Microeconomics

Chapter 1
Technology

Fall 2026

Technology

- The technology of a firm describes how it transforms **inputs to produce outputs**.
- To study firms' choices we need ways to summarize their production possibilities.
- One way to summarize this is the production function: $y = f(\mathbf{x})$, where y is output and $\mathbf{x}$ is a vector of inputs.
- We often assume a certain functional form for $f(\cdot)$, such as Cobb-Douglas, CES, etc.
- This chapter:
  - Discusses more general ways to describe a firm's technology.
  - Introduces assumptions on the technology that are in line with the well-known production functions.

Production plan

- Suppose the firm has n possible goods that can serve as inputs and/or outputs.
- Let j be any good, y_j^i are inputs, y_j^o outputs
- Let us define that $y_j^i > 0$ if the firm uses good j as an input. Hence, $y_j^i = 0$ if the firm does not use good j as an input. Similar for y_j^o with output.
- **Net output** for good j: $y_j = y_j^o - y_j^i$.
- **Production plan**: a list of net outputs of various goods, which is described by the vector $\mathbf{y}$ in R^n.

$$\mathbf{y} = \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} = \begin{pmatrix} y_1^o - y_1^i \\ \vdots \\ y_n^o - y_n^i \end{pmatrix} = \begin{pmatrix} y_1^o \\ \vdots \\ y_n^o \end{pmatrix} - \begin{pmatrix} y_1^i \\ \vdots \\ y_n^i \end{pmatrix}$$

Production possibilities set

- **Production possibilities set**: the set of technologically feasible production plans.
- This is denoted by Y, and is a subset of R^n:

$$Y = \{\mathbf{y} \text{ in } R^n : \mathbf{y} \text{ is technologically feasible}\}$$

- What is technological feasible depends on the time period
  - Some inputs may be fixed in the short run.
  - So only the production that is compatible with this is feasible.
  - Production plans can be either **immediately** feasible (long-run) or **eventually** feasible (short-run)

Short-run production possibilities set

- Let $\mathbf{z}$ be a vector of the short-run constraints on the inputs such that z is in R^n
- Then the short run production possibility set y is denoted by $Y(\mathbf{z})$
- $Y(\mathbf{z})$ is the feasible plans consistent with the constraints $\mathbf{z}$
- For example, let there be n restrictions and consider that $y_n = \bar{y}_n$ in the short term:

$$Y(\bar{y}_n) = \{\mathbf{y} \text{ in } R^n : \mathbf{y} \text{ is technologically feasible and } y_n = \bar{y}_n\}$$

Example of a short-run production possibilities set

- Let $y = f(l, k)$
- Production plan: $(y, -l, -k)$
- Let l be the **variable** input and k the **fixed** input
- Then the short-run production possibilities set

$$Y(\bar{k}) = \{(y, -l, -k) \text{ is in Y} : k = \bar{k}\}$$

Input requirement set

- Let a firm produce one output y, and this output is not used as an input and vice versa.
- Then $\mathbf{y}$ can be described by $(y, -\mathbf{x})$
  - $\mathbf{x}$ is a vector of (positive) inputs
  - that produces y units of output.

$$\mathbf{y} = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix} = \begin{pmatrix} y_1^o - 0 \\ 0 - y_2^i \\ \vdots \\ 0 - y_n^i \end{pmatrix} = \begin{pmatrix} y_1 \\ -x_1 \\ \vdots \\ -x_n \end{pmatrix} = \begin{pmatrix} y \\ -\mathbf{x} \end{pmatrix}$$

- With $(y, -\mathbf{x})$ we can define the input requirement set.
- **Input requirement set**: the set of all input bundles $\mathbf{x}$ that can produce at least y units of outputs.

$$V(y) = \{\mathbf{x} \text{ in } R_+^n : (y, -\mathbf{x}) \text{ is in } Y\}$$

Isoquant

- Consider a function $y = f(\mathbf{x})$, a **level set** $L(\bar{y})$ for a fixed value $y = \bar{y}$ is described by:

$$L(\bar{y}) = \{\mathbf{x} : f(\mathbf{x}) = \bar{y}\}$$

- (recall Chapter 7)
- An isoquant is an other example of a **level set**.
- **Isoquant**: all input bundles $\mathbf{x}$ that produce exactly y units of output.

$$Q(y) = \{\mathbf{x} \text{ in } R^n_+ : \mathbf{x} \text{ is in } V(y) \text{ but not in } V(y') \text{ for } y' > y\}$$

Production function

- **Production function**: picks out the maximum output as a function of the inputs.

$$f(\mathbf{x}) = \{y \text{ in } R\text{: } y \text{ is the maximum output associated with } -\mathbf{x} \text{ in } Y\}$$

- Most efficient use of resources (vs. all feasible ones in PPS)

Example: Cobb-Douglas technology

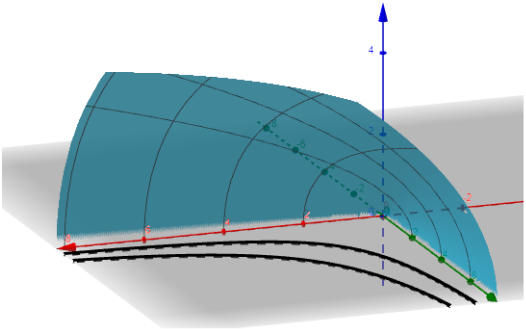
$$Y = \{(y, -x_1, -x_2) \in \mathbb{R}^3 : x_1^a x_2^{1-a} \geq y\}$$

$$V(y) = \{(x_1, x_2) \in \mathbb{R}^2_+ : x_1^a x_2^{1-a} \geq y\}$$

$$Q(y) = \{(x_1, x_2) \in \mathbb{R}^2_+ : x_1^a x_2^{1-a} = y\}$$

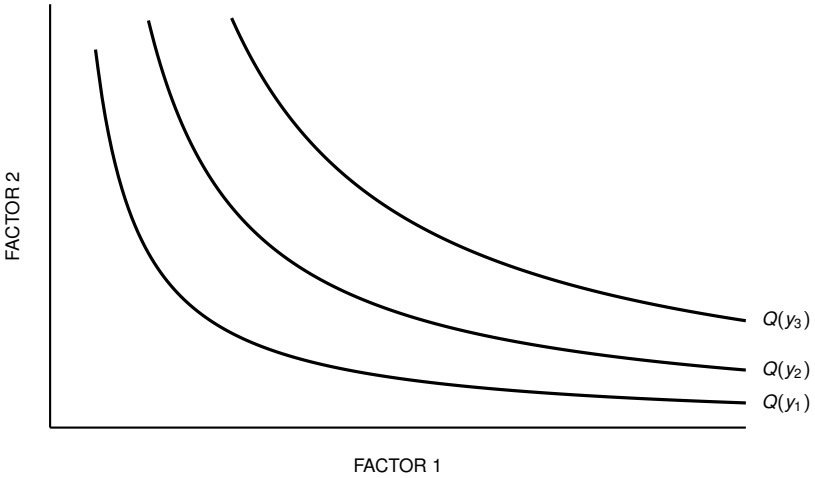
$$f(x_1, x_2) = x_1^a x_2^{1-a}.$$

The Cobb-Douglas function in 3D



The figure shows $y = x_1^{0.5} x_2^{0.5}$. y is blue axis, x_1 is red axis, and x_2 is green axis. The dark lines show the 2D isoquants with $y = 2$ and $y = 3$. The Cobb-Douglas function is **concave** and so it has a **convex input requirement set**.

Example: Cobb-Douglas technology



Example: Leontief technology

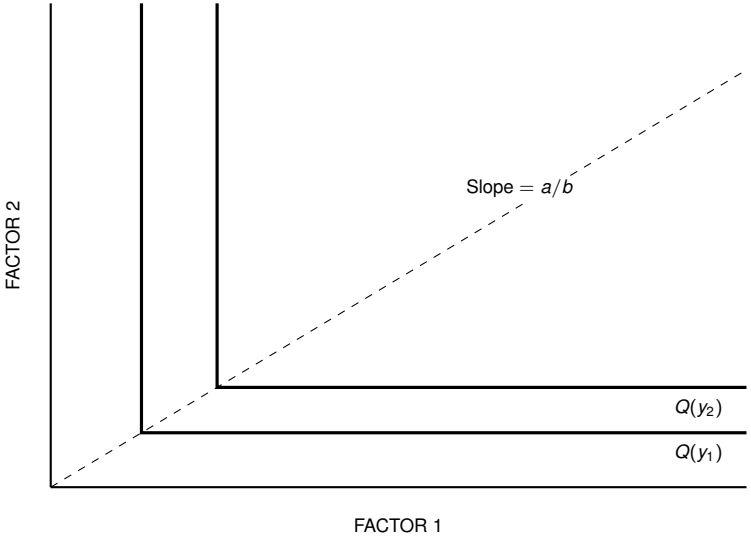
$$Y = \{(y, -x_1, -x_2) \in \mathbb{R}^3 : \min(ax_1, bx_2) \geq y\}$$

$$V(y) = \{(x_1, x_2) \in \mathbb{R}^2_+ : \min(ax_1, bx_2) \geq y\}$$

$$Q(y) = \{(x_1, x_2) \in \mathbb{R}^2_+ : \min(ax_1, bx_2) = y\}$$

$$f(x_1, x_2) = \min(ax_1, bx_2).$$

Example: Leontief technology



Exercise

Consider a Leontief production function $y = \min(ax_1, bx_2)$. Draw the short-run production possibilities set $Y(z)$ while $x_1 = z$. Hence, draw a graph that contains the following set:

$$Y(z) = \{(y, -x_2) \text{ in } R^2\text{: } \min(ax_1, bx_2) \geq y \text{ , } x_1 = z\}$$

Activity analysis: Example of an input requirement set

- Suppose that we can produce output y by using two inputs x_1 and x_2.
  - **Technique A:** one unit of x_1 and two units of x_2 produce one unit of y.
  - **Technique B:** two units of x_1 and one unit of x_2 produce one unit of y.
- The input requirement set can be written as:

$$V(1) = \{(1,2),(2,1)\}$$

- Can we write the input requirement set for $y = 2$ (and any y) units of output as follows?:

$$V(2) = \{(2,4),(4,2)\}$$
$$V(y) = \{(y,2y),(2y,y)\}$$

- Incomplete, does not include mixtures of technique A and B. Let y_A and y_B be the amount produced via technique A and B, respectively. Then:

$$V(2) = \{(2,4),(3,3),(4,2)\}$$
$$V(y) = \{(y_A + 2y_B, 2y_A + y_B) : y = y_A + y_B\}$$

Example of an input requirement set

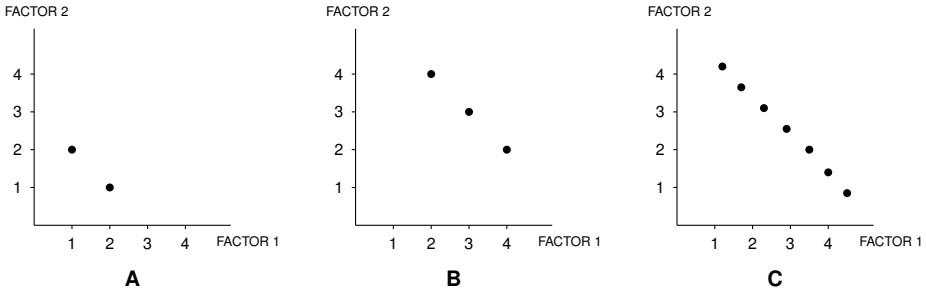


Figure. Input requirement sets. Panel A depicts $V(1)$, panel B depicts $V(2)$, and panel C depicts $V(y)$ for a larger value of y.

Monotonic input requirement set

- Same example as previous slides. Suppose $\mathbf{x} = (3, 2)$. We can produce one unit of output with $(1, 2)$ via technique A and have a leftover input of $(2, 0)$.
- If **free disposal** of inputs is allowed, it is reasonable to assume that if we can produce y with $\mathbf{x}$, and $\mathbf{x}' \geq \mathbf{x}$, we can also produce y with $\mathbf{x}'$.
- Thus, the input requirement sets should be monotonic in the following sense

  Monotonicity: if $\mathbf{x}$ is in $V(y)$, and $\mathbf{x}' \geq \mathbf{x}$, then $\mathbf{x}'$ is in $V(y)$.

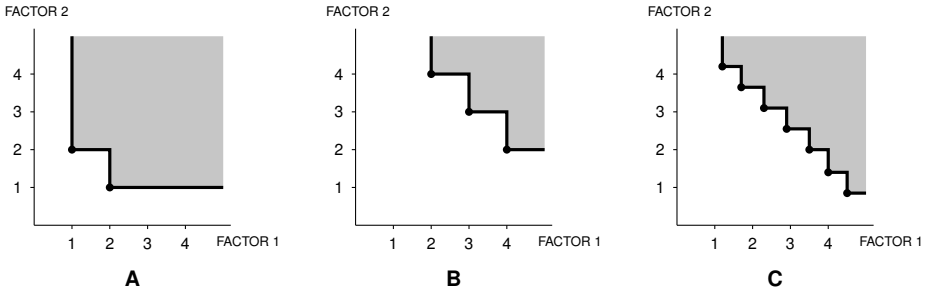


Figure. Monotonicity. Here are the same three input requirement sets if we also assume monotonicity.

Monotonicity of production sets

- Appropriate assumption also for production sets:
- If $\mathbf{y}$ is in Y, and $\mathbf{y}' \leq \mathbf{y}$, then $\mathbf{y}'$ must also be in Y.

Convex input requirement set

- Now assume we want to produce 100 units of output.
- If we replicate technique A or B 100 times (without mixture), the input requirement set is:
$$V(100) = \{(100, 200), (200, 100)\}$$
- Again, consider the possibility of a mixture of techniques, e.g., using technique A 25 times and B 75 times. In this case, $0.25 \times (100, 200) + 0.75 \times (200, 100) = (175, 125)$ is also in $V(100)$. More generally,
$$t \begin{pmatrix} 100 \\ 200 \end{pmatrix} + (1-t) \begin{pmatrix} 200 \\ 100 \end{pmatrix} = \begin{pmatrix} t100 + (1-t)200 \\ t200 + (1-t)100 \end{pmatrix}$$
may be in $V(100)$ for $0 \leq t \leq 1$ in case a mixture of techniques is possible.
- **Convexity:** if $\mathbf{x}$ and $\mathbf{x}'$ are in $V(y)$, and $V(y)$ is a convex set, then $t\mathbf{x} + (1-t)\mathbf{x}'$ is in $V(y)$ for all $0 \leq t \leq 1$.

Convex input requirement set

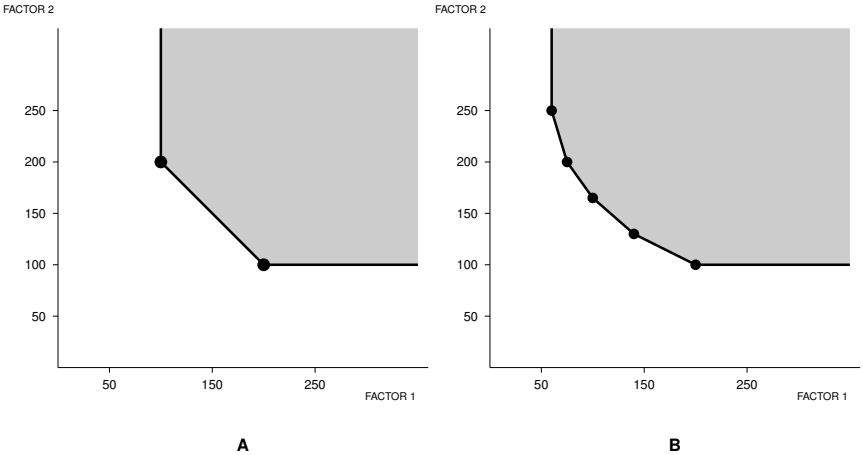


Figure. Convex input requirement sets. If $\mathbf{x}$ and $\mathbf{x}'$ can produce y units of output, then any weighted average $t\mathbf{x} + (1 - t)\mathbf{x}'$ can also produce y units of output. Panel *A* depicts a convex input requirement set with two underlying activities; panel *B* depicts a convex input requirement set with many activities.

Parametric representations of technology

We summarize the monotonic and convex input set by a “smoothed” input set. This looks like an isoquant from a Cobb-Douglas function.

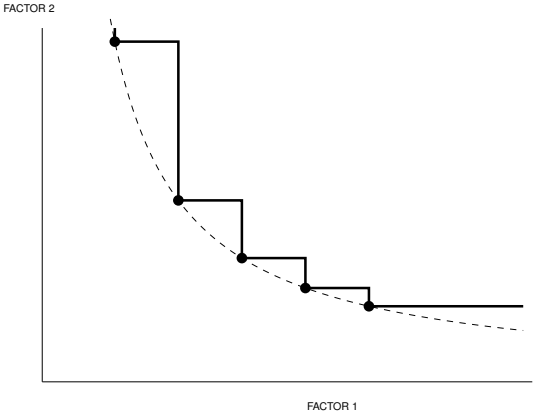


Figure. Smoothing an isoquant. An input requirement set and a “smooth” approximation to it.

Do not take these functional forms literally. The engineering data describes the production plans, and we pick the functional form that best describes this data.

Concave and convex **functions** and convex **sets**

- Let $\mathbf{x}^1$ and $\mathbf{x}^2$ be two points in the domain of function f. Define the following convex combinations:

$$\begin{aligned}\mathbf{x}^t &= t\mathbf{x}^1 + (1-t)\mathbf{x}^2 \\ y^t &= tf(\mathbf{x}^1) + (1-t)f(\mathbf{x}^2)\end{aligned}$$

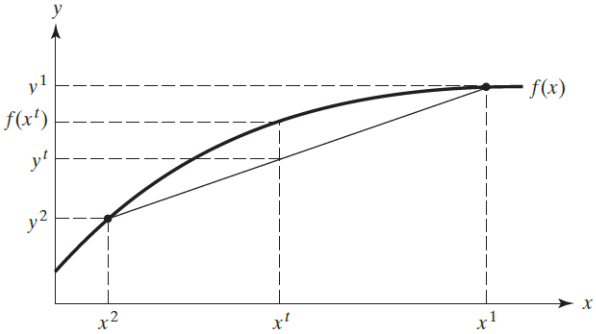
- **Concave function:** f is a concave function if for all $\mathbf{x}^1$ and $\mathbf{x}^2$ we have that for $0 \leq t \leq 1$

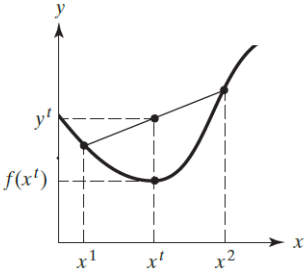
$$f(\mathbf{x}^t) \geq y^t$$

- **Convex function:** f is a convex function if for all $\mathbf{x}^1$ and $\mathbf{x}^2$ we have that for $0 \leq t \leq 1$

$$f(\mathbf{x}^t) \leq y^t$$

Concave and convex **functions** and convex **sets**





Relationship between concave and convex functions and convex sets:

- f is concave (left) $\leftrightarrow$ points on and below the graph form a convex set
- f is convex (right) $\leftrightarrow$ points on and above the graph form a convex set

Concave and convex **functions** and convex **sets**

Upper contour set for function f: let $f(\bar{\mathbf{x}}) = \bar{y}$ for some $\mathbf{x} = \bar{\mathbf{x}}$, then

$$\{\mathbf{x}\colon f(\mathbf{x}) \geq \bar{y}\}$$

Lower contour set for function f: let $f(\bar{\mathbf{x}}) = \bar{y}$ for some $\mathbf{x} = \bar{\mathbf{x}}$, then

$$\{\mathbf{x}\colon f(\mathbf{x}) \leq \bar{y}\}$$

Further relationships between concave and convex functions and convex sets:

- f is concave $\leftrightarrow$ upper contour set is a convex set
- f is convex $\leftrightarrow$ lower contour set is a convex set

Let $f(\mathbf{x})$ be a production function, then we defined the upper contour set as the input requirement set. It follows that a **(quasi)concave production function** $f(\mathbf{x})$ like a Cobb-Douglas has a **convex input requirement set**.

Marginal productivity

Consider a setting with two inputs, so that $f(\mathbf{x}) = f(x_1, x_2)$.

Marginal productivity of input 1 or 2: how much does output change if we change input 1 or 2.

$$MP_i = \frac{\partial f(\mathbf{x})}{\partial x_i}, \quad \text{for } i = 1, 2.$$

The technical rate of substitution

- **Technical rate of substitution:** how easy (or difficult) is it for a firm to change between the usage of x_1 and x_2 while keeping output y constant?
- Let $x_2(x_1)$ be the isoquant at constant output $y = \bar{y}$, then:

$$TRS = \frac{\partial x_2(x_1)}{\partial x_1}.$$

- The function $x_2(x_1)$ satisfies the identity by definition $f(x_1, x_2(x_1)) = \bar{y}$
- Differentiating the above identity, we obtain:

$$\frac{df(\mathbf{x})}{dx_1} = \frac{\partial f(\mathbf{x})}{\partial x_1} + \frac{\partial f(\mathbf{x})}{\partial x_2}\frac{\partial x_2(x_1)}{\partial x_1} = 0.$$

- Rearranging, we can get an expression for the TRS:

$$TRS = \frac{\partial x_2(x_1)}{\partial x_1} = -\frac{\frac{\partial f(\mathbf{x})}{\partial x_1}}{\frac{\partial f(\mathbf{x})}{\partial x_2}} = -\frac{MP_1}{MP_2}.$$

- The TRS for the firm is analogous to the marginal rate of substitution (MRS) from Chapter 7.

Exercise

1. What is the technical rate of substitution of the following Cobb-Douglas production function?

$$f(\mathbf{x}) = x_1^{\alpha} x_2^{1-\alpha}$$

2. Draw the graph of an isoquant that has a technical rate of substitution of zero. What does this mean for the production function?

The elasticity of substitution

Elasticity of substitution (σ): the percentage change in the input ratio divided by the percentage change in the TRS, with output being held fixed.

$$\sigma = \frac{\Delta(\frac{x_2}{x_1})}{\frac{x_2}{x_1}} \Big/ \frac{\Delta TRS}{TRS}$$

Rewrite, take the limit of this expression as Δ goes to zero, and use the logarithmic derivative:

$$\sigma = \frac{TRS}{\frac{x_2}{x_1}} \frac{\Delta(\frac{x_2}{x_1})}{\Delta TRS} = \frac{TRS}{\frac{x_2}{x_1}} \frac{\partial(\frac{x_2}{x_1})}{\partial TRS} = \frac{\partial ln(\frac{x_2}{x_1})}{\partial ln(TRS)}$$

$\Rightarrow$ In general **elasticities** are the percentage change in y due to the percentage change in x, and can be calculated by the logarithmic derivative.

Exercise

What is the elasticity of substitution of the following Cobb-Douglas production function?

$$f(\mathbf{x}) = x_1^{\alpha} x_2^{1-\alpha}$$

Returns to scale

- It sounds reasonable that if we scale the inputs by some amount t, we also will produce t times as much output. After all, we can replicate what we did initially t times.
  Constant returns to scale: $f(t\mathbf{x}) = tf(\mathbf{x})$ for all $t \geq 0$.
- Does scaling the inputs allows for more efficient means of production?
  Increasing returns to scale: $f(t\mathbf{x}) > tf(\mathbf{x})$ for all $t > 1$.
- Does scaling the inputs allows for less efficient means of production?
  Decreasing returns to scale: $f(t\mathbf{x}) < tf(\mathbf{x})$ for all $t > 1$.
- Examples:

Homogeneous technologies

- **Homogenous functions:** a function $f(\mathbf{x})$ is homogenous of degree k if $f(t\mathbf{x}) = t^k f(\mathbf{x})$. Special cases:
  - Degree zero: $f(t\mathbf{x}) = f(\mathbf{x})$
  - Degree one: $f(t\mathbf{x}) = tf(\mathbf{x})$
  - A function f is homogeneous of degree 1 $\Leftrightarrow$ A function f has constant returns to scale.
- A function $g : R \to R$ is a **positive monotonic transformation** when $h(\mathbf{x}') > h(\mathbf{x})$ (strictly increasing)implies $g(h(\mathbf{x}')) > g(h(\mathbf{x}))$.
  - Note that $h(\mathbf{x}') = y'$ and $h(\mathbf{x}) = y$ are numbers in R, so it is simply a transformation that preserves the original order of y.

Homothetic technologies

- **Homothetic functions:** a monotonic transformation of a function that is homogeneous of degree 1.
- A function $f(\mathbf{x})$ is **homothetic** $\Leftrightarrow f(\mathbf{x}) = g(h(\mathbf{x}))$, where h is homogeneous of degree 1 and g is a monotonic function.

Homogeneous and homothetic technologies

Figure. Homogeneous and homothetic functions. Panel *A* depicts a function that is homogeneous of degree 1. If x and x' can both produce y units of output, then $2x$ and $2x'$ can both produce $2y$ units of output. Panel *B* depicts a homothetic function. If x and x' produce the *same* level of output, y, then $2x$ and $2x'$ can produce the *same* level of output, but not necessarily $2y$.

For homogeneous and homothetic functions the isoquants are "blown up" versions of a single isoquant. Moreover, for either of these functions the TRS is independent of the scale of production.

Exercise

1. Prove the following statement: if $f(\mathbf{x})$ is homogeneous of degree $k \geq 1$, then $\frac{\partial f(\mathbf{x})}{\partial x_i}$ is of homogeneous degree $k - 1$.

2. Use your answer to the previous question to show that the TRS of a production function with constant returns to scale is independent of the scale of production.

Homework exercises

Exercises: 1.2, 1.3, 1.8, 1.10 (show only that Y must be convex) and exercises on the slides.

All homework exercises are relevant for the exam.