

Mathematics 2

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Lecturers:

Alberto López Martín
Joshué Helí Ricalde Guerrero



**Lisbon School
of Economics
& Management**
Universidade de Lisboa

Lecture 3: Topology in $\mathbb{R}^n$.¹

¹References: [1], [2], [3]. See also the *Recommended Bibliography* section in the course's website.

We introduce some fundamental topological concepts. We focus only on the finite-dimensional Euclidean space: $\mathbb{R}^n$ for some given (fixed) positive integer n .

These concepts provide a rigorous, theoretical foundation for the notions of **limits**, **continuity**, and **differentiability**.

Namely, we formally define the notions of:

- ▶ **open** and **closed** sets;
- ▶ **interior**, **closure** and **boundary**;
- ▶ **accumulation** (or **limit**) **points**;
- ▶ **neighbourhoods** (or n -**dimensional balls**);
- ▶ **boundedness** and **compactness**.

Definition 1

Let $x, y \in \mathbb{R}^n$ two arbitrary points. The ***Euclidean distance*** between x and y is defined as

$$d(x, y) = \sqrt{\sum_{i=1}^n (x_i - y_i)^2}.$$

Some **important** properties of the Euclidean distance:

1. **Identity of indiscernibles:**

$$d(x, y) = 0 \iff x = y.$$

2. **Symmetry:**

$$d(x, y) = d(y, x).$$

3. **Triangle inequality:**

$$d(x, z) \leq d(x, y) + d(y, z).$$

Remark 2

Even though the notion of distance can be extended beyond Definition 1, the Euclidean distance continues to be a very helpful tool in engineering and applied mathematics in general.

Moreover: Definition 1 retains the natural intuition that the shortest distance between two points is a straight line. This is also what motivates the next definition.

Definition 3

Let $a \in \mathbb{R}^n$ and let $\varepsilon > 0$ be given. The ε -**neighbourhood of** a , or the **open ball centered at a of radius ε** is defined as

$$B_\varepsilon(a) = \{x \in \mathbb{R}^n : d(x, a) < \varepsilon\}.$$

Similarly, the **closed ball centered at a of radius ε** is

$$\overline{B}_\varepsilon(a) = \{x \in \mathbb{R}^n : d(x, a) \leq \varepsilon\}.$$

Intuitively, the neighborhood of a point a are all of the points in space surrounding it from every direction, up to a certain radius.

Geometrically, a 1-dimensional ball corresponds to an interval, with the center as its middle-point. In two dimensions a ball corresponds to a full disc, and in three dimensions to a solid sphere (possibly without the 'crust').

Furthermore: distances can produce **norms**, which measure the **length** of multidimensional objects (vectors).

Definition 4

Let $x \in \mathbb{R}^n$ be given. The ***Euclidean norm*** of x is defined as its distance from the origin:

$$\|x\| = d(x, 0).$$

Observe: It is immediate from Definition 4 that

$$\|x - y\| = d(x, y).$$

Consequently, an open ball can also be written as

$$B_\varepsilon(a) = \{x \in \mathbb{R}^n : \|x - a\| < \varepsilon\}.$$

Before we alluded to a 'crust' when interpreting the geometry of a neighborhood. We now make this notion rigorous.

Definition 5

Let $A \subseteq \mathbb{R}^n$ and let $a \in \mathbb{R}^n$ be given.

- If there exists $\varepsilon > 0$ (i.e. a positive radius) such that

$$B_\varepsilon(a) \subseteq A,$$

we then say that a ***is an interior point of*** A .

Definition 5 (cont.)

Let $A \subseteq \mathbb{R}^n$ and let $a \in \mathbb{R}^n$ be given.

- If there exists $\varepsilon > 0$ such that

$$B_\varepsilon(a) \subseteq \mathbb{R}^n \setminus A$$

we then say that a **is an exterior point of** A .

Definition 5 (cont.)

Let $A \subseteq \mathbb{R}^n$ and let $a \in \mathbb{R}^n$ be given.

► If for every $\varepsilon > 0$,

$$B_\varepsilon(a) \cap A \neq \emptyset \quad \text{and} \quad B_\varepsilon(a) \cap (\mathbb{R}^n \setminus A) \neq \emptyset, \quad (1)$$

we then say that a **is a boundary point of** A .²

²The symbol $\emptyset$ in (1) represents the *set with no elements*, $\{\}$, or *empty set*.

Accordingly, the **interior**, the **exterior** and the **boundary** of a given set $A \subseteq \mathbb{R}^n$ are defined as the collection of **interior**, **exterior** and **boundary points** of A , respectively:

$$\text{int } A = \{a \in \mathbb{R}^n : a \text{ is an interior point of } A\},$$

$$\text{ext } A = \{a \in \mathbb{R}^n : a \text{ is an exterior point of } A\},$$

$$\text{bd } A = \{a \in \mathbb{R}^n : a \text{ is boundary point of } A\}.$$

Remark 6

The sets

$\text{int } A,$

$\text{ext } A,$

$\text{bd } A,$

are pairwise disjoint,

$$\text{int } A \cap \text{ext } A = \emptyset,$$

$$\text{int } A \cap \text{bd } A = \emptyset,$$

$$\text{ext } A \cap \text{bd } A = \emptyset.$$

As a result, it holds that

$$\text{int } A \cup \text{ext } A \cup \text{bd } A = \mathbb{R}^n. \quad (2)$$

In other words: every point in $\mathbb{R}^n$ can be classified as an interior, exterior, or boundary point relative to A .

Definition 7

A set $A \subseteq \mathbb{R}^n$ is said to be **open** if

$$A = \text{int } A.$$

Equivalently, every point $a \in A$ has an open ball contained entirely in A :

$$\forall a \in A, \quad \exists \varepsilon > 0 : B_\varepsilon(a) \subseteq A.$$

Definition 8

The **closure** of A is

$$\text{cl } A = \text{int } A \cup \text{bd } A.$$

Consequently, a set A is said to be **closed** if

$$A = \text{cl } A.$$

That is to say, a closed set contains all of its boundary points.

Why should we care about the *openness* or *closedness* of sets?

As alluded earlier, these are the building blocks for the formal definition of **limits**, which in turn define **continuity** and, eventually, **differentiability**.

Remark 9

Openness and closedness are not mutually exclusive.

There exist sets which are **both open and closed** (sometimes referred to as ***clopen***).

At the same time, there also exist sets which are **neither open nor closed**.

Example 10

The set $\mathbb{R}$ is both open and closed in $\mathbb{R}$.

Proof.

Suggested exercise.³



³*Hint:* Prove that both $\mathbb{R}$ and the empty set $\emptyset$ are open according to Definition 7. Conclude using the decomposition (2).

Example 11

The set of rational numbers $\mathbb{Q}$ is neither open nor closed in $\mathbb{R}$.

Unlike Example 10, it may be less evident how Example 11 follows from the definitions. However, it does motivate the concepts we introduce next.

Observe that, implicitly, decomposition (2) states that a **set** $A \subseteq \mathbb{R}$ **is closed if and only if its complement** $A^c = \mathbb{R} \setminus A$ **is open.**

Thus, a natural question to ask is: *If I cannot ascertain the openness/closedness of a set or its complement directly, is there a way for me to close them?*

Proposition 12

The closure of a set A is the smallest closed set containing A . In particular, if C is closed and $A \subseteq C$, then

$$\text{cl } A \subseteq C.$$

Equivalently,

$$\text{cl } A = \bigcap \{C : C \text{ is closed and } A \subseteq C\}.$$

Definition 13

Let $a \in \mathbb{R}^n$ be given.

- We say $a \in A$ is an **accumulation point** of A if every neighbourhood of a contains a point of A different from a :

$$\forall \varepsilon > 0, \quad B_\varepsilon(a) \cap (A \setminus \{a\}) \neq \emptyset.$$

The set of accumulation points is called the **derived set of** A and is denoted by A' .

- Conversely, we say $a \in A$ is an **isolated point** of A if there exists $\varepsilon > 0$ such that

$$B_\varepsilon(a) \cap A = \{a\}.$$

Proposition 14

The closure of A can be characterised by its accumulation and isolated points:

$$\text{cl } A = A' \cup \{\text{isolated points of } A\}.$$

In other words, the **closure of a set consists precisely of all points which are either accumulation points of A or isolated points belonging to A .**

Furthermore: the closure (thus, the openness/closedness) of a set corresponds to the **local structure of its points** (where are they **accumulated**, where are they **isolated**).

Example 15

Let

$$A = \{(x, y) \in \mathbb{R}^2 : y \geq x\}.$$

Then, A is **closed**.

Proof.

Suggested exercise.



Example 16

Let

$$A = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 4\},$$

Then, A is **closed**.

Proof.

Suggested exercise.⁴



⁴*Hint:* Observe that A corresponds to a disk of radius 2 centered at the origin.

Example 17

Let

$$A = \{(x, y) \in \mathbb{R}^2 : y \leq x \text{ and } x^2 + y^2 < 4\}.$$

Then, A is **not closed**.

Proof.

Suggested exercise.⁵



⁵*Hint:* Notice that now A corresponds to a *fraction* of the disk of radius 2 centered at the origin. Which sections are missing?

We end with two important topological concepts that will prove useful down the line, wherever **continuity**, **differentiation** and **integration** are involved.

Definition 18

Let $A \subseteq \mathbb{R}^n$ be given.

1. We say A is **bounded** if there exists $\varepsilon > 0$ such that

$$A \subseteq B_\varepsilon(0).$$

That is to say, the *entire set can be contained in a sufficiently large ball centered at the origin.*

2. We say A is **compact** if it is both **closed and bounded**.

Example 15 (cont.)

Let

$$A = \{(x, y) \in \mathbb{R}^2 : y \geq x\}.$$

Then, A is **unbounded**.

Proof.

Suggested exercise.



Example 16 (cont.)

Let

$$A = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 4\}.$$

Then, A is **bounded** and, consequently, **compact**.

Proof.

Suggested exercise.



Example 17 (cont.)

Let

$$A = \{(x, y) \in \mathbb{R}^2 : y \leq x \text{ and } x^2 + y^2 < 4\}.$$

Then, A is **bounded** but **not compact**.

Proof.

Suggested exercise.



Summary

- ▶ Euclidean distance provides the basic metric structure of $\mathbb{R}^n$.
- ▶ Open balls define neighbourhoods.
- ▶ Points can be classified as interior, exterior or boundary points.
- ▶ Accumulation and isolated points describe the local structure of a set.
- ▶ Open sets satisfy

$$A = \text{int } A.$$

- ▶ Closed sets satisfy

$$A = \text{cl } A = \text{ext } A \cup \text{bd } A.$$

- ▶ A compact set is closed and bounded.

References

- [1] Tom M Apostol. *Calculus, Volume I: One-Variable Calculus, with an Introduction to Linear Algebra*. John Wiley & Sons, 1967.
- [2] Tom M Apostol. *Calculus, Volume II: Multi-Variable Calculus and Linear Algebra, with Applications to Differential Equations*. John Wiley & Sons, 1989.
- [3] Robert Gardner Bartle and Donald R. Sherbert. *Introduction to real analysis*. 4th ed. Hoboken, NJ: Wiley, 2011.