited from State prices

For cashflows x in cashflow space \underline{C} with pricing function $price(x)$:

- No arbitrage exists if $price(x) > 0$ when:
 - $x \geq 0$ in all states, and
 - $x > 0$ in at least one state

Interpretation

You cannot form a portfolio for free that pays positively in some states without ever incurring a loss.

Key Implications

- Positive payoffs must have positive prices
- Cannot create free portfolios that:
 - Have potential positive payoffs
 - Never make a loss
- If $x \geq y$ for all states and $x > y$ in some states, then $price(x) > price(y)$
- If one payoff dominates another, the price of the dominant payoff must be higher.

Stochastic Discount Factors and Arbitrage

Connection to SDFs

- Absence of arbitrage \iff existence of positive SDF ($m > 0$)
- Having $m > 0$ implies no arbitrage
- Not all m satisfying $price(x) = E[mx]$ need be positive

Market Completeness

- Complete market with no arbitrage:
 - Unique $m > 0$ exists
 - $price(x) = E[mx]$
- Incomplete market:
 - Choosing unique $m > 0$ completes the market
 - Ensures no arbitrage

Important Note

Existence of positive SDF does not guarantee $price(x)$ is defined for all payoffs outside \underline{C} .
Extending to larger asset sets may introduce arbitrage.

SDF Assumption 1: Free Portfolio Formation

Mathematical Expression:

$$x_1, x_2 \in \underline{C} \Rightarrow ax_1 + bx_2 \in \underline{C} \quad (2)$$

for any real a, b , where \underline{C} is the space of cashflows.

Interpretation:

- A portfolio can be constructed using any weighting of assets.
- This assumption allows investors to trade underlying assets and form portfolios that replicate any contingent claim.
- Payoffs from any asset can be matched by a portfolio of contingent claim assets.
- If free portfolio formation exists, then an SDF (Stochastic Discount Factor) can also exist.
- If investors can form a portfolio, it will have a price.

SDF Assumption 2: Law of One Price

Mathematical Expression:

$$\text{price}(ax_1 + bx_2) = a \cdot \text{price}(x_1) + b \cdot \text{price}(x_2) \quad (3)$$

Interpretation:

- The law of one price means the value of a portfolio equals the sum of its contents.
- For example, if $x = y + z$, then:

$$E[mx] = E[my] + E[mz]$$

- This ensures all cashflows can be priced by:

$$\text{price}(x) = E[mx]$$

- Supply and demand ensure the price of a portfolio matches the price of its constituent assets, leading to the assumption of a common price.
- The existence of an SDF (Stochastic Discount Factor) implies the law of one price.
- Conversely, the law of one price also implies the existence of (at least one) SDF.

Combining the Assumptions

Free Portfolio Formation + Law of One Price:

Key Result:

- Given free portfolio formation (Eq. 2) and the law of one price (Eq. 3), there exists a Stochastic Discount Factor (SDF) $m > 0 \in \underline{C}$ such that:

$$\text{price}(x) = E[mx]$$

is unique for all $x \in \underline{C}$ if and only if there are no arbitrage opportunities.

Implications:

- Together, these assumptions imply that a payoff of 0 exists and has a price of 0.
- SDFs are based solely on these two assumptions, avoiding the need for other assumptions such as utility functions or complete markets.
- Existence of an SDF implies a linear pricing function, which in turn implies the law of one price (and vice versa).

No Arbitrage in Complete Markets

Main Result

In complete markets, no arbitrage implies that:

- There exists a unique discount factor $m > 0$ such that: $p = E(mx)$
- Equivalently, the Law of One Price holds: there exists a portfolio x^* with $p = E(x^*x)$
- In a complete market, x^* is unique.

Proof (by Contradiction)

Suppose $x^* \leq 0$ in some states.

- Construct a payoff that yields 1 in those states and 0 elsewhere.
- This payoff is strictly positive.
- Its price is given by:

$$\sum_{s: x^*(s) < 0} \pi(s)x^*(s)$$

- Since $x^*(s) < 0$ and $\pi(s) > 0$, the price is negative.
- This contradicts the no-arbitrage assumption.

Existence of Discount Factor

Assumptions

- **A1:** Portfolio formation is unrestricted.
- **A2:** Law of One Price — payoff pricing is linear.

Payoff Space

The payoff space X consists of all possible portfolios or combinations of original payoffs:

$$X = \{c^T x\}$$

where c is a vector of portfolio weights.

Implications

Given portfolio formation (A1) and the Law of One Price (A2), a unique discount factor x^* exists in X :

$$p(x) = E(x^* x) \quad \text{for all } x \in X$$

Mimicking Portfolio

Any discount factor m can be represented as: $m = x^* + \varepsilon$ with $E(x\varepsilon) = 0$

Complete Markets: No orthogonal component ($E(x\varepsilon) \neq 0$).

Theorem Does Not Say

Non-Uniqueness of $m > 0$

The theorem does not say that $m > 0$ is unique. Uniqueness only holds in a complete market.

Pricing Assets

Any line through x^* perpendicular to X also prices assets:

$$p = E[(m + \varepsilon)x] \quad \text{if} \quad E[\varepsilon x] = 0$$

All such $m + \varepsilon$ are in the same orthant.

Positive Discount Factor

The theorem guarantees the existence of a positive m , but not that every discount factor m must be positive.

Implications of Market Completeness

Market Completeness:

- If markets are complete, the orthogonal component ε does not exist:

$$E[x\varepsilon] = 0 \implies \varepsilon = 0$$

- This means:
 - There is only **one unique SDF**.
 - All assets can be priced using m^* alone.

Market Incompleteness:

- When markets are incomplete, the presence of ε introduces flexibility:
- Different assets rely on different combinations of m^* and ε for pricing.

Incomplete Market and Discount Factors

Incomplete Market

Many discount factors price assets. Any m of the form:

$$m = x^* + \varepsilon \quad \text{with} \quad E(\varepsilon x) = 0$$

We want to show at least one of these is positive, but it may not be x^* .

No Arbitrage Condition

No arbitrage implies the existence of a strictly positive discount factor $m > 0$:

$$p = E(mx) \quad \forall x \in X$$

Key Insight

Discount factors other than x^* are not in X , so we cannot use the same argument.

Decomposing the SDF

SDF Representation:

- The SDF m can be decomposed into:

$$m = m^* + \varepsilon$$

- Where:
 - m^* : Correlated component with the asset.
 - ε : Uncorrelated component satisfying $E[\varepsilon x] = 0$.

Pricing Multiple Assets:

- A single SDF can price multiple assets by utilizing the breakdown:
 - m^* correlates with one asset.
 - ε prices other assets.
- Different assets have different breakdowns between m^* and ε .

Extensions and Implications

Function Extension

We can extend the function defined on X to all payoffs R^S without implying arbitrage opportunities:

$p(x)$ defined over all R^S

Many Positive Discount Factors

Typically, there are many positive discount factors, implying many contingent claim economies consistent with observed preferences.

Strictly Positive Discount Factors

Strictly positive discount factors can be interpreted as possible contingent claims prices.

Note that:-

- An incomplete set of prices \Rightarrow payoffs (generated by a complete market) if there is no arbitrage on observed prices.
- Arbitrage is a weak characterization of market equilibrium.
- Continuous random variables work just as well as R^S

Proof of No Arbitrage

Constructing M

Join $(-p(x), x)$ to get vectors in \mathbb{R}^{s+1} :

$$M = \{(-p(x), x) \mid x \in X\}$$

M is a linear space:

$$m_1, m_2 \in M \Rightarrow am_1 + bm_2 \in M.$$

No Arbitrage Implication

Elements of M cannot consist entirely of positive elements. If x is positive, $-p(x)$ is negative.

Separating Hyperplane

Create a linear function $F : \mathbb{R}_+^{s+1} \rightarrow \mathbb{R}$:

$$F(-p, x) = 0 \quad \text{for } (-p, x) \in M$$

$$F(-p, x) > 0 \quad \text{for } (-p, x) \in \mathbb{R}_+^{s+1} \setminus \{0\}$$

Riesz Representation

Represent F as an inner product:

$$F(-p, x) = -p + E(mx)$$

Since $F(-p, x) \geq 0$, m must be positive.

SDF in Continuous State Space: Integrals not sums

State Price Density:

- In continuous state spaces, state prices and probabilities form a continuum.
- The SDF is redefined in terms of densities:

$$m(\omega) = \frac{\text{State Price Density}}{\text{Probability Density}} = \frac{\Psi(\omega)}{\Pr(\omega)}$$

Continuous Expectation:

- In continuous state space the expectation is calculated as an integral:

$$E[mx] = \int_A \Pr(\omega)m(\omega)x(\omega)d\omega$$

- Here, A is the set of all possible states $\omega \in A$.

Key Insights:

- In a complete market, every possible financial claim has a price.
- In a continuous state space, markets are typically incomplete:
 - There are infinitely many possible states.
 - However, there are only finitely many assets.
- Modelling can "complete" the market by supplying market-consistent prices where none exist.
 - Probabilities are assigned to each state, often using a probability distribution.
 - Example: In the Black-Scholes model, markets are completed by assuming log-normal asset return distributions.
- Completing a market is equivalent to selecting one Stochastic Discount Factor (SDF) from all possible SDFs.

Time

Introducing Time to the SDF Framework

Extending the Notation

- Each realized sample ω produces an SDF process $m_t(\omega)$, varying over time t .
- Cashflows $x_t(\omega)$ depend on both time t and state ω .
- Dropping ω the pricing equation for multi-period cashflows becomes:

$$\text{price}(x) = x_0 = \sum_t E[m_t x_t].$$

Simplified Pricing Notation

- Dropping ω and filtering to time s gives the conditional pricing equation:

$$x_s = \frac{E_s[m_t x_t]}{m_s},$$

where $E_s[\cdot]$ represents the expectation given information up to time s .

Sidenote: Filtrations in Probability Spaces

Definition: Filtration

- A filtration $\{\mathcal{F}_t\}_{t \geq 0}$ is an increasing family of σ -algebras on a probability space $(\Omega, \mathcal{F}, \Pr)$.
- Each \mathcal{F}_t represents the information available up to time t .
- Formally, $\mathcal{F}_s \subseteq \mathcal{F}_t \subseteq \mathcal{F}$ for $s \leq t$.

Importance in Asset Pricing

- Filtrations ensure that expectations, such as $E_t[\cdot]$, are conditioned on the correct information set \mathcal{F}_t .
- They provide the mathematical foundation for modeling the evolution of stochastic processes, including the SDF m_t and cashflows x_t .

Remarks:

- Filtrations allow for a rigorous treatment of dynamic pricing and no-arbitrage conditions.
- In practice, \mathcal{F}_t is often generated by observable market variables or underlying state processes.
- Notation allows visibility of data to be rigorously defined.

Sidenote: Martingales in Probability Theory

Definition: Martingale

- A stochastic process $\{X_t\}_{t \geq 0}$ on a probability space $(\Omega, \mathcal{F}, \Pr)$ is a martingale with respect to a filtration $\{\mathcal{F}_t\}_{t \geq 0}$ if:
 1. X_t is \mathcal{F}_t -measurable for all t .
 2. $E[|X_t|] < \infty$ for all t .
 3. $E[X_t | \mathcal{F}_s] = X_s$ for $s \leq t$ (the conditional expectation property).

Relevance to Finance

- Martingale properties simplify modeling of fair games and risk-neutral measures.
- Used extensively in pricing derivatives and risk management.

Remarks:

- Martingales ensure stochastic processes have no bias or predictable trends.
- They are central to modern financial mathematics, particularly in continuous-time models.

Martingale Property of Stochastically Discounted Prices

Key Insight: Martingale Property

- Rearranging the pricing equation gives:

$$m_s x_s = E_s[m_t x_t].$$

- This shows that the discounted price of any cashflow is a martingale.
- The martingale property ensures no-arbitrage in continuous time.

Pricing Multiple Cashflows

- For m as the SDF, the price at time s becomes:

$$x_s = \frac{1}{m_s} E_s \left[\sum_{j=s+1}^t m_j x_j \right].$$

- Extending this to $t \rightarrow \infty$ requires restrictions on m to ensure the summation is well-defined.

Discounted Prices and No-Arbitrage Condition

[Harrison and Pliska, 1981] showed that absence of arbitrage involving asset y implies the existence of an intrinsic discount process β . Where β makes βy into a martingale.

No-Arbitrage Condition:

The Stochastic Discount Factor (SDF) must satisfy the intrinsic discount process:

$$m_s x_s = \frac{1}{\beta_s} E_s [\beta_t m_t x_t] \quad (4)$$

The market value at time s of a stochastic cashflow y due at time $t > s$

Key Insight: No Arbitrage over time is different to over space

- The SDF is constructed ensuring that all discounted processes of assets are martingales.
- For each asset, the current discounted price equals the expected value of the future discounted price.
- Only the ratio of the SDF at two time points affects the price x_s .
- Multiplying the SDF process by a constant does not change the pricing model.

Note: E_s represents the expectation based on information available up to time s (filtration).

Scaling the SDF for Simplification

Simplifying the SDF:

- To simplify modeling, the SDF is scaled by the intrinsic discount process:

$$\hat{m}_t = \beta_t m_t$$

- Revised market value of x at time s :

$$x_s = \frac{1}{\hat{m}_s} E_s [\hat{m}_t x_t]$$

Continuous Time Pricing:

- Dropping the hat notation leads to the basic pricing equation:

$$m_s x_s = E_s [m_t x_t]$$

- Each SDF m now acts as a state price.

Conditions for Finite Prices

- To avoid bubbles, the following condition must hold:

$$\lim_{h \rightarrow \infty} E_t[m_{t+h}x_{t+h}] \rightarrow 0.$$

- Cashflows must be mean-summable:

$$E \left(\sum_{t=0}^{\infty} |x_t| \right) < \infty.$$

- If m is also mean-summable, then the price $E \left(\sum_{t=0}^{\infty} m_t x_t \right)$ is well-defined and finite.
- [Duffie, 2001] gives a fundamental introduction and greater depth and [Giotto and Ortu, 1997] give a mathematically rigorous description of the equivalence between SDFs and state prices.

Avoiding Arbitrage

- All state prices must be positive to prevent arbitrage.
- Since probabilities are non-negative, SDFs must be strictly positive.
- Standard convention: $m_0 = 1$ for initial SDF value. (But not for multicurrency models)

Exercise 2.1: Numerical Calculation of SDF

1. Consider an incomplete market with $S = 100$ equally likely states and $n = 3$ assets. Let the payoff matrix be $\mathbf{X} \in \mathbb{R}^{3 \times 100}$ and the observed price vector be:

$$\mathbf{P} = \begin{bmatrix} 0.98 \\ 1.02 \\ 0.95 \end{bmatrix}.$$

Generate some simulation data and using the sample moments, calculate:

- 1.1 The portfolio weights θ that form the SDF:

$$\theta = \left(\frac{1}{S} \sum_{s=1}^S \mathbf{x}_s \mathbf{x}_s^T \right)^{-1} \mathbf{P}$$

- 1.2 The SDF vector $\mathbf{M} = \theta^T \mathbf{X}$, representing the discount factor for each of the 100 states.
- 1.3 Verify that the pricing equation holds for each asset i :

$$P_i = \frac{1}{S} \sum_{s=1}^S X_{i,s} M_s$$

Exercise 2.1: Computing the SDF in Python: Calculation

```
import numpy as np
def calculate_sdf(payoffs, prices):
    """
    payoffs: (N x S) matrix (assets x states)
    prices: (N x 1) vector of prices
    """
    S = payoffs.shape[1]

    # Second moment matrix
    EXXT = (payoffs @ payoffs.T) / S

    # Portfolio weights
    theta = np.linalg.inv(EXXT) @ prices

    # Stochastic Discount Factor
    M = theta.T @ payoffs
    return M, theta
```

Exercise 2.1: Computing the SDF in Python: Testing

```
np.random.seed(42)
N, S = 3, 100

X = 1.05 + np.random.normal(0, 0.2, (N, S))
P = np.array([[0.98], [1.02], [0.95]])

M, theta = calculate_sdf(X, P)

expected_val = np.mean(X * M, axis=1)

if np.allclose(input_prices.flatten(), expected_val):
    print("\nSUCCESS: E[X * M] equals the input prices.")
else:
    print("\nFAILURE: Calculation mismatch.")
```

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The End