EXAM - ÉPOCA NORMAL - 15/06/2023
EN-2023 - STOCHASTIC CALCULUS

1(a) $Y_t = f(t, \underbrace{B_t^{(1)}}_{x_1}, \underbrace{B_t^{(2)}}_{x_2}, \underbrace{U_t}_{x_3}) = B_t^{(1)} B_t^{(2)} + 4 (B_t^{(2)})^2 + \underbrace{6\int_0^t s B_s^{(1)} ds}_{=U_t} - Kt - ct^2 B_t^{(1)}$

$dU_t = 6t B_t^{(1)} dt$, By Itô formula, since $f \in C^{1,2}$,

$$dY_t = \underbrace{(-K - 2ctB_t^{(1)})}_{\frac{\partial f}{\partial t}} dt + \underbrace{(B_t^{(2)} - ct^2)}_{\frac{\partial f}{\partial x_1}} dB_t^{(1)} + \underbrace{(B_t^{(1)} + 8B_t^{(2)})}_{\frac{\partial f}{\partial x_2}} dB_t^{(2)}$$

$$+ \frac{1}{2} \underbrace{(8)}_{\frac{\partial^2 f}{\partial x_2^2}} dt + \underbrace{dU_t}_{6tB_t^{(1)}dt} = (-K - 2ctB_t^{(1)} + 4 + 6tB_t^{(1)}) dt +$$

$$+ (B_t^{(2)} - ct^2) dB_t^{(1)} + (B_t^{(1)} + 8B_t^{(2)}) dB_t^{(2)}$$

So, $Y_t = \int_0^t (-K - 2csB_s^{(1)} + 4 + 6sB_s^{(1)}) ds +$

$$+ \underbrace{\int_0^t (B_s^{(2)} - cs^2) dB_s^{(1)}}_{\text{stochastic integral is a martingale}} + \int_0^t \underbrace{(B_s^{(1)} + 8B_s^{(2)}) dB_s^{(2)}}_{\text{stochastic integral is a martingale}}$$

So, Y_t is a martingale if and only if

$$-K - 2csB_s^{(1)} + 4 + 6sB_s^{(1)} = 0 \Leftrightarrow \begin{cases} K = 4 \\ 2c = 6 \end{cases} \Leftrightarrow \begin{cases} K = 4 \\ c = 3 \end{cases}$$

Note that the processes $B_t^{(2)} - ct^2 \in L^2_{a,T}$ and $B_t^{(1)} + 8B_t^{(2)} \in L^2_{a,T}$ (are polynomial functions of B_t and t).

1(b) $X_t = B^{(1)}_{2t} - B^{(1)}_t \sim N(0, 2t-t) = N(0,t)$ 2

is a Gaussian process

$\mathbb{E}[X_t] = 0$

$\mathrm{Cov}[X_t, X_s] = \mathbb{E}[X_t X_s] = \mathbb{E}\left[(B^{(1)}_{2t} - B^{(1)}_t)(B^{(1)}_{2s} - B^{(1)}_s)\right]$

$= \mathbb{E}\left[B^{(1)}_{2t} B^{(1)}_{2s}\right] - \mathbb{E}\left[B^{(1)}_{2t} B^{(1)}_s\right] - \mathbb{E}\left[B^{(1)}_t B^{(1)}_{2s}\right] + \mathbb{E}\left[B^{(1)}_t B^{(1)}_s\right]$

$= \min\{2t, 2s\} - \min\{2t, s\} - \min\{t, 2s\} + \min\{t, s\}$

By the covariance function of $B^{(1)}$

$$= \begin{cases} 2t - 2t - t + t = 0 & \text{if } 2t < s \\ 2t - s - t + t = 2t - s & \text{if } t < s \text{ and } 2t > s \\ 2s - s - 2s + s = 0 & \text{if } 2s < t \\ 2s - s - t + s = 2s - t & \text{if } s < t \text{ and } 2s > t \end{cases}$$

Since the covariance function of X is different from the covariance function of the SBm, X is not a Standard Brownian motion.

2) $\exp(2B_T) = ?$ The process $V_t \in \mathcal{L}^2_{aT}$ exists by the Itô representation theorem.

Let $X_t = \exp(2B_t - 2t) = f(t, B_t)$. By Itô formula,

$$dX_t = \underbrace{-2\exp(2B_t - 2t)}_{\frac{\partial f}{\partial t}} dt + \underbrace{2\exp(2B_t - 2t)}_{\frac{\partial f}{\partial x}} dB_t + \frac{1}{2} \times \underbrace{4\exp(2B_t - 2t)}_{\frac{\partial^2 f}{\partial x^2}} dt$$

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Therefore,

$$dX_t = 2\exp(2B_t - 2t)\, dB_t$$

$$\Leftrightarrow \underbrace{X_T - X_0}_{\exp(0)=1} = \int_0^T 2\exp(2B_t-2t)\, dB_t \quad \Leftrightarrow \underbrace{X_T}_{\exp(2B_T-2T)} = 1 + \int_0^T 2\exp(2B_t-2t)\,dB_t$$

$$\Leftrightarrow \exp(2B_T)\exp(-2T) = 1 + \int_0^T 2\exp(2B_t-2t)\,dB_t$$

$$\Leftrightarrow \exp(2B_T) = e^{2T} + \int_0^T 2e^{2T}\exp(2B_t-2t)\,dB_t$$

For $4B_T^2 + 5B_T$, let us consider $f(B_t) = 4B_t^2 + 5B_t = Y_t$ and apply the Itô formula to f:

$$dY_t = \underbrace{(8B_t+5)}_{\frac{\partial f}{\partial x}}\, dB_t + \frac{1}{2}\cdot\underbrace{(8)}_{\frac{\partial^2 f}{\partial x^2}}\, dt \Rightarrow$$

$$\Rightarrow \underbrace{Y_T - Y_0}_{4B_T^2+5B_T-0} = 4T + \int_0^T (8B_t+5)\,dB_t$$

Therefore, $\exp(2B_T) + 4B_T^2 + 5B_T = \underbrace{e^{2T} + 4T}_{f(T)} + \int_0^T \underbrace{(2e^{2T}\exp(2B_t-2t) + 8B_t + 5)}_{H_t}\, dt$ //

(4) 3 a) $|f(t,x)| = |t^4 \ln(1+x^2)| \le T^4 \ln(1+|x|)^2$

$= 2T^4 \ln(1+|x|) \le 2T^4(1+|x|)$

Likewise $\ln(1+|x|) \le 1+|x| \Leftrightarrow 1+|x| \le e^{1+|x|} \le 1+1+|x|+\frac{(1+|x|)^2}{2!}+\dots$

Therefore $f(t,x)$ satisfies the linear growth condition.

$|g(t,x)| = |t^2 \exp(\sin(X_t))| \le T^2 e^{1} \le T^2 e \le T^2 e(1+|x|)$

satisfies the linear growth condition

$$\left|\frac{\partial f}{\partial x}(t,x)\right| = \left|t^4 \frac{2x}{1+x^2}\right| \le \begin{cases} T^4\, 2|x| \le T^4 2 & \text{if } |x| \le 1 \\ T^4 \frac{2|x|}{|x|^2} = T^4 \frac{2}{|x|} \le T^4 2 & \text{if } |x| > 1 \end{cases}$$

so $\left|\frac{\partial f}{\partial x}(t,x)\right| \le T^4 \times 2$ is bounded and f is differentiable with respect to x $\Rightarrow$ f satisfies the Lipschitz property with respect to x.

$\left|\frac{\partial g}{\partial x}(t,x)\right| = |t^2 e^{\sin(X_t)} \times \cos(X_t)| \le T^2 e \Rightarrow g$ satisfies the Lipschitz property with respect to x.

By the existence and uniqueness theorem, exists one unique solution in $[0,T]$.

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3(b) $dX_t = \frac{1}{4} X_t\,dt + \frac{1}{2} X_t\,dB_t \Leftarrow \alpha = 1$

Let $Z_t = \underbrace{\ln(X_t)}_{h(X_t)}$. By Itô formula,

$$dZ_t = \underbrace{\frac{1}{X_t}}_{\frac{\partial h}{\partial x}} dX_t + \frac{1}{2}\underbrace{\left(-\frac{1}{X_t^2}\right)}_{\frac{\partial^2 h}{\partial x^2}} (dX_t)^2 = \frac{1}{\cancel{X_t}}\left(\frac{1}{4}\cancel{X_t}\,dt + \frac{1}{2}\cancel{X_t}\,dB_t\right)$$

$$+ \frac{1}{2}\left(-\frac{1}{\cancel{X_t^2}}\right)\left(\frac{1}{4}\cancel{X_t^2}\,dt\right) \iff dZ_t = \left(\frac{1}{4} - \frac{1}{8}\right)dt + \frac{1}{2}\,dB_t$$

$Z_t = Z_0 + \frac{1}{8}t + \frac{1}{2}B_t$ and $X_t = e^{Z_t} = \underbrace{x}_{X_0}\, e^{\frac{1}{8}t + \frac{1}{2}B_t}$ //

If $\alpha = -2$: $\begin{cases} dX_t = \frac{1}{4X_t^2}\,dt + \frac{1}{2}X_t\,dB_t \\ X_0 = x > 0 \end{cases}$

Integrating factor: $F_t = \exp\left(\int_0^t \frac{1}{2}\,dB_s - \frac{1}{2}\int_0^t \left(\frac{1}{2}\right)^2 ds\right) =$

$= \exp\left(\frac{1}{2}B_t - \frac{1}{8}t\right) \Rightarrow F_t^{-1} = e^{\frac{1}{8}t - \frac{1}{2}B_t}$

$X_t = F_t Y_t \iff Y_t = F_t^{-1} X_t = e^{-\frac{1}{2}B_t + \frac{1}{8}t} X_t$

$f(t,x) = \frac{1}{4}x^{-2} \Rightarrow dY_t = (F_t)^{-1} f(t, F_t Y_t) = e^{\frac{1}{8}t - \frac{1}{2}B_t}\,\frac{1}{4}F_t^{-2} Y_t^{-2}\,dt$

$\iff \begin{cases} dY_t = \frac{1}{4} e^{\frac{3}{8}t - \frac{3}{2}B_t} Y_t^{-2}\,dt \\ Y_0 = x \end{cases} \iff \begin{cases} Y_t^2\,dY_t = \frac{1}{4} e^{\frac{3}{8}t - \frac{3}{2}B_t} \\ Y_0 = x \end{cases}$.

$\iff \left[\frac{Y_s^3}{3}\right]_0^t = \int_0^t \frac{1}{4} e^{\frac{3}{8}s - \frac{3}{2}B_s}\,ds \iff$

$\iff \frac{1}{3}Y_t^3 - \frac{1}{3}x^3 = \frac{1}{4}\int_0^t e^{\frac{3}{8}s - \frac{3}{2}B_s}\,ds \iff Y_t = \left(x^3 + \frac{3}{4}\int_0^t e^{\frac{3}{8}s - \frac{3}{2}B_s}\,ds\right)^{\frac{1}{3}}$

$\Rightarrow X_t = F_t Y_t = e^{\frac{1}{2}B_t - \frac{1}{8}t}\left(x^3 + \frac{3}{4}\int_0^t e^{\frac{3}{8}s - \frac{3}{2}B_s}\,ds\right)^{\frac{1}{3}}$. //

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⑥ 4(a) $Z_s = F(s, X_s)$. By the Itô formula,

$$dZ_s = \frac{\partial F}{\partial s} ds + \frac{\partial F}{\partial x} dX_s + \frac{1}{2}\frac{\partial^2 F}{\partial x^2}(dX_s)^2 =$$

$$= \frac{\partial F}{\partial s} ds + \frac{\partial F}{\partial x}(2X_s ds + 6X_s dB_s) + \frac{1}{2}\frac{\partial^2 F}{\partial x^2}(36X_s^2 ds) =$$

$$= \underbrace{\left(\frac{\partial F}{\partial s} + 2X_s\frac{\partial F}{\partial x} + 18X_s^2\frac{\partial^2 F}{\partial x^2}\right)}_{\text{by the PDE } = -X_s} ds + 6X_s\left(\frac{\partial F}{\partial x}\right) dB_s =$$

$$= -X_s ds + 6X_s \frac{\partial F}{\partial x}(s, X_s) dB_s.$$ Integrating between t and T:

$$\int_t^T dF(s, X_s) = \int_t^T -\cancel{X_s}\, ds + \int_t^T 6X_s \frac{\partial F}{\partial x}(s,X_s) dB_s \Leftrightarrow$$

$$\Leftrightarrow F(T, X_T) - F(t, X_t) = -\int_t^T X_s ds + 6\int_t^T X_s \frac{\partial F}{\partial x}(s, X_s) dB_s$$

$$\Rightarrow \mathbb{E}_{t,x}[F(T,X_T)] - \underbrace{\mathbb{E}_{t,x}[F(t,X_t)]}_{=F(t,x)} = -\int_t^T \mathbb{E}_{t,x}[X_s] ds + 6\underbrace{\mathbb{E}_{t,x}\left[\int_t^T X_s \frac{\partial F}{\partial x}(s,X_s) dB_s\right]}_{0}.$$

$$\Leftrightarrow F(t,x) = \mathbb{E}_{t,x}[\underbrace{F(T,X_T)}_{h_1(X_T^4)}] + \int_t^T \mathbb{E}_{t,x}[X_s] ds$$

$$\Leftrightarrow F(t,x) = \mathbb{E}_{t,x}\left[h_1(X_T^4)\right] + \int_t^T \mathbb{E}_{t,x}[X_s] ds. \quad //$$

4(h) In order to apply the formula let us calculate E_t

X_s: $\begin{cases} dX_s = 2X_s ds + 6X_s dB_s \\ X_t = x \end{cases}$ $\Rightarrow$ solution is the G.B.m.

$$X_s = x\, e^{(2-\frac{1}{2}6^2)(s-t)+6(B_s-B_t)} \Leftrightarrow$$

$$\Leftrightarrow X_s = x\, e^{-16(s-t)+6(B_s-B_t)}$$

By the formula, $F(t,x) = E_{t,x}\Big[\ln\big(\overbrace{x^4 e^{-64(T-t)+24(B_T-B_t)}}^{X_T^4}\big)\Big] +$

$$+\int_t^T E_{t,x}\left[x\, e^{-16(s-t)+6(B_s-B_t)}\right]ds =$$

$$= E_{t,x}\left[\ln(x^4) + (-64(T-t)+24(B_T-B_t))\right] + \int_t^T x\, e^{-16(s-t)}\, e^{18(s-t)}\, ds$$

$$= \ln(x^4) - 64(T-t) + x\left[\frac{e^{2(s-t)}}{2}\right]_t^T =$$

$$= \ln(x^4) - 64(T-t) + \frac{x}{2}\left(e^{2(T-t)} - 1\right). \quad //$$

5) Under Q, we have:

$\begin{cases} dS_t = r S_t dt + 6 S_t dW_t \\ S_0 = S_0 \end{cases}$, where W is a S.B.m. under Q and r is the risk-free interest rate

Solution is the G.B.m.:

$$S_t = S_0\, e^{(r-\frac{1}{2}6^2)t + 6W_t}$$

⑧ The price at time 0 is

$$F(0,S_0) = e^{-rT}\,\mathbb{E}_Q\left[\Phi(S_T)\right] =$$

$$= e^{-rT}\,\mathbb{E}_Q\left[\frac{1}{K}S_T^4 + \ln(S_T) + K\,\mathbb{1}_{\{M_1 < S_T < M_2\}}(S_T)\right] =$$

$$= e^{-rT}\left\{\frac{1}{K}\mathbb{E}_Q\left[S_0^4 e^{4(r-\frac{1}{2}\sigma^2)T + 4\sigma W_T}\right] + \mathbb{E}_Q\left[\ln(S_0) + (r-\frac{1}{2}\sigma^2)T + \sigma W_T\right]\right.$$

$$\left. + K\,\mathbb{P}_Q\left[M_1 < S_T < M_2\right]\right\} = e^{-rT}\left\{\frac{1}{K}S_0^4 e^{(r-\frac{1}{2}\sigma^2)4T} e^{\frac{1}{2}4^2\sigma^2 T} + \right.$$

$$\left. + \ln(S_0) + (r-\frac{1}{2}\sigma^2)T + K\,\mathbb{P}_Q\left(\ln(M_1) < \ln(S_T) < \ln(M_2)\right)\right\}$$

$$= e^{-rT}\left\{\frac{S_0^4}{K} e^{(4r+6\sigma^2)T} + \ln(S_0) + (r-\frac{1}{2}\sigma^2)T + \right.$$

$$\left. + K\,\mathbb{P}_Q\left[\underbrace{\frac{\ln(M_1) - \ln(S_0) - (r-\frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}}_{\equiv d_1} < \underset{\sim N(0,1)}{Z} < \underbrace{\frac{\ln(M_2) - \ln(S_0) - (r-\frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}}_{\equiv d_2}\right]\right\}$$

$$= e^{-rT}\left\{\frac{S_0^4}{K} e^{(4r+6\sigma^2)T} + \ln(S_0) + (r-\frac{1}{2}\sigma^2)T + \right.$$

$$\left. + K\left(N[d_2] - N[d_1]\right)\right\}$$, where N is the cumulative distribution function of the $N(0,1)$ distribution

6. Let $n > m$ and consider

$$\left\| Z_t^{(n)} - Z_t^{(m)} \right\|_{L^2(\mathbb{P})} = \left\| \sum_{k=m}^{n-1} \left(Z_t^{(k+1)} - Z_t^{(k)} \right) \right\|_{L^2(\mathbb{P})} \leq$$

$$\leq \sum_{k=m}^{n-1} \left\| Z_t^{(k+1)} - Z_t^{(k)} \right\|_{L^2(\mathbb{P})} = \sum_{k=m}^{n-1} \left\{ \mathbb{E}\left[\left| Z_t^{(k+1)} - Z_t^{(k)} \right|^2 \right] \right\}^{\frac{1}{2}} \leq$$

By the norm triangular inequality property

$$\leq \sum_{k=m}^{\infty} \left(\frac{C^{k+1} t^{k+1}}{(k+1)!} \right)^{\frac{1}{2}}$$

$\rightarrow$ This series is convergent because:

$$\frac{a_{k+1}}{a_k} = \frac{\left(\frac{C^{k+2} t^{k+2}}{(k+2)!} \right)^{\frac{1}{2}}}{\left(\frac{C^{k+1} t^{k+1}}{(k+1)!} \right)^{\frac{1}{2}}} = \left(\frac{Ct}{k+2} \right)^{\frac{1}{2}} \xrightarrow[k \to \infty]{} 0$$

Therefore $\displaystyle\sum_{k=m}^{\infty} \left(\frac{C^{k+1} t^{k+1}}{(k+1)!} \right)^{\frac{1}{2}} \xrightarrow[m \to \infty]{} 0.$

So, $\left\| Z_t^{(n)} - Z_t^{(m)} \right\|_{L^2(\mathbb{P})} \longrightarrow 0$ as $n, m \to \infty$ and the sequence is a Cauchy sequence in $L^2(\mathbb{P})$. Since this space is a complete space, a Cauchy sequence is a convergent sequence. Therefore $Z_t^{(n)}$ converges to a limit as $n \to \infty$.

So, $Z_t^{(n)} \xrightarrow[n \to \infty]{} Z_t$ in $L^2(\mathbb{P})$.

Every sequence that converges in $L^2(\mathbb{P})$ has a subsequence that converges a.s. Therefore, exists a subsequence $Z_t^{(n_k)} \longrightarrow Z_t$ a.s.