List 3 - Lebesgue integral

1. Let $(\Omega, \mathcal{F}, \mu)$ be a measure space and $\varphi, \varphi' : \Omega \to \mathbb{R}$ be simple functions and $a, a' \geq 0$. Show that if $\varphi \leq \varphi'$, then
$$\int \varphi \; \mathrm{d}\mu \leq \int \varphi' \; \mathrm{d}\mu.$$

2. Consider a simple function $\varphi = \sum_{j=1}^{N} c_j \mathcal{X}_{A_j}$ (not necessarily non-negative). Show that:

   (a) $\varphi.\mathcal{X}_A = \displaystyle\sum_{j=1}^{N} c_j \mathcal{X}_{A_j \cap A}$.

   (b) $\displaystyle\int_A \varphi \, d\mu = \sum_{j=1}^{N} c_j \mu(A_j \cap A)$.

3. Let μ be the counting measure. Consider $A = \{a_1, a_2, a_3\} \in \mathcal{F}$ and a measurable function f.

   (a) Is $\mathcal{X}_A f$ a simple function?

   (b) Compute $\int_A f \, d\mu$.

4. Show that if F_1 and F_2 are anti-derivatives of f, then $F_1 - F_2$ is a constant function.

5. Given measures μ_1, μ_2 and $c_1, c_2 \geq 0$, let $\mu = c_1\mu_1 + c_2\mu_2$. Take any function f which is simultaneously μ_1-integrable and μ_2-integrable. Show that f is also μ-integrable and that
$$\int f \, d\mu = c_1 \int f \, d\mu_1 + c_2 \int f \, d\mu_2.$$
Hint: First prove it for simple functions and then use the Monotone Convergence Theorem.

6. Let $\mathcal{A} \subset \mathcal{F}$ a σ-subalgebra, f, g are $\mathcal{F}$-measurable functions and h is $\mathcal{A}$-measurable. Are the following propositions true? If not, write examples that contradict the statements.

   (a) If $\int_B f \, d\mu = \int_B g \, d\mu$ for every $B \in \mathcal{F}$, then $f = g$ a.e.

   (b) If $\int_A f \, d\mu = \int_A h \, d\mu$ for every $A \in \mathcal{A}$, then $f = h$ a.e.

7. Use the Dominated Convergence Theorem to determine the limits:

   (a) $\displaystyle\lim_{n\to+\infty} \int_{[0,\pi]} \frac{\sqrt[n]{x}}{1+x^2} \, d\delta_0(x)$

   (b) $\displaystyle\lim_{n\to+\infty} \int_{]0,\pi]} \frac{\sqrt[n]{x}}{1+x^2} \, d\delta_0(x)$

   (c) $\displaystyle\lim_{n\to+\infty} \int_{[0,\pi]} \frac{\sqrt[n]{x}}{1+x^2} \, dx$

   (d) $\displaystyle\lim_{n\to+\infty} \int_{\mathbb{R}} e^{-|x|} \cos^n(x) \, dx$

(e) $\lim\limits_{n \to +\infty} \int_{[0,+\infty[} \frac{r^n}{1+r^{n+2}} d\delta_1(r)$

(f) $\lim\limits_{n \to +\infty} \int_{[0,+\infty[} \frac{r^n}{1+r^{n+2}} dr$