Universidade de Lisboa
Instituto Superior de Economia e Gestão
BsC in Economics, Finance and Management

Mathematics II – 1st Semester - 2023/2024

Repeat Assessment - 1st of February 2024

Duration: $(120+\varepsilon)$ minutes, $|\varepsilon| \leq 30$

Version A

Name: ....

Student ID #: ....

Part I

- Complete the following sentences in order to obtain true propositions. The items are independent from each other.
- There is no need to justify your answers.

(a) *(4)* If $A : \mathbb{R}^2 \to \mathbb{R}^2$ is a linear map given by $A(x,y) = (3x+4y, -7y)$ then the eigenvalues of A^{-1} are ..... and ......

(b) *(7)* The (maximal) domain of $f : D_f \to \mathbb{R}$ defined by

$$f(x,y) = \frac{\sqrt{y-x^2}}{\ln(x^2+y^2)}$$

is the set

$D_f =$ ....

and its **planar representation** in the cartesian plane (x,y) is:

(c) *(7)* The **symmetric** matrix associated to the quadratic form in $\mathbb{R}^3$:

$$Q(x,y,z) = (x+3y)^2 + 5z^2$$

is

$$\begin{pmatrix} ..... & ..... & ..... \\ ..... & ..... & .... \\ ..... & ..... & .... \end{pmatrix}$$

and Q may be classified as .... defined.

Since Q is positively homogeneous of degree ...., Euler's identity says that:

$$x\frac{\partial Q}{\partial x}(x,y,z) + y\frac{\partial Q}{\partial y}(x,y,z) + z\frac{\partial Q}{\partial z}(x,y,z) = \text{..................}$$

(d) *(7)* With respect to the set

$$\Omega = \left\{(x,y) \in \mathbb{R}^2 : x \in [1,2] \ \wedge \ \frac{1}{x} \leq y\right\}$$

we may conclude that $\left(\frac{3}{2}, 3\right)$ is an .... point of Ω and

$\partial\Omega = fr(\Omega) =$ ....

Since Ω is unbounded, then Ω is **not** ....

(e) *(4)* With respect to the continuous map $f : \mathbb{R}^2 \to \mathbb{R}$, its directional derivatives along the non-null vector (v_1, v_2) at $(0,0)$ are given by

$$D_{(v_1,v_2)}f(0,0) = \frac{v_1^2}{1+v_2^2}.$$

Hence we may conclude that:

$$\frac{\partial f}{\partial x}(0,0) = \text{................} \quad \text{and} \quad \frac{\partial f}{\partial y}(0,0) = \text{................}$$

(f) *(4)* If $u_n = \left(\ln\left(1 - \frac{1}{2n+1}\right)^{n+\frac{1}{2}} ; \frac{1}{n}\right)$, $n \in \mathbb{N}$ and $f(x,y) = -2x + \cos y$, then

$$\lim_{n\to+\infty} f(u_n) = \text{......}$$

(g) *(6)* If $f(x,y) = \dfrac{x^2y}{x^2-y^2}$ where $x \neq \pm y$, then

$$.........= \lim_{(x,y)\to(0,0),y=x^2+x} f(x,y) \neq \lim_{(x,y)\to(0,0),y=2x} f(x,y) = ...,$$

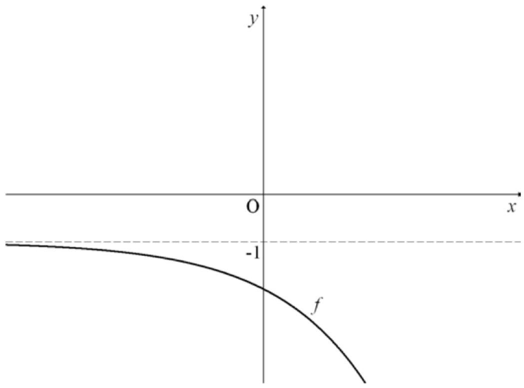
which means that f

....

(h) *(6)* If $f(x,y) = x^2y$, $x(t) = e^{3t}$ and $y(t) = \sin t$, by the *Chain rule* we get:

$\dfrac{df}{dt}(t) =$ ....

(i) *(4)* Let $g : \mathbb{R}^2 \to \mathbb{R}$ be defined by $g(x,y) = -y^2 + \cos x$ and $f : \mathbb{R} \to \mathbb{R}$ be a map whose graphical representation is given by:



Then

$$\lim_{(x,y)\to(0,-\infty)} \frac{-3}{(f \circ g)(x,y)} = ...$$

(j) *(4)* With respect to a map $f : \mathbb{R}^2 \to \mathbb{R}$, one knows that

$$\nabla f(x,y) = (-2x\sin(x^2+y); -\sin(x^2+y)).$$

If $f(x,y)$ does not have constant terms in both components, then

$$f(x,y) = ...$$

(k) *(3)* The **differential of order 2** of the map $f(x,y) = 1 + 3x + y^2$ at the point $(0,0)$ is given by

$$D_2f(0,0)(h_1,h_2) = ...h_1^2 + ...h_1h_2 + ...h_2^2$$

(l) *(4)* With respect to the C^2 map $f : \mathbb{R}^2 \to \mathbb{R}$, we know that $\nabla f(3,2) = (0,0)$ and

$$H_f(3,2) = \begin{pmatrix} -5 & \ldots\ldots \\ 0 & \ldots\ldots \end{pmatrix}.$$

Then, we conclude that $(3,2)$ is a local **maximizer** of f.

(m) *(4)* Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by $f(x,y) = \sin(2x+y)$. Then:

$$\frac{\partial^{200} f}{\partial x^{200}}(x,y) = \ldots\ldots\ldots\ldots$$

(n) *(4)* The following equality holds:

$$\int_{-1}^{1} \int_{0}^{\sqrt{1-x^2}} 1 \text{ dy dx} = \ldots\ldots$$

(o) *(4)* The map $y(x) = \cos(4x)$, $x \in \mathbb{R}$, is a solution of the IVP $\begin{cases} y'' = \ldots\ldots \\ y(\ldots\ldots) = 0 \end{cases}$.

(p) *(4)* The .... law (associated to a given population of size p that depends on the time $t \geq 0$) states that

$$p' = kp, \qquad k \in \mathbb{R}.$$

If $p(0) = 10$ and $k = -2$, then the **solution** of the previous differential equation is given **explicitly** by

...., where $t \in \mathbb{R}_0^+$.

(q) *(3)* Assuming that y depends on x, any solution of the differential equation $y' = e^{-x}$ is monotonically ....

(r) *(6)* For $b \in \mathbb{R}$, consider the following differential equation (P depends on $t \in \mathbb{R}$):

$$P'' + bP' + 5P = 15$$

The solution P is periodic on t if and only if ....

If $b = 2$, then $\lim_{t \to +\infty} P(t) = \ldots\ldots\ldots$

Part II

- Give your answers in exact form. For example, $\frac{\pi}{3}$ is an exact number while 1.047 is a decimal approximation for the same number.
- In order to receive credit, you must show all of your work. If you do not indicate the way in which you solve a problem, you may get little or no credit for it, even if your answer is correct.

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1. For $\alpha \in \mathbb{R}$, consider the following matrix $\mathbf{A} = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 4 & 0 \\ 0 & 0 & \alpha \end{bmatrix}$.
   (a) Classify the quadratic form $Q(X) = X^T \mathbf{A} X$, $X \in \mathbb{R}^3$, as function of α.
   (b) If $\alpha = 5$, find the eigenspace associated to the eigenvalue 5 and indicate its geometrical multiplicity.

2. Consider the map $f(x,y) = \begin{cases} \dfrac{x(x+y)}{\sqrt{x^2+y^2}} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$.
   (a) **Show** that f is continuous in $\mathbb{R}^2$.
   (b) Find the directions $(v_1, v_2) \in \mathbb{R}^2 \backslash \{(0,0)\}$ along with there exists directional derivative of f at $(0,0)$. In these cases, compute it.

3. Consider the map $f : \mathbb{R}^2 \to \mathbb{R}$ defined by

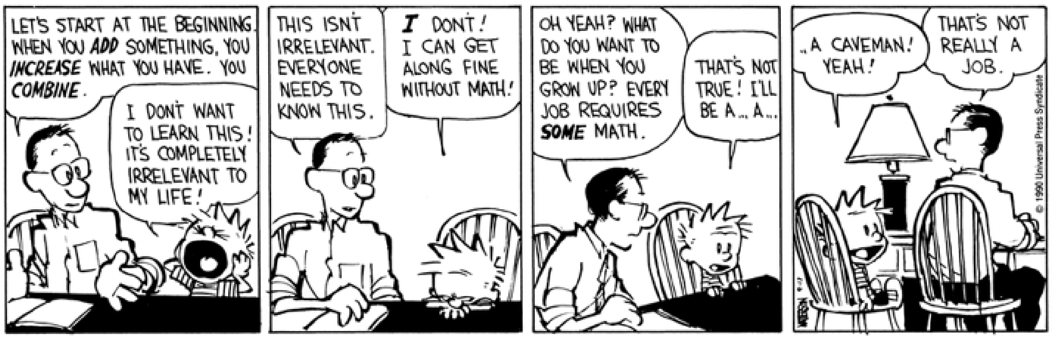
$$f(x,y) = y(x+2)^2 + xy^2 - yx^2$$

   (a) Identify the critical points of f and show that they are saddle-points.
   (**Suggestion:** first simplify the expression of $f(x,y)$)
   (b) Consider the set $M = \{(x,y) \in \mathbb{R}^2 : x^2 + (y+2)^2 \leq 10\}$.

   Without performing computations:
   (i) show that $f|_M$ has a global maximum and a global minimum.
   (ii) show that the global extrema of $f|_M$ lie on the boundary of M.

4. Consider the set $\Omega = \{(x, y) \in \mathbb{R}^2 : x^2 \leq y \leq -3x \wedge x \leq -1\}$.
   (a) Sketch the set Ω in the cartesian plane (x, y).
   (b) Compute
$$\iint_\Omega \frac{e^{\frac{y}{x}}}{x} \text{ dx dy}.$$

5. Consider the following IVP (y is a function of x):
$$\begin{cases} x^4 y' + 4x^3 y = \cos x \\ y(\pi) = \pi \end{cases}$$
Write the solution y of the IVP, identifying its maximal domain.



Credits:

I	II.1(a)	II.1(b)	II.2(a)	II.2(b)	II.3(a)	II.3(b)	II.4(a)	II.4(b)	II.5
85	10	10	10	15	15	15	5	15	20