Lisbon School of Economics & Management
Universidade de Lisboa

STATISTICS I

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Pratical Class 5

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https://doity.com.br/estatistica-aplicada-a-nutricao

https://basiccode.com.br/produto/informatica-basica/

Exercises: 1, 2, 5, 6, 8, 9

Expected Value, Variance and Quantiles for a Random Variable: Exercises

Chapter 3

1

1. Let X be a random variable that takes on the values 0, 1, 2, and 3 with probabilities $\frac{1}{10}, \frac{3}{10}, \frac{2}{10}, \frac{4}{10}$.

   (a) Find $E(X)$ and $E(X^2)$.

   (b) Use the results of part *(a)* to determine the value of $E[(X - \mu_X)^2]$.

   (c) Use the definition to calculate σ_X.

Exercise 1 a)

$$D_x = \} 0, 1, 2, 3 \{$$

$$f_x(x) = \begin{cases} \frac{1}{10} & (x = 0) \\ \frac{3}{10} & (x = 1) \\ \frac{2}{10} & (x = 2) \\ \frac{4}{10} & (x = 3) \end{cases}$$

Exercise 1 a)

a)

$$E(X) = \sum_{x=0}^{3} x\, f_X(x) = 0 \times \frac{1}{10} + 1 \times \frac{3}{10} + 2 \times \frac{2}{10} + 3 \times \frac{4}{10} =$$

$$= \frac{3}{10} + \frac{4}{10} + \frac{12}{10} = \frac{19}{10}$$

$$E(X^2) = \sum_{x=0}^{3} x^2 f_X(x) = 0^2 \times \frac{1}{10} + 1^2\, \frac{3}{10} + 2^2 \times \frac{2}{10} + 3^2 \times \frac{4}{10} =$$

$$= \frac{3}{10} + \frac{8}{10} + \frac{36}{10} = \frac{47}{10}$$

Exercise 1 b) and c)

b)

$$E[(X-\mu_x)^2] = \underbrace{Var(X)}_{\sigma_x^2} = E(X^2) - E(X)^2 =$$

$$= \frac{47}{10} - \left(\frac{19}{10}\right)^2 = \frac{470}{100} - \frac{361}{100} = \frac{109}{100}$$

c)

$$\sigma_x = \sqrt{\sigma_x^2} = \sqrt{\frac{109}{100}} =$$

2. Let X be a continuous random variable and f_X its density function

$$f_X(x) = \begin{cases} 1/3, & 0 < x < 1 \\ \frac{4}{45}x, & 1 < x < 4 \end{cases}.$$

(a) Compute the expected value and the variance of X.

(b) Compute the expected value of Y that is given by

$$Y = g(X) = \begin{cases} 0, & X < 1 \\ 1, & X \geq 1 \end{cases}.$$

(c) Compute the expected value of $Z = 2Y - 1$.

Exercise 2 a)

$$E(X) = \int_0^4 x f_X(x)\,dx = \int_0^1 \frac{x}{3}\,dx + \int_1^4 \frac{4}{45}x^2\,dx =$$

$$= \frac{1}{3}\left[\frac{x^2}{2}\right]_0^1 + \frac{4}{45}\left[\frac{x^3}{3}\right]_1^4 = \frac{1}{3}\left(\frac{1}{2}-0\right) + \frac{4}{45}\left(\frac{4^3}{3}-\frac{1^3}{3}\right) =$$

$$= \frac{1}{6} + \frac{4}{45}\left(\frac{64}{3}-\frac{1}{3}\right) = \frac{1}{6} + \frac{4}{45}\left(\frac{63}{3}\right) = \frac{1}{6} + \frac{252}{135}$$

$$= \frac{135}{810} + \frac{1512}{810} = \frac{1647}{810} = \frac{61}{30}$$

$$\mathrm{Var}(X) = E(X^2) - E(X)^2 = \frac{52}{9} - \left(\frac{61}{30}\right)^2 = \frac{493}{300}$$

<u>Auxiliary calculation</u>:

$$E(X^2) = \int_0^4 x^2 f_X(x)\,dx = \int_0^1 x^2 \cdot \frac{1}{3}\,dx + \int_1^4 x^2 \cdot \frac{4}{45}x\,dx = \frac{1}{3}\int_0^1 x^2\,dx + \frac{4}{45}\int_1^4 x^3\,dx =$$

$$= \frac{1}{9}\left[x^3\right]_0^1 + \frac{4}{180}\left[x^4\right]_1^4 = \frac{1}{9}(1^3-0^3) + \frac{1}{45}(4^4-1^4) = \frac{1}{9} + \frac{255}{45} =$$

$$= \frac{1}{9} + \frac{51}{9} = \frac{52}{9}$$

Exercise 2 b)

b)

$$\boxed{E[g(X)] = \int_{-\infty}^{+\infty} g(x)\, f_X(x)\, dx}$$

$$Y = g(X) = \begin{cases} 0, & X < 1 \\ 1, & X \geqslant 1 \end{cases}$$

$$E(Y) = \int_0^4 g(x)\, f_X(x)\, dx = \int_0^1 0\, f_X(x)\, dx + \int_1^4 1\, f_X(x)\, dx =$$

$$= \underbrace{\int_0^1 0\, dx}_{0} + \int_1^4 \frac{4}{45} x\, dx = \frac{4}{45}\left[\frac{x^2}{2}\right]_1^4 =$$

$$= \frac{4}{90}(4^2 - 1^2) = \frac{2}{45} \times 15 = \frac{30}{45} = \frac{6}{9} = \frac{2}{3}$$

Exercise 2 c)

$$E(Z) = E(2Y-1) = 2E(Y) - 1 = 2 \times \frac{2}{3} - 1 = \frac{4}{3} - 1 = \frac{1}{3}$$

Alternative solution:

Let: $Y = g_1(X) = \begin{cases} 0, & X < 1 \\ 1, & X \geqslant 1 \end{cases}$ and

$Z = g_2(Y) = 2Y - 1$

$Y = \begin{cases} 0 \to Z(0) = -1 \\ 1 \to Z(1) = 1 \end{cases}$

Then $Z = \underbrace{g_2 \circ g_1}_{g}(X) = g_2(g_1(X)) = g(X) \begin{cases} -1 & \text{if } 0 < X < 1 \\ 1 & \text{if } 1 \leqslant X < 4 \end{cases}$

$$E(Z) = E(g(X)) = \int_0^4 g(x) f_X(x)\,dx = \int_0^1 (-1) f_X(x)\,dx + \int_1^4 1\, f_X(x)\,dx =$$

$$= \int_0^1 -\frac{1}{3}\,dx + \int_1^4 \frac{4}{45} x\,dx = -\frac{1}{3}[x]_0^1 + \frac{4}{45}\left[\frac{x^2}{2}\right]_1^4 =$$

$$= -\frac{1}{3}(1-0) + \frac{4}{90}(4^2 - 1^2) = -\frac{1}{3} + \frac{2}{45}(15) = -\frac{1}{3} + \frac{30}{45} =$$

$$= -\frac{5}{15} + \frac{10}{15} = \frac{5}{15} = \frac{1}{3}$$

3. The demand of a certain product, in Kg, in a random day is well represented by the random variable X with density function

$$f_X(x) = \begin{cases} 1/5, & 0 < x < 5 \\ 0, & \text{otherwise} \end{cases}.$$

The firm that sells this product has a profit of 5 euros per Kg sold and a loss of 2 euros per Kg that is not sold.

(a) How many Kg of the product should the firm have in stock to maximize the expected profit?

(b) Assume now that X is a discrete random variable, with a probability function

$$f_X(x) = \frac{1}{6}, \quad \text{for } x = 0, 1, 2, 3, 4, 5.$$

What is the expected profit?

Exercise 3 a)

$X \equiv$ Demand (in kg)

$$f_X(x) = \begin{cases} \frac{1}{5} & (0 < x < 5) \\ 0 & (\text{otherwise}) \end{cases}$$

$Y =$ Profit (in €)

$Q =$ Quantity in stock $(Q \in \mathbb{R})$

$$Y = 5\min(Q, X) - 2\max(0, Q - X) =$$

$$= \begin{cases} 5X - 2(Q - X) \\ 5Q \end{cases} = \begin{cases} 7X - 2Q & (0 \leq X < Q) \\ 5Q & (Q \leq X \leq 5) \end{cases} = g(X; Q)$$

Exercise 3 a)

First let's find an expression for the expected profit:

$$E(Y) = \int_0^5 g(x)\, f_X(x)\, dx = \int_0^Q (7x - 2Q)\,\frac{1}{5}\, dx + \int_Q^5 (5Q)\,\frac{1}{5}\, dx$$

$$= \frac{1}{5}\int_0^Q 7x - 2Q\, dx + \int_Q^5 Q\, dx\, dx =$$

$$= \frac{7}{10}\left[x^2\right]_0^Q - \frac{2}{5}Q\left[x\right]_0^Q + Q\left[x\right]_Q^5 =$$

$$= \frac{7}{10}Q^2 - \frac{2}{5}Q^2 + (5-Q)Q = \frac{3}{10}Q^2 - Q^2 + 5Q$$

$$= -\frac{7}{10}Q^2 + 5Q$$

Exercise 3 a)

Now let's find the value of Q that maximizes $E(Y)$.

First order condition:

$$\frac{d}{dQ} E(Y) = \frac{d}{dQ}\left(-\frac{7}{10}Q^2 + 5Q\right) = -\frac{7}{5}Q + 5 = 0 \iff$$

$$\iff \frac{7}{5}Q = 5 \iff Q = \frac{25}{7}$$ → critical point of $E(Y)$

Second order condition:

$$\frac{d^2}{dQ^2} E(Y) = \frac{d}{dQ}\left(-\frac{7}{5}Q + 5\right) = -\frac{7}{5} < 0$$

$Q = \frac{25}{7}$ is a maximizer of $E(Y)$. Since it is the only critical point, it is the global maximizer.

Conclusion: $Q = \frac{25}{7}$

Exercise 3 b)

$$f_X(x) = \frac{1}{6} \qquad (0, 1, 2, 3, 4, 5)$$

$$Y = \begin{cases} 7X - 2Q & (0 \leq X < Q) \\ 5\,Q & (Q \leq X \leq 5) \end{cases} = g(X; Q)$$

$$E(Y) = \sum_{x=0}^{5} g(X;Q)\, f_X(x) = \sum_{x=0}^{Q-1} (7x - 2Q)\frac{1}{6} + \sum_{x=Q}^{5} 5Q\,\frac{1}{6}$$

$$= \frac{7}{6}\sum_{x=0}^{Q-1} x - \frac{1}{3}\sum_{x=0}^{Q-1} Q + \frac{5}{6}\sum_{x=Q}^{5} Q =$$

$$= \frac{7}{6}\left(\frac{1}{2}(Q-1)Q\right) - \frac{1}{3}Q^2 + \frac{5}{6}(6 - Q)Q =$$

$$= \frac{7}{12}(Q^2 - Q) - \frac{1}{3}Q^2 + \frac{5}{6}(6Q - Q^2) =$$

$$= \frac{7}{12}Q^2 - \frac{7}{12}Q - \frac{1}{3}Q^2 + \frac{30}{6}Q - \frac{5}{6}Q^2 =$$

$$= -\frac{7}{12}Q^2 + \frac{53}{12}Q \rightarrow \text{function of } Q$$

Exercise 3 b)

Conclusion:

Q	0	1	2	3	4	5
$E(Y)$	0	$3.8(3)$	6.5	8	$8.(3)$	7.5

Auxiliary calculations:

arithmetic series

$$\sum_{x=0}^{Q-1} x = \sum_{x=1}^{Q-1} x = \frac{1}{2}(Q-1)\,Q$$

$$\sum_{x=0}^{Q-1} Q = \underbrace{Q + Q + Q + \cdots + Q}_{Q \text{ times}} = Q^2$$

$$\sum_{x=Q}^{5} Q = \underbrace{Q + Q + \cdots + Q}_{5-Q+1 \text{ times} = 6-Q \text{ times}} = (6-Q)\,Q$$

4. Let X be a continuous random variable and f_X its distribution. Prove that the expected value of

$$Y = \begin{cases} a, & X < 0 \\ b, & X \geq 0 \end{cases},$$

is $E(Y) = aP(X < 0) + bP(X \geq 0)$.

Exercise 4

X is a continuous r.v.; $X \sim f_X(x)$.

$$Y = \begin{cases} a, & X < 0 \\ b, & X \geqslant 0 \end{cases} = g(X)$$

$$E(Y) = E[g(X)] = \int_{-\infty}^{+\infty} g(x) f_X(x)\, dx =$$

$$= \int_{-\infty}^{0} a f_X(x)\, dx + \int_{0}^{+\infty} b f_X(x)\, dx =$$

$$= a\, P(X<0) + b\, \underbrace{P(X \geqslant 0)}_{= P(X>0) \text{ because } X \text{ is a continuous r.v.}}, \quad QED$$

5. Find $E(X)$, $E(X^2)$ and σ_X^2 for the random variable X that has probability density function

$$f_X(x) = \begin{cases} \frac{x}{2} & \text{for } 0 < x < 2 \\ 0 & \text{elsewhere} \end{cases}.$$

Exercise 5

$$f_x(x) = \begin{cases} \frac{x}{2} & (0 < x < 2) \\ 0 & (\text{elsewhere}) \end{cases}$$

$$E(X) = \int_{-\infty}^{+\infty} x \, f_x(x) \, dx = \int_0^2 x \frac{x}{2} \, dx = \frac{1}{2} \left[\frac{x^3}{3} \right]_0^2 =$$

$$= \frac{1}{6} (2^3 - 0^3) = \frac{8}{6} = \frac{4}{3}$$

$$E(X^2) = \int_{-\infty}^{+\infty} x \, f_x(x) \, dx = \int_0^2 x^2 \frac{x}{2} \, dx = \frac{1}{2} \left[\frac{x^4}{4} \right]_0^2 =$$

$$= \frac{1}{8} (16 - 0) = 2$$

$$\sigma_x^2 = E(X^2) - E(X)^2 = 2 - \left(\frac{4}{3}\right)^2 = \frac{18}{9} - \frac{16}{9} = \frac{2}{9}$$

6. Let X be a discrete random variable such that

$$f_X(x) = \begin{cases} 1/2, & x = 0 \\ 1/3, & x = 1 \\ 1/6, & x = 2 \end{cases}$$

Compute γ_1.

Exercise 6

$$f_X(x) = \begin{cases} \frac{1}{2} & (x = 0) \\ \frac{1}{3} & (x = 1) \\ \frac{1}{6} & (x = 2) \end{cases}$$

$$\gamma_1 = \frac{\mu_3}{\sigma_X^3} = \frac{E[(X - \mu_X)^3]}{\sigma_X^3}$$

$$\mu_X = E(X) = \sum_{x=0}^{2} x \, f_X(x) = \frac{1}{2} \times 0 + \frac{1}{3} \times 1 + \frac{1}{6} \times 2 =$$

$$= \frac{1}{3} + \frac{2}{6} = \frac{2}{3}$$

Exercise 6

$$E[\underbrace{(X-\mu_X)^3}_{g(X)}] = \sum_{x=0}^{2}(x-\mu_x)^3 f_X(x) = \left(0-\frac{2}{3}\right)^3\frac{1}{2} + \left(1-\frac{2}{3}\right)^3\frac{1}{3} + \left(2-\frac{2}{3}\right)^3\frac{1}{6} =$$

$$= \left(-\frac{2}{3}\right)^3\frac{1}{2} + \left(\frac{1}{3}\right)^3\frac{1}{3} + \left(\frac{4}{3}\right)^3\frac{1}{6} =$$

$$= -\frac{8}{27}\times\frac{1}{2} + \frac{1}{27}\times\frac{1}{3} + \frac{64}{27}\times\frac{1}{6} =$$

$$= -\frac{8}{54} + \frac{1}{81} + \frac{64}{162} =$$

$$= -\frac{24}{162} + \frac{2}{162} + \frac{64}{162} = \frac{42}{162} = \frac{7}{27}$$

Exercise 6

$$\sigma_X^2 = E(X^2) - E(X)^2 = 1 - \left(\frac{2}{3}\right)^2 = 1 - \frac{4}{9} = \frac{5}{9}$$

$$E(X^2) = \sum_{x=0}^{2} x^2 f_X(x) = \frac{1}{2} \times 0^2 + \frac{1}{3} \times 1^2 + \frac{1}{6} \times 2^2 =$$

$$= \frac{1}{3} + \frac{4}{6} = 1$$

Conclusion:

$$\gamma_1 = \frac{\frac{7}{27}}{\left(\sqrt{\frac{5}{9}}\right)^3} \simeq 0.626099$$

7. Let X be a continuous random variable such that

$$f_X(x) = \begin{cases} x, & 0 < x < 1 \\ 1/2, & 1 < x < 2 \\ 0, & \text{otherwise} \end{cases}$$

Compute γ_2.

Exercise 7

$$f_x(x) = \begin{cases} x, & 0 < x < 1 \\ 1/2, & 1 < x < 2 \\ 0, & (\text{otherwise}) \end{cases}$$

$$\gamma_2 = \frac{E\left[(X-\mu_X)^4\right]}{Var(X)^2} = \frac{\mu_4}{\sigma_X^4}. = \frac{\frac{24594}{207360}}{\left(\frac{35}{144}\right)^2} = \frac{12297}{6125} \simeq 2.00767$$

$$\mu_x = \int_{-\infty}^{+\infty} x\, f_x(x)\, dx = \int_0^1 x^2\, dx + \int_1^2 \frac{x}{2}\, dx =$$

$$= \left[\frac{x^3}{3}\right]_0^1 + \frac{1}{2}\left[\frac{x^2}{2}\right]_1^2 = \frac{1}{3} + \frac{1}{4}(4-1) =$$

$$= \frac{1}{3} + \frac{3}{4} = \frac{4}{12} + \frac{9}{12} = \frac{13}{12}$$

$$Var(x) = E(X^2) - E(X)^2 = \frac{17}{12} - \left(\frac{13}{12}\right)^2 = \frac{17}{12} - \frac{169}{144} = \frac{204-169}{144} = \frac{35}{144}$$

$$E(x^2) = \int_{-\infty}^{+\infty} x^2 f_x(x)\, dx = \int_0^1 x^3\, dx + \int_1^2 \frac{x^2}{2}\, dx =$$

$$= \left[\frac{x^4}{4}\right]_0^1 + \frac{1}{2}\left[\frac{x^3}{3}\right]_1^2 = \frac{1}{4}(1-0) + \frac{1}{6}(8-1) =$$

$$= \frac{1}{4} + \frac{7}{6} = \frac{6}{24} + \frac{28}{24} = \frac{34}{24} = \frac{17}{12}$$

Exercise 7

$$E[(X-\mu_X)^4] = \int_{-\infty}^{+\infty} (x-\mu_X)^4 f_X(x)\,dx = \underbrace{\int_0^1 \left(x-\frac{13}{12}\right)^4 x\,dx}_{(*)} + \underbrace{\int_1^2 \left(x-\frac{13}{12}\right)^4 \frac{1}{2}\,dx}_{(*)} =$$

$$= \frac{11\,173}{207360} + \frac{13421}{207360} = \frac{24\,594}{207360}$$

see auxiliary calculation

$$(*)\ \int_0^1 \left(x-\frac{13}{12}\right)^4 x\,dx = \int_0^1 \left(x^4 - \frac{13}{3}x^3 + \frac{169}{24}x^2 - \frac{2197}{432}x + \frac{28561}{20736}\right) x\,dx$$

$$= \int_0^1 x^5 - \frac{13}{3}x^4 + \frac{169}{24}x^3 - \frac{2197}{432}x^2 + \frac{28561}{20736}x\,dx$$

$$= \left[\frac{x^6}{6}\right]_0^1 - \frac{13}{3}\left[\frac{x^5}{5}\right]_0^1 + \frac{169}{24}\left[\frac{x^4}{4}\right]_0^1 - \frac{2197}{432}\left[\frac{x^3}{3}\right]_0^1 + \frac{28561}{20736}\left[\frac{x^2}{2}\right]_0^1$$

$$= \frac{1}{6} - \frac{13}{3}\left(\frac{1}{5}\right) + \frac{169}{24}\left(\frac{1}{4}\right) - \frac{2197}{432}\left(\frac{1}{3}\right) + \frac{28561}{20736}\left(\frac{1}{2}\right)$$

$$= \frac{11\,173}{207360}$$

Exercise 7

$$(*)\ \int_1^2 \left(x-\frac{13}{12}\right)^4 \frac{1}{2}\,dx = \int_1^2 \left(x^4 - \frac{13}{3}x^3 + \frac{169}{24}x^2 - \frac{2197}{432}x + \frac{28561}{20736}\right)\frac{1}{2}\,dx$$

$$= \frac{1}{2}\int_1^2 x^4 - \frac{13}{3}x^3 + \frac{169}{24}x^2 - \frac{2197}{432}x + \frac{28561}{20736}\,dx$$

$$= \frac{1}{2}\left(\left[\frac{x^5}{5}\right]_1^2 - \frac{13}{3}\left[\frac{x^4}{4}\right]_1^2 + \frac{169}{24}\left[\frac{x^3}{3}\right]_1^2 - \frac{2197}{432}\left[\frac{x^2}{2}\right]_1^2 + \frac{28561}{20736}\left[x\right]_1^2\right)$$

$$= \frac{1}{10}(2^5-1) - \frac{13}{24}(2^4-1) + \frac{169}{144}(2^3-1) - \frac{2197}{1728}(2^2-1) + \frac{28561}{20736}(2-1) =$$

$$= \frac{13421}{207360}$$

Auxiliary calculation:

$$\left(x-\frac{13}{12}\right)^4 \overset{\uparrow}{=} \sum_{K=0}^{4}\binom{4}{K}x^{4-K}\left(-\frac{13}{12}\right)^K = \underbrace{\binom{4}{0}}_{1}x^4\left(-\frac{13}{12}\right)^0 + \underbrace{\binom{4}{1}}_{4}x^3\left(-\frac{13}{12}\right)^1 + \underbrace{\binom{4}{2}}_{6}x^2\left(-\frac{13}{12}\right)^2 + \underbrace{\binom{4}{3}}_{4}x^1\left(-\frac{13}{12}\right)^3 + \underbrace{\binom{4}{4}}_{1}\underbrace{x^0}_{1}\left(\frac{13}{12}\right)^4 =$$

Binomial theorem (Newton)

$$= x^4 - 4x^3\frac{13}{12} + 6x^2\left(\frac{13}{12}\right)^2 - 4x\left(\frac{13}{12}\right)^3 + \left(\frac{13}{12}\right)^4 =$$

$$= x^4 - \frac{52}{12}x^3 + \frac{1014}{144}x^2 - \frac{8788}{1728}x + \frac{28561}{20736} =$$

$$= x^4 - \frac{13}{3}x^3 + \frac{169}{24}x^2 - \frac{2197}{432}x + \frac{28561}{20736}$$

Exercise 7

Note: In the last integral, instead of the Binomial theorem we could have used integration by substitution, which would have been more convenient:

$$\int_1^2 \left(x-\frac{13}{12}\right)^4 \frac{1}{2}\,dx = \frac{1}{2}\int_1^2 \underbrace{\left(x-\frac{13}{12}\right)}_{u}^4 \underbrace{dx}_{dx=du=1} = \frac{1}{2}\int_{-\frac{1}{12}}^{\frac{11}{12}} u^4\,du = \frac{1}{10}\left[u^5\right]_{-\frac{1}{12}}^{\frac{11}{12}} = \frac{1}{10}\left(\left(\frac{11}{12}\right)^5 - \left(-\frac{1}{12}\right)^5\right) = \frac{13421}{207360}$$

$$u = x - \frac{13}{12} \quad \begin{cases} x = 2 \rightarrow u = \frac{11}{12} \\ x = 1 \rightarrow u = -\frac{1}{12} \end{cases}$$

8. Let X be a random variable that has probability density function

$$f(x) = \begin{cases} x/2 & \text{for } 0 < x \leq 1 \\ 1/2 & \text{for } 1 < x \leq 2 \\ (3-x)/2 & \text{for } 2 < x < 3 \\ 0 & \text{elsewhere} \end{cases}$$

(a) Find $E(X)$, the median and the mode of X.

(b) Find $E(X^2)$.

(c) Use the results of part (a) and (b) to determine $E(X^2 - 5X + 3)$.

(d) Compute the standard deviation.

Exercise 8 a)

$$f(x) = \begin{cases} x/2 & \text{for } 0 < x \leq 1 \\ 1/2 & \text{for } 1 < x \leq 2 \\ (3-x)/2 & \text{for } 2 < x < 3 \\ 0 & \text{elsewhere} \end{cases}$$

a)

$$E(X) = \int_{-\infty}^{+\infty} x f_x(x)\, dx = \int_0^1 x \frac{x}{2}\, dx + \int_1^2 x \frac{1}{2}\, dx + \int_2^3 x \frac{3-x}{2}\, dx =$$

$$= \frac{1}{2}\int_0^1 x^2\, dx + \frac{1}{2}\int_1^2 x\, dx + \frac{1}{2}\int_2^3 3x - x^2\, dx =$$

$$= \frac{1}{2}\left[\frac{x^3}{3}\right]_0^1 + \frac{1}{2}\left[\frac{x^2}{2}\right]_1^2 + \frac{1}{2}\left[\frac{3}{2}x^2 - \frac{x^3}{3}\right]_2^3 =$$

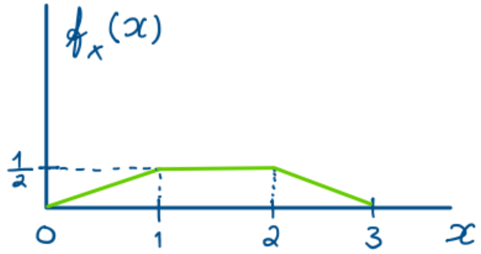
$$= \frac{1}{6} + \frac{3}{4} + \frac{1}{2}\left(\frac{3}{2}3^2 - \frac{3^3}{3} - \left(\frac{3}{2}2^2 - \frac{2^3}{3}\right)\right) =$$

$$= \frac{1}{6} + \frac{3}{4} + \frac{1}{2}\left(\frac{27}{2} - 9 - 6 + \frac{8}{3}\right) =$$

$$= \frac{1}{6} + \frac{3}{4} + \frac{27}{4} - \frac{15}{2} + \frac{8}{6} = \frac{4}{24} + \cancel{\frac{18}{24}} + \cancel{\frac{162}{24}} - \cancel{\frac{180}{24}} + \frac{32}{24} =$$

$$= \frac{36}{24} = \frac{6}{4} = \frac{3}{2}$$

Exercise 8 a)



The median of X is a value $m \in \mathbb{R}$: $F_X(m) = \frac{1}{2}$

$$\int_0^1 f_X(x)\,dx = \int_0^1 \frac{x}{2}\,dx = \frac{1}{2}\left[\frac{x^2}{2}\right]_0^1 = \frac{1}{4} < \frac{1}{2}$$

$$\int_0^2 f_X(x)\,dx = \frac{1}{4} + \int_1^2 \frac{1}{2}\,dx = \frac{1}{4} + \frac{1}{2}[x]_1^2 =$$

$$= \frac{1}{4} + \frac{1}{2} = \frac{3}{4} > \frac{1}{2} \rightarrow 1 < m < 2$$

Exercise 8 a)

$$F_X(m) = \frac{1}{2} \Leftrightarrow \int_0^m f_X(x)\,dx = \frac{1}{2} \Leftrightarrow$$

$$\Leftrightarrow \frac{1}{4} + \int_1^m \frac{1}{2}\,dx = \frac{1}{2} \Leftrightarrow \qquad \frac{2}{4}$$

$$\Leftrightarrow \frac{1}{2}[x]_1^m = \frac{1}{2} - \frac{1}{4} \Leftrightarrow$$

$$\Leftrightarrow \frac{1}{2}(m-1) = \frac{1}{4} \Leftrightarrow$$

$$\Leftrightarrow m - 1 = \frac{1}{2} \Leftrightarrow m = \frac{3}{2}$$

- The mode of X, $mo(X)$, is a value such that $f_X(mo(X)) \geq f_X(x)\ \forall x \in \mathbb{R}$
- $mo(X)$ does not have to be unique.

By looking at the plot we can easily see that in this case we have $mo(X) = x : x \in (1,2)$.

Another expression for the modes of X is: $mo(X) = \alpha + 2(1-\alpha)$, $\alpha \in (0,1)$

Exercise 8 b)

$$E(X^2) = \int_{-\infty}^{+\infty} x^2 f_X(x) \, dx = \int_0^1 x^2 \frac{x}{2} \, dx + \int_1^2 x^2 \frac{1}{2} \, dx + \int_2^3 x^2 \frac{3-x}{2} \, dx =$$

$$= \frac{1}{2} \int_0^1 x^3 \, dx + \frac{1}{2} \int_1^2 x^2 \, dx + \frac{1}{2} \int_2^3 3x^2 - x^3 \, dx =$$

$$= \frac{1}{2} \left[\frac{x^4}{4} \right]_0^1 + \frac{1}{2} \left[\frac{x^3}{3} \right]_1^2 + \frac{1}{2} \left[x^3 - \frac{x^4}{4} \right]_2^3 =$$

$$= \frac{1}{8} + \frac{7}{6} + \frac{1}{2} \left(27 - \frac{81}{4} - (8 - 4) \right) =$$

$$= \frac{1}{8} + \frac{7}{6} + \frac{1}{2} \left(27 - \frac{81}{4} - 4 \right) =$$

$$= \frac{1}{8} + \frac{7}{6} + \frac{27}{2} - \frac{81}{8} - 2 =$$

$$= -\frac{80}{8} + \frac{7}{6} + \frac{81}{6} - \frac{12}{6} = -\frac{80}{8} + \frac{76}{6} =$$

$$= -\frac{480}{48} + \frac{608}{48} = \frac{128}{48} = \frac{16}{6} = \frac{8}{3}$$

$\div 8$

Exercise 8 c)

$$E(X^2 - 5X + 3) = E(X^2) - 5E(X) + 3 =$$

$$= \frac{8}{3} - 5\left(\frac{3}{2}\right) + 3 =$$

$$= \frac{8}{3} - \frac{15}{2} + 3 =$$

$$= \frac{16}{6} - \frac{45}{6} + \frac{18}{6} = -\frac{11}{6}$$

Exercise 8 d)

d)

$$\sigma_x^2 = E(X^2) - E(X)^2 = \frac{8}{3} - \left(\frac{3}{2}\right)^2 = \frac{8}{3} - \frac{9}{4} =$$

$$= \frac{32}{12} - \frac{27}{12} = \frac{5}{12}$$

$$\sigma_x = \sqrt{\sigma_x^2} = \sqrt{\frac{5}{12}}$$

9. Find the expected value, the median and the mode of the discrete random variable X having the probability distribution $f_X(x) = |x - 2|/7$, $x = -1, 0, 1, 3$.

Exercise 9

$$f_x(x) = \frac{|x-2|}{7} \quad x = -1, 0, 1, 3$$

$$f_x(-1) = \frac{|-3|}{7} = \frac{3}{7} \qquad f_x(1) = \frac{1}{7}$$

$$f_x(0) = \frac{|-2|}{7} = \frac{2}{7} \qquad f_x(3) = \frac{1}{7}$$

$$\epsilon(x) = \sum_{x=-1}^{3} x \, f_x(x) \, dx =$$

$$= (-1)\frac{3}{7} + 0 \times \frac{2}{7} + 1 \times \frac{1}{7} + 3 \times \frac{1}{7} =$$

$$= -\frac{3}{7} + \frac{1}{7} + \frac{3}{7} = \frac{1}{7}$$

Exercise 9

$$mo(X) = \arg\max_{x \in \mathbb{R}} f_X(x) = -1$$

$$me(X) = \min\{x \in \mathbb{R} : F_X(x) \geq 0.5\} = 0$$

Auxiliary calculations:

$$F_X(-1) = f_X(-1) = \frac{3}{7} < 0.5$$

$$F_X(0) = f_X(-1) + f_X(0) = \frac{5}{7} > 0.5$$

Thanks!

Questions?