



List 5 - Independence

1. Prove that:
 - (a) If A_1 and A_2 are independent, then A_1^c and A_2 are also independent.
 - (b) Any full probability event is independent of any other event. The same for any zero probability event.
 - (c) Two disjoint events are independent iff at least one of them has zero probability.
 - (d) Consider two events $A_1 \subset A_2$. They are independent iff A_1 has zero probability or A_2 has full probability.
2. Suppose that A and C are independent events as well as B and C with $A \cap B = \emptyset$. Show that $A \cup B$ and C are also independent.
3. Give examples of probability measures $\mathbb{P}_1$ and $\mathbb{P}_2$, and of events A_1 and A_2 such that $\mathbb{P}_1(A_1 \cap A_2) = \mathbb{P}_1(A_1)\mathbb{P}_1(A_2)$ but $\mathbb{P}_2(A_1 \cap A_2) \neq \mathbb{P}_2(A_1)\mathbb{P}_2(A_2)$. Notice that the definition of independence depends on the probability measure.
4. Show that any random variable is independent of a constant random variable.
5. Let X and Y be independent random variables and f and g $\mathcal{B}$ -measurable functions on $\mathbb{R}$. Prove that:
 - (a) $f(X)$ and $g(Y)$ are independent.
 - (b) $\mathbb{E}(f(X)g(Y)) = \mathbb{E}(f(X))\mathbb{E}(g(Y))$ if $\mathbb{E}(|f(X)|), \mathbb{E}(|g(Y)|) < +\infty$.
6. Suppose that the random variables X and Y have only values in $\{0, 1\}$. Show that if $\mathbb{E}(XY) = \mathbb{E}(X)\mathbb{E}(Y)$, then X, Y are independent.
7. Construct an example of two uncorrelated random variables that are not independent.
8. Show that if $\text{Var}(X) \neq \text{Var}(Y)$, then $X + Y$ and $X - Y$ are not independent.
9. Prove that:
 - (a) If $\mathcal{G} \subset \mathcal{F}_1$ and $\mathcal{F}_1, \mathcal{F}_2$ are independent σ -algebras, then $\mathcal{G}$ and $\mathcal{F}_2$ are also independent.
 - (b) Two random variables X, Y are independent iff $\sigma(X)$ and $\sigma(Y)$ are independent.