Chapter 1

Basics of Probability

Exercise 1.1 (Reading off a distribution)**.** *A fair six-sided die is rolled once, and X denotes the number shown.*

(a) Write down the range of X and its probability mass function.

(b) Compute $\mathbb{P}(X \leqslant 4)$ and $\mathbb{P}(X > 2)$.

Exercise 1.2 (Checking the properties of a cdf)**.** *Let*

$$F(x) = \begin{cases} 0, & x < 0, \\ x/2, & 0 \leqslant x < 1, \\ 1, & x \geqslant 1. \end{cases}$$

(a) Verify that F satisfies the defining properties of a cdf (Proposition 1.1.6).

(b) Is F the cdf of a continuous, discrete, or mixed random variable? Justify your answer.

Exercise 1.3 (A simple pmf)**.** *Let X take the values $0, 1, 2$ with $\mathbb{P}(X = 0) = 0.5$, $\mathbb{P}(X = 1) = 0.3$, $\mathbb{P}(X = 2) = 0.2$.*

(a) Check that this defines a valid probability mass function.

(b) Write down F_X and sketch its graph (describe it in words).

Exercise 1.4 (Recognising a density)**.** *Let $f(x) = cx$ for $0 \leqslant x \leqslant 2$ and $f(x) = 0$ otherwise.*

(a) Find the constant c for which f is a valid probability density function.

(b) Compute $\mathbb{P}(X \leqslant 1)$ for a random variable X with this density.

Exercise 1.5 (Expected value of a simple discrete variable). *Let X be as in Exercise 1.3, i.e.* $\mathbb{P}(X=0)=0.5$, $\mathbb{P}(X=1)=0.3$, $\mathbb{P}(X=2)=0.2$.

(a) Compute $E(X)$.

(b) Compute $E(X^2)$ and $\mathrm{Var}(X)$.

Exercise 1.6 (Expected value of a Bernoulli variable). *Let $X \sim \mathrm{Bernoulli}(p)$, i.e.* $\mathbb{P}(X=1)=p$ *and* $\mathbb{P}(X=0)=1-p$, *for some* $p \in (0,1)$.

(a) Compute $E(X)$ directly from the definition.

(b) Compute $E(X^2)$, and deduce $\mathrm{Var}(X)$.

Exercise 1.7 (Joint pmf and marginals). *Two random variables X, Y each take values in $\{0,1\}$, with joint pmf given by*

$$\mathbb{P}(X=0, Y=0)=0.4, \quad \mathbb{P}(X=0, Y=1)=0.1,$$

$$\mathbb{P}(X=1, Y=0)=0.2, \quad \mathbb{P}(X=1, Y=1)=0.3.$$

(a) Find the marginal pmf of X and of Y.

(b) Are X and Y independent?

Exercise 1.8 (Checking independence from a joint density). *Let (X,Y) have joint density $f(x,y)=4xy$ for $0 \leqslant x \leqslant 1$, $0 \leqslant y \leqslant 1$, and $f(x,y)=0$ otherwise.*

(a) Find the marginal densities f_X and f_Y.

(b) Show that X and Y are independent.

Exercise 1.9 (Conditional expectation given a die roll parity). *Let X be the outcome of a fair die roll, and let A be the event that X is even.*

(a) Find the conditional pmf of X given A.

(b) Compute $E(X \mid A)$.

Exercise 1.10 (Conditional expectation under independence). *Let X and Y be independent random variables, with $E(X)=3$.*

(a) What is $E(X \mid Y)$? Justify your answer using the definition/properties of conditional expectation.

(b) Deduce $E(XY)$ in terms of $E(Y)$.

Exercise 1.11 (Variance of a Bernoulli variable, revisited). *Let* $X \sim \text{Bernoulli}(p)$.

(a) For which value of $p \in (0, 1)$ *is* $\text{Var}(X) = p(1 - p)$ *maximised? What is the maximum value?*

(b) Explain, in words, why the variance is small when p *is close to* 0 *or* 1.

Exercise 1.12 (Median of a uniform distribution). *Let* $X \sim \text{Uniform}(a, b)$, *with density* $f_X(x) = 1/(b - a)$ *for* $a \leqslant x \leqslant b$.

(a) Find F_X, *and hence the quantile function* $\pi_p = F_X^{-1}(p)$ *for* $p \in (0, 1)$.

(b) Compute the median of X, *and check it agrees with the midpoint of* (a, b).

Exercise 1.13 (Poisson probabilities). *Let* $X \sim \text{Poisson}(\lambda)$, *with pmf* $\mathbb{P}(X = k) = e^{-\lambda}\lambda^k/k!$, $k = 0, 1, 2, \ldots$

(a) With $\lambda = 2$, *compute* $\mathbb{P}(X = 0)$, $\mathbb{P}(X = 1)$, *and* $\mathbb{P}(X \leqslant 1)$.

(b) Using $E(X) = \lambda$ *(from the catalogue), what is the expected number of events when* $\lambda = 2$?

Exercise 1.14 (Exponential probabilities). *Let* $X \sim \text{Exp}(\theta = 4)$, *so* $S_X(x) = e^{-x/4}$ *for* $x \geqslant 0$.

(a) Compute $\mathbb{P}(X > 4)$ *and* $\mathbb{P}(X > 8)$.

(b) Using memorylessness (Example 1.9.2), compute $\mathbb{P}(X > 8 \mid X > 4)$ *without further calculation, and check it against the ratio* $\mathbb{P}(X > 8)/\mathbb{P}(X > 4)$.

Exercise 1.15 (Moment generating function of a Bernoulli variable). *Let* $X \sim \text{Bernoulli}(p)$.

(a) Compute the moment generating function $M_X(r) = E(e^{rX})$.

(b) Differentiate M_X *at* $r = 0$ *to recover* $E(X) = p$.

Exercise 1.16 (pgf of a sum of independent Poisson variables). *Let* $X \sim \text{Poisson}(\lambda)$ *and* $Y \sim \text{Poisson}(\mu)$ *be independent, and let* $S = X + Y$. *The probability generating function of a* $\text{Poisson}(\lambda)$ *variable is* $P_X(z) = E(z^X) = e^{\lambda(z-1)}$.

(a) Using the pgf of a sum of independent random variables, find $P_S(z)$.

(b) Identify the distribution of S *from its pgf.*

Exercise 1.17 (Mean residual life of the exponential). *Let* $X \sim \mathrm{Exp}(\theta)$*.*

(a) Using memorylessness, or Proposition 1.9.1 directly, compute the mean residual life $e_X(d)$ *for* $d \geqslant 0$*.*

(b) Interpret the result: how does $e_X(d)$ *depend on* d*?*

Exercise 1.18 (Limited expected value of a uniform variable). *Let* $X \sim \mathrm{Uniform}(0, 10)$*, and let* $u = 6$*.*

(a) Using Proposition 1.9.6, compute $E(X \wedge u)$*.*

(b) Compare $E(X \wedge u)$ *with* $E(X)$*, and explain why* $E(X \wedge u) < E(X)$*.*

Exercise 1.19 (Comparing tails via a limiting ratio). *Let* $X \sim \mathrm{Exp}(\theta_1)$ *and* $Y \sim \mathrm{Exp}(\theta_2)$ *with* $\theta_1 > \theta_2$*.*

(a) Using Definition 1.10.2, show that X *has a heavier tail than* Y*.*

(b) Does this match the intuition that a larger θ *means a "more spread out" distribution?*

Exercise 1.20 (The exponential as a tail benchmark). *Let* $X \sim \mathrm{Exp}(\theta)$*.*

(a) Using Definition 1.10.3, show directly that X *itself is* not *heavy-tailed (as one would hope, since the exponential is the benchmark).*

(b) Relate this to Proposition 1.10.4 and the (constant) hazard rate of X*.*

Further exercises.

Exercise 1.21 (A mixed random variable: shock magnitude). *A system is subject to random shocks. With probability* $p_0 = 0.3$ *no shock occurs in a given period, and the magnitude* X *recorded for that period is* 0*; with probability* 0.7 *a shock occurs, and its magnitude, given that a shock occurs, follows a* $\mathrm{Gamma}(\alpha = 2, \theta = 5)$ *law.*

(a) Write F_X *using the decomposition of Theorem 1.2.1/Definition 1.2.3, and confirm* X *is a mixed random variable with a jump of size* 0.3 *at the origin.*

(b) Compute $E(X)$ *and* $\mathrm{Var}(X)$*.*

(c) Find the median of X *(numerically).*

Exercise 1.22 (Hazard rate of a Weibull-type model)**.** *Let $X \geqslant 0$ be a continuous random variable with survival function*

$$S_X(x) = \exp\Big[-\Big(\frac{x}{\theta}\Big)^{\tau}\Big], \qquad x \geqslant 0,$$

for fixed constants $\theta > 0$ (scale) and $\tau > 0$ (shape). This is the Weibull *model, a common generalisation of the exponential distribution used, e.g., for the time to failure of a component.*

(a) Compute the density f_X and the hazard rate $h_X(x)$.

(b) Show that h_X is increasing on $(0, \infty)$ if $\tau > 1$, decreasing if $\tau < 1$, and constant if $\tau = 1$ (in which case $X \sim \mathrm{Exp}(\theta)$).

(c) Using Proposition 1.10.4, classify X as light- or heavy-tailed for $\tau > 1$ and for $\tau < 1$.

Exercise 1.23 (Quantile function of the Pareto distribution)**.** *Let $X \sim \mathrm{Pareto}(\alpha, \theta)$.*

(a) Show that F_X is continuous and strictly increasing on $(0, \infty)$, and find the quantile function $\pi_p = F_X^{-1}(p)$, $p \in (0, 1)$.

(b) Take $\alpha = 3$, $\theta = 1000$. Compute $\pi_{0.95}$ numerically.

Exercise 1.24 (Characterisation of the exponential by memorylessness)**.** *Let $X \geqslant 0$ be a continuous random variable with $S_X(x) > 0$ for all $x \geqslant 0$, and suppose X is* ***memoryless****, i.e.*

$$S_X(x + d) = S_X(x)\, S_X(d), \qquad \text{for all } x, d \geqslant 0.$$

Prove that $X \sim \mathrm{Exp}(\theta)$ for some $\theta > 0$ (thus Example 1.9.2 identifies all *continuous memoryless nonnegative random variables, not merely one example of one).*

Exercise 1.25 (Moments of the Gamma distribution via the cumulant generating function)**.** *Let $X \sim \mathrm{Gamma}(\alpha, \theta)$.*

(a) Show that the moment generating function of X is

$$M_X(r) = (1 - \theta r)^{-\alpha}, \qquad r < \frac{1}{\theta},$$

and hence that the cumulant generating function is $\kappa_X(r) = -\alpha \ln(1 - \theta r)$.

(b) *Use κ_X and Proposition 1.8.3 to recover $E(X) = \alpha\theta$ and $\mathrm{Var}(X) = \alpha\theta^2$.*

Exercise 1.26 (A CLT approximation for an aggregate quantity)**.** *A large population of $n = 100$ independent units is observed over one year. The loss amount of each unit is independent, identically distributed, with an $\mathrm{Exp}(\theta = 500)$ law (in some fixed monetary unit), and S_{100} denotes the total loss of the population.*

(a) *Using the Central Limit Theorem (Theorem 1.8.9), find a normal approximation to $\mathbb{P}(S_{100} > 55{,}000)$.*

(b) *Find (approximately) the smallest t such that $\mathbb{P}(S_{100} > t) \leqslant 0.05$.*

Exercise 1.27 (Limited expected value of an exponential variable)**.** *Let $X \sim \mathrm{Exp}(\theta)$. Show that $E(X \wedge u) = \theta\big(1 - e^{-u/\theta}\big)$, and check directly (using Example 1.9.2) that $E(X) = E(X \wedge u) + e_X(u)S_X(u)$ is consistent with Proposition 1.9.7.*

Exercise 1.28 (Mean residual life of the uniform distribution)**.** *Let $X \sim \mathrm{Uniform}(0, b)$, $b > 0$.*

(a) *Show that the hazard rate of X is $h_X(x) = \dfrac{1}{b-x}$ for $0 < x < b$, and that it is increasing.*

(b) *Compute the mean residual life function $e_X(d)$ for $0 \leqslant d < b$, directly from Proposition 1.9.1, and verify that it is decreasing — consistent with part (a) and Proposition 1.10.5 (applied in its "increasing hazard $\Rightarrow$ decreasing mean excess" form).*

(c) *Take $b = 10$, $d = 4$. What is $e_X(4)$?*

Exercise 1.29 (Mean residual life of a limited variable)**.** *Let $X \geqslant 0$ be continuous with $S_X(0) = 1$, and let $0 \leqslant d < u$. Recall from Definition 1.9.5 the limited variable $X \wedge u$. Prove that*

$$e_{X\wedge u}(d) := E\big[(X \wedge u) - d \mid X \wedge u > d\big] = \frac{\displaystyle\int_d^u S_X(t)\,dt}{S_X(d)}.$$

(This generalises both Proposition 1.9.1, to which it reduces as $u \to \infty$, and Proposition 1.9.6, which is the special case $d = 0$.)

Exercise 1.30 (Pareto versus Gamma tails). *Using Definition 1.10.2 and the densities of Section 1.7, show that the Pareto distribution has a heavier tail than the Gamma distribution, i.e. that* $\lim_{x\to\infty} f_{\text{Pareto}}(x)/f_{\text{Gamma}}(x) = +\infty$ *for any choice of parameters. (*Hint: *the Pareto density decays polynomially,* $f(x) \sim \alpha\theta^\alpha x^{-\alpha-1}$*, while the Gamma density decays exponentially,* $f(x) \sim c\,x^{\alpha-1}e^{-x/\theta}$*; compare the two using the ratio's logarithm.)*

Exercise 1.31 (Hazard rate of the Pareto distribution). *Show that the hazard rate of* $X \sim \text{Pareto}(\alpha,\theta)$ *is* $h_X(x) = \alpha/(x+\theta)$*, which is decreasing in* x*: consistent with the Pareto being heavy-tailed.*

Exercise 1.32 (Hazard rate of the Gamma distribution). *Let* $X \sim \text{Gamma}(\alpha,\theta)$*. By studying* $1/h_X(x) = \theta^\alpha\Gamma(\alpha)e^{x/\theta}x^{1-\alpha}\int_x^\infty \dots$ *— or, more directly, by examining* $\frac{d}{dx}\ln f_X(x) = \frac{\alpha-1}{x} - \frac{1}{\theta}$ *together with the shape of* S_X *— show that the hazard rate of the Gamma distribution is decreasing for* $\alpha < 1$ *and increasing for* $\alpha > 1$ *(with* $\alpha = 1$*, the exponential case, giving the constant hazard rate of Example 1.6.12).*

Exercise 1.33 (Mean excess loss function of the Gamma distribution). *Using Exercise 1.32 and Proposition 1.10.5, describe the qualitative behaviour of the mean excess loss function of a* $\text{Gamma}(\alpha,\theta)$ *distribution for* $\alpha < 1$ *and* $\alpha > 1$*, and use Proposition 1.10.7 to find* $\lim_{d\to\infty} e_X(d)$ *in both cases.*