Universidade de Lisboa
Lisbon School of Economics and Management

PhD in Economics
Advanced Mathematical Economics
2023-2024

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Exam: **Re-sit Exam** (Época Recurso) solutions
1.2.2024 (18.00h-21.00h)

Instructions:

- This is only an abridged version of the solution. The assessment of the answer take into consideration the provides proof.

1. `[2 points (1,1)]` Consider the scalar ODE

$$\frac{y^{'}(x)\,x}{y(x)} = \mu, \text{ for } x \in X \subseteq \mathbb{R}$$

where μ is a constant.

(a) **Prove that the general solution follows a power law.**
The solution is $y(x) = y(x_0)\left(\frac{x}{x_0}\right)^{\mu}$, for $y(x_0) > 0$

(b) **Consider the ODE jointly with the constraint $\int_{x_0}^{\infty} y(x)dx = 1$. Find the solution to the problem, specifying the conditions on the parameter μ such that the solution exists.**
The solution is $y(x) = -(1+\mu)\,x_0^{-(1+\mu)}\,x^{\mu}$, for $\mu < -1$ and $x_0 > 0$.

2. `[4 points (2,1,1)]` Consider the planar ODE $\dot{\mathbf{y}} = \mathbf{A}\mathbf{y}$ where $\mathbf{y} \in \mathbb{R}^2$, and

$$\mathbf{A} = \begin{pmatrix} -3 & -2 \\ 2 & 1 \end{pmatrix}.$$

(a) **Solve the ODE.** The solution is

$$y_1(t) = e^{-t}\left(y_1(0) - 2\,t\,(y_1(0) + y_2(0))\right)$$
$$y_2(t) = e^{-t}\left(y_2(0) + 2\,t\,(y_1(0) + y_2(0))\right)$$

(b) **Assume the initial value $\mathbf{y}(0) = \left(\frac{1}{4}, \frac{1}{4}\right)^{\top}$. Find the solution to the initial-value problem.** The solution is

$$y_1(t) = e^{-t}\left(\frac{1}{4} - t\right)$$
$$y_2(t) = e^{-t}\left(\frac{1}{4} + t\right)$$

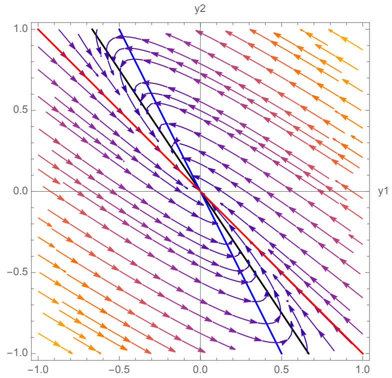

Figure 1: Stable node with multiplicity

(c) **Draw the phase diagram, and characterize it.**

3. `[5 points (1,2,2)]` Consider the following ODE, depending on a parameter $\lambda \in (-\infty, \infty)$:

$$\dot{y}_1 = \lambda\, y_2 + y_1^2,$$
$$\dot{y}_2 = y_1 - y_2^2.$$

(a) **Find possible bifurcation points.**
The vector field is $\mathbf{F}(\mathbf{y}) = (\lambda\, y_2 + y_1^2, y_1 - y_2^2)^\top$ has the trace and the determinant of the Jacobian $T(\mathbf{y}) = 2\,(y_1 - y_2)$ and $D(\mathbf{y}) = -(\lambda + 4\, y_1\, y_2$. Then solving jointly $\mathbf{F}(\mathbf{y}) = \mathbf{0}$ and $D(\mathbf{y}) = 0$ we find a bifurcation point $(\mathbf{y}, \lambda) = (0, 0, 0)$.

(b) **Find the steady states, for different values of λ, and characterize them.**
We have three cases:

i. if $\lambda < 0$ there are two steady states $\mathbf{y}^1 = (0, 0)$ that is a center, and $\mathbf{y}^2 = \left((-\lambda)^{\frac{2}{3}}, (-\lambda)^{\frac{1}{3}}\right)$ that is a saddle point.

ii. if $\lambda = 0$ we have one steady state $\mathbf{y} = (0, 0)$ that is a degenerate saddle-node.

iii. if $\lambda > 0$ there are two steady states $\mathbf{y}^1 = (0, 0)$ that is a saddle point, and $\mathbf{y}^2 = \left(\lambda^{\frac{2}{3}}, -\lambda^{\frac{1}{3}}\right)$ that is an unstable node.

(c) **Draw the phase diagrams for all relevant cases. Discuss the existence of invariant orbits that do not occur in linear ODEs.**

4. `[5 points (2,2,1)]` Consider the problem for a monopolist that seeks to distribute its product across a market spanning locations $X = [-1, 1]$. Let the firm be located at location $x = 0$. The quantity sold in location x is $q(x)$ and the price is $p(x) = \bar{p} - \phi\, q(x)$, where both $\bar{p}$ and ϕ are positive constants. For simplicity, assume that the firm has zero production costs, but it incurs into adjustment costs when it changes the quantity sold to any location x, that is $q'(x) \equiv \frac{dq(x)}{dx}$. The firm's objective is to maximize the aggregate profits from selling to all locations, by solving the distributional problem:

$$\max_{u(\cdot)} \int_{-1}^{1} p(x)\, q(x) - \frac{1}{2}\,(u(x))^2\, dx$$

subject to

$$q'(x) = u(x), \text{ for } x \in [-1, 1].$$

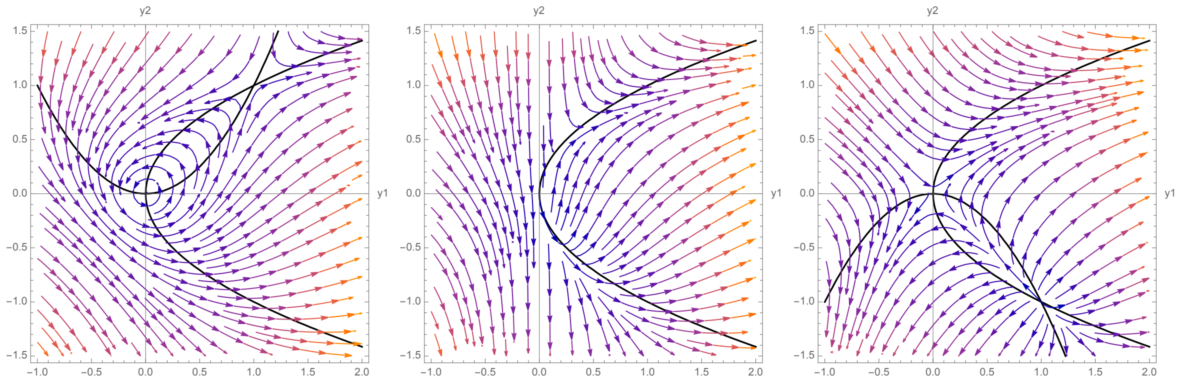

Figure 2: Phase diagrams for $\lambda < 0$, $\lambda = 0$ and $\lambda > 0$.
Non-linear invariant curves: homoclinic for $\lambda < 0$ and heteroclinic for $\lambda > 0$

(a) **Assume that the quantities to be sold at the two boundaries of the market are free, that is $q(-1)$ and $q(1)$ are free. Solve the firm's distributional problem.** The MHDS

$$q'(x) = u(x), \text{ for } x \in [-1, 1]$$
$$u'(x) = 2\,\phi\, q(x) - \bar{p} \text{ for } x \in [-1, 1]$$

Solving the MHDS with the side conditions $u(-1) = u(1) = 0$ yields the solution the optimal sales

$$q(x) = \frac{\bar{p}}{2\,\phi} \text{ for each } x \in [-1, 1]$$

(b) **Assume instead the quantities to be sold at the two boundaries of the market are fixed to be zero, that is $q(-1) = q(1) = 0$. Solve the new distributional problem.** Solving the MHDS with the side conditions $q(-1) = q(1) = 0$ yields the optimal sales

$$q(x) = \frac{\bar{p}}{2\,\phi}\left(1 + \frac{\left(e^{\lambda} - e^{-\lambda}\right)\left(e^{\lambda\, x} - e^{-\lambda\, x}\right)}{e^{-2\,\lambda} - e^{2\,\lambda}}\right) \text{ for each } x \in [-1, 1]$$

(c) **Compare the solutions you have found in (a) and (b), and provide an intuition.** In the first case the sales are the same across the space, and in the second they are heterogeneous over space. The first case is natural because there are no space-specific heterogeneous price and cost variables. The second case is natural because the solution is forced to be space-dependent.

5. `[4 points (1,2,1)]` Consider the diffusion equation

$$dX(t) = \sigma\, dW(t), \text{ for } t \in [0, \infty),$$

where $(W(t))_{t\in\mathbb{R}_+}$ is a standard Brownian motion, and $\sigma > 0$.

(a) **Let $X(0) = x_0 > 0$ be known. Find the solution for $X(t)$.** The solution is $X(t) = x_0 + \sigma\, W(t)$ for $t \in [0, \infty)$.

(b) **Let $p(t, x) = \mathbb{P}[X(t) = x | X(0) = x_0]$. Find the solution to the Fokker-Planck equation.**
The FPK equation is

$$\partial_t p(t, x) = \frac{\sigma^2}{2}\,\partial_{xx} p(t, x).$$

Using Fourier transforms yield $P(t,\omega) = P(0,\omega)e^{-(2\pi\omega)\,a}$ for $a = \frac{\sigma^2\,t}{2}$. Transforming back yields $p(t,x) = \delta(x - x_0) * g(t,x)$ where $g(t,x) = (2\,\pi\,a)^{-\frac{1}{2}}\,e^{-\frac{x^2}{4\,a}}$. Therefore

$$p(t,x) = \frac{1}{\sqrt{2\,\pi\,\sigma^2\,t}}\,e^{-\frac{(x-x_0)^2}{2\,\sigma^2\,t}},\ \text{for } t \in [0,\infty)$$

(c) **Find $\mathbb{E}[X(t)|X(0) = x_0]$ and $\mathbb{V}[X(t)|X(0) = x_0]$. Characterize the dynamic and statistic behavior of the process $(X(t))_{t\in\mathbb{R}_+}$.**

We have

$$\mathbb{E}[X(t)|X(0) = x_0] = x_0,\ \mathbb{V}[X(t)|X(0) = x_0] = x_0^2\left(e^{\sigma^2\,t} - 1\right)$$

The process is stationary but non-ergodic: $\lim_{t\to\infty} mathbbV[X(t)|X(0) = x_0] = \infty$.