



List 4 - Distributions

1. Suppose that $f: \mathbb{R} \rightarrow \mathbb{R}$ is a $\mathcal{B}(\mathbb{R})$ -measurable function and α is the distribution of a random variable X . Find the distribution of $f \circ X$.
2. Show that $\text{Var}(X) = 0$ iff $\mathbb{P}(X = \mathbb{E}(X)) = 1$.
3. Consider $X_1, \dots, X_n$ integrable random variables. Show that if $\text{Cov}(X_i, X_j) = 0$, $i \neq j$, then

$$\text{Var}\left(\sum_{i=1}^n X_i\right) = \sum_{i=1}^n \text{Var}(X_i).$$

4. Let X be a random variable and $\lambda > 0$. Prove the *Chebyshev's inequalities*:

- (a) For $k \in \mathbb{N}$,

$$\mathbb{P}(|X| \geq \lambda) \leq \frac{1}{\lambda^k} \mathbb{E}(|X|^k).$$

When $k = 1$ this is also called *Markov inequality*.

- (b) If $\mathbb{E}(X)$ is finite,

$$\mathbb{P}(|X - \mathbb{E}(X)| \geq \lambda) \leq \frac{\text{Var}(X)}{\lambda^2}.$$

5. Find the distribution of a simple function in the form

$$X = \sum_{j=1}^N c_j \mathcal{X}_{A_j}$$

and compute its moments.

6. Determine the distribution functions of the following probability measures on $\mathbb{R}$ (δ_a is the Dirac measure at a and m_A is the Lebesgue measure on A):

(a) $\mu = \frac{1}{3}\delta_2 + \frac{2}{3}\delta_3$

(b) $\mu = \frac{1}{3}\delta_2 + \frac{1}{3}\delta_3 + \frac{1}{3}m_{[0,1]}$

7. Show that for $-\infty \leq a \leq b \leq +\infty$ we have

(a) $\alpha(\{a\}) = F(a) - F(a^-)$

(b) $\alpha(]a, b]) = F(b^-) - F(a)$

(c) $\alpha([a, b]) = F(b^-) - F(a^-)$

(d) $\alpha([a, b]) = F(b) - F(a^-)$

(e) $\alpha(]-\infty, b]) = F(b^-)$

(f) $\alpha([a, +\infty[) = 1 - F(a^-)$

(g) $\alpha(]a, +\infty[) = 1 - F(a)$

8. Compute the distribution function of the following distributions:

- (a) The Dirac distribution δ_a at $a \in \mathbb{R}$.
- (b) The Bernoulli distribution $p\delta_a + (1-p)\delta_b$ with $0 \leq p \leq 1$ and $a, b \in \mathbb{R}$.
- (c) The uniform distribution on a bounded interval $I \subset \mathbb{R}$

$$m_I(A) = \frac{m(A \cap I)}{m(I)}, \quad A \in \mathcal{B}(\mathbb{R}),$$

where m is the Lebesgue measure.

- (d) $\alpha = c_1\delta_a + c_2m_I$ on $\mathcal{B}(\mathbb{R})$ where $c_1, c_2 \geq 0$ and $c_1 + c_2 = 1$.

$$(e) \alpha = \sum_{n=1}^{+\infty} \frac{1}{2^n} \delta_{-1/n}.$$

9. Find the distribution functions of the following discrete distributions $\alpha(A) = P(X \in A)$, $A \in \mathcal{B}(\mathbb{R})$:

$$(a) \text{ Degenerate (or Dirac or atomic) distribution: } \alpha(A) = \begin{cases} 1, & a \in A \\ 0, & \text{o.c.} \end{cases}, \text{ where } a \in \mathbb{R}.$$

$$(b) \text{ Binomial distribution with } n \in \mathbb{N}: \alpha(\{k\}) = C_k^n p^k (1-p)^{n-k}, \quad 0 \leq k \leq n.$$

$$(c) \text{ Poisson distribution with } \lambda > 0: \alpha(\{k\}) = \frac{\lambda^k}{k!e^\lambda}, \quad k \in \mathbb{N} \cup \{0\}.$$

This describes the distribution of 'rare' events with rate λ .

$$(d) \text{ Geometric distribution with } 0 < p < 1: \alpha(\{k\}) = (1-p)^k p, \quad k \in \mathbb{N} \cup \{0\}.$$

This describes the distribution of the number of unsuccessful attempts preceding a success with probability p .

$$(e) \text{ Negative binomial distribution: } \alpha(\{k\}) = C_k^{n+k-1} (1-p)^k p^n, \quad k \in \mathbb{N} \cup \{0\}.$$

This describes the distribution of the number of accumulated failures before n successes.

10. Find the distribution functions of the following absolutely continuous distributions $\alpha(A) = P(X \in A) = \int_A f(x) dx$, $A \in \mathcal{B}(\mathbb{R})$ where f is the density function:

$$(a) \text{ Uniform distribution on } [a, b]: f(x) = \begin{cases} \frac{1}{b-a}, & x \in [a, b] \\ 0, & \text{o.c.} \end{cases}$$

$$(b) \text{ Exponential distribution: } f(x) = e^{-x}, \quad x \geq 0$$

$$(c) \text{ The two-sided exponential distribution: } f(x) = \frac{1}{2} e^{-|x|}, \quad x \in \mathbb{R}$$

$$(d) \text{ The Cauchy distribution: } f(x) = \frac{1}{\pi} \frac{1}{1+x^2}, \quad x \in \mathbb{R}$$

11. Let $X_n \sim U([0, 1])$ and $Y_n = -\frac{1}{\lambda} \log(1 - X_n)$, with $\lambda > 0$. Show that $Y_n \sim \text{Exp}(\lambda)$.