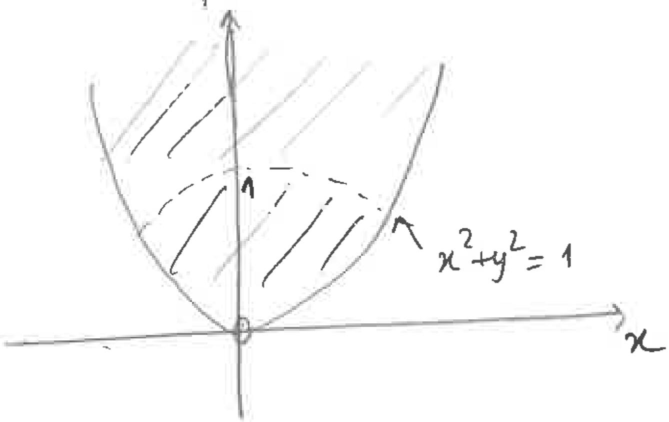
Mathematics II – Exam 01/02/2024 – Version A ①

Part I

a) $1/3$; $-1/7$

b) $D_f = \{(x,y) \in \mathbb{R}^2 : y - x^2 \geq 0 \wedge x^2+y^2 > 0 \wedge x^2+y^2 \neq 1\}$



c) $\begin{pmatrix} 1 & 3 & 0 \\ 3 & 9 & 0 \\ 0 & 0 & 5 \end{pmatrix}$

- semi-positively defined
- degree 2
- $2\,Q(x,y,z)$

d) interior / accumulation / adherent

$\partial\Omega = \{(x,y) \in \mathbb{R}^2 : (x=1 \wedge y \geq 1) \vee (x=2 \wedge y \geq \frac{1}{2}) \vee$
$\vee \left(y = \frac{1}{x} \wedge x \in [1,2]\right)\}$

Compact

e) 1, 0;

f) 2

g) $-\frac{1}{2}$; 0; f is not continuous at (0,0).

h) $\frac{df}{dt}(t) = 2xy\Big|_{(x(t),y(t))} 3e^{3t} + x^2\Big|_{(x(t),y(t))} \cos t =$

$= 2 \cdot e^{3t} \cdot \sin t \cdot 3e^{3t} + e^{6t} \cdot \cos t$

$= 6(e^{3t})^2 \sin t + e^{6t} \cos t$

$= e^{6t}(6 \sin t + \cos t)$

i) 3

j) $f(x,y) = \cos(x^2+y)$

k) 0, 0, 2

l) $\begin{pmatrix} -5 & 0 \\ 0 & -3 \end{pmatrix}$ for instance.

m) $\frac{\partial^{200} f}{\partial x^{200}}(x,y) = 2^{200} \sin(2x+y)$

n) $\pi/2$

o) $\begin{cases} y'' = -16y \\ y(\pi/8) = 0 \end{cases}$ OR $\begin{cases} y'' = -16\cos(4x) \\ y(\pi/8) = 0 \end{cases}$

p) Malthus, $p(t) = 10e^{-2t}$, $t \in \mathbb{R}_0^+$

q) increasing

r) $b = 0, 3$.

Part II

1 a) $A = \begin{pmatrix} 1 & 2 & 0 \\ 2 & 4 & 0 \\ 0 & 0 & \alpha \end{pmatrix}$

$p(\lambda) = (1-\lambda)(4-\lambda)(\alpha-\lambda) - (\alpha-\lambda)\cdot 4$

$= (\alpha-\lambda)\left((1-\lambda)(4-\lambda) - 4\right) =$

$= (\alpha-\lambda)\left(\cancel{4} - 5\lambda + \lambda^2 - \cancel{4}\right)$

$= (\alpha-\lambda)\ \lambda \cdot (\lambda-5)$

eigenvalues $\{0, 5, \alpha\}$

Using Eigenvalue method, we know that

if $\alpha \geq 0$, then Q is S.P.D

if $\alpha < 0$, then Q is undefined

b) $\alpha = 5$

$P(\lambda) = -\lambda(\lambda-5)^2$ eigenvalues: 0, 5

$$\begin{pmatrix} -4 & 2 & 0 \\ 2 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \begin{cases} -4x + 2y = 0 \\ 2x - y = 0 \\ \end{cases} \Leftarrow \boxed{y = 2x}$$

$E_5 = \langle (1,2,0), (0,0,1) \rangle$ $\quad g.m.(5) = 2$

2a)

If $(x,y) \neq (0,0)$, f is continuous because $f(x,y)$ is the ratio of continuous maps where the denominator does not vanish.

If $(x,y) = (0,0)$, then

$|x| \leq \sqrt{x^2+y^2}$

$$0 \leq \left| \frac{x(x+y)}{\sqrt{x^2+y^2}} \right| = \frac{|x| \cdot |x+y|}{\sqrt{x^2+y^2}} \leq |x+y|$$

By Squeezing theorem, it follows that

$$\lim_{(x,y)\to(0,0)} \frac{x(x+y)}{\sqrt{x^2+y^2}} = 0 = f(0,0).$$

$\therefore f$ is continuous at $(0,0)$

2 b)

$$D_{(v_1,v_2)} f(0,0) = \lim_{t\to 0} \frac{f((0,0)+t(v_1,v_2)) - f(0,0)}{t} =$$

$$= \lim_{t\to 0} \frac{f(tv_1, tv_2)}{t} = \lim_{t\to 0} \frac{\cancel{t} v_1 (tv_1 + tv_2)}{\cancel{t}\sqrt{(tv_1)^2 + (tv_2)^2}}$$

$$= \lim_{t\to 0} \frac{t}{|t|} \cdot \frac{v_1^2 + v_1 v_2}{\sqrt{v_1^2+v_2^2}} = \begin{cases} \dfrac{v_1^2+v_1v_2}{\sqrt{v_1^2+v_2^2}} & t>0 \\ -\dfrac{v_1^2+v_1v_2}{\sqrt{v_1^2+v_2^2}} & t<0 \end{cases}$$

$\therefore$ this limit exists iff $v_1^2 + v_1 v_2 = 0 \Leftrightarrow$
(and is 0)

$\Leftrightarrow v_1 (v_1 + v_2) = 0$

$\Leftrightarrow v_1 = 0 \ \vee \ v_1 = -v_2$

- $(0, v_2), (v_1, -v_1), \ v_1, v_2 \neq 0$

3)

a) $f(x,y) = y(x+2)^2 + xy^2 - yx^2$

$= y(x^2+4x+4) + xy^2 - yx^2$

$= \cancel{yx^2} + 4xy + 4y + xy^2 - \cancel{yx^2}$

$\nabla f(x,y) = (4y + y^2;\ 4x + 4 + 2xy)$

$\nabla f(x,y) = \bar{0} \Leftrightarrow \begin{cases} 4y+y^2=0 \\ 4x+4+2xy=0 \end{cases} \Leftrightarrow \begin{cases} y(4+y)=0 \\ \text{——} \end{cases}$

$\Leftrightarrow \begin{cases} y=0 \\ x=-1 \end{cases} \vee \begin{cases} y=-4 \\ x=1 \end{cases}$

Critical points $(-1,0)$ and $(1,-4)$.

$Hf(x,y) = \begin{pmatrix} 0 & 4+2y \\ 4+2y & 2x \end{pmatrix}$

$Hf(-1,0) = \begin{pmatrix} 0 & 4 \\ 4 & -2 \end{pmatrix}$ $\Delta_1 = 0$, $\Delta_2 = -16 < 0$

$Hf(1,-4) = \begin{pmatrix} 0 & -4 \\ -4 & 2 \end{pmatrix}$ $\Delta_1 = 0$, $\Delta_2 = -16 < 0$

$\Rightarrow$ Hf are undefined of F $\Downarrow$ $(-1,0), (1,-4)$ are saddle.

b)

i) M is compact (M is a closed and bounded disk)

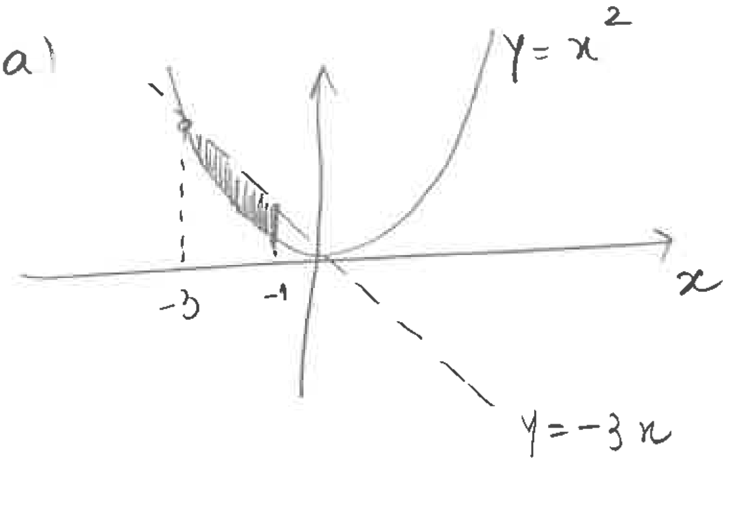
f is continuous (f is polynomial)

then, using Weierstrass' theorem, we may conclude that $f|_M$ has a global max and a global min.

ii) $M = \underbrace{\text{int}(M)}_{\text{open}} \,\dot\cup\, \partial M$

If the global extrema lie on int M, then they would be detected on (a). Since the candidates for extrema are saddles, then they should lie on ∂M.

4) a)



b) $\iint_{\Omega} \frac{1}{x} e^{\frac{y}{x}} \, dx\, dy =$

$$\underset{\Omega \text{ is type I}}{=} \int_{-3}^{-1} \int_{x^2}^{-3x} \frac{1}{x} e^{\frac{y}{x}} \, dy\, dx =$$

$$= \int_{-3}^{-1} \left[e^{y/x} \right]_{y=x^2}^{y=-3x} dx =$$

$$= \int_{-3}^{-1} e^{-3} - e^{x} \, dx =$$

$$= \left[e^{-3} x - e^{x} \right]_{x=-3}^{x=-1} = -e^{-3} - e^{-1} - (-3e^{-3} - e^{-3})$$

$$= \cancel{-e^{-3}} - e^{-1} + 3e^{-3} + \cancel{e^{-3}} = 3e^{-3} - e^{-1} = \frac{3}{e^3} - \frac{1}{e}$$

5

$$x^4 y' + 4x^3 y = \cos x$$

$x \neq 0$ $\quad : x^4$

$$y' + \frac{4}{x} y = \frac{\cos x}{x^4}$$

Integrating factor: $\mu(x) > 0$, $\forall x \in \mathbb{R}^+$

$$\mu(x) \cdot \frac{4}{x} = \mu'(x) \Leftrightarrow \frac{\mu'(x)}{\mu(x)} = \frac{4}{x}$$

$$\Rightarrow \frac{d}{dx} \ln \mu(x) = \frac{4}{x}$$

$$\Rightarrow \ln \mu(x) = 4 \ln x$$

$$\Rightarrow \boxed{\mu(x) = x^4}$$

$$\frac{d}{dx}\left[x^4 \cdot y \right] = \frac{\cos x}{\cancel{x^4}} \cdot \cancel{x^4}$$

$$\Leftrightarrow y(x) = \frac{\sin x + C}{x^4}, \quad C \in \mathbb{R}$$

Since $y(\pi) = \pi$, then $\pi = \frac{0 + C}{\pi^4} \Leftrightarrow C = \pi^5$

$$\therefore y(x) = \frac{\sin x + \pi^5}{x^4}$$

$\underbrace{D = \mathbb{R}^+}$

biggest open set containing π