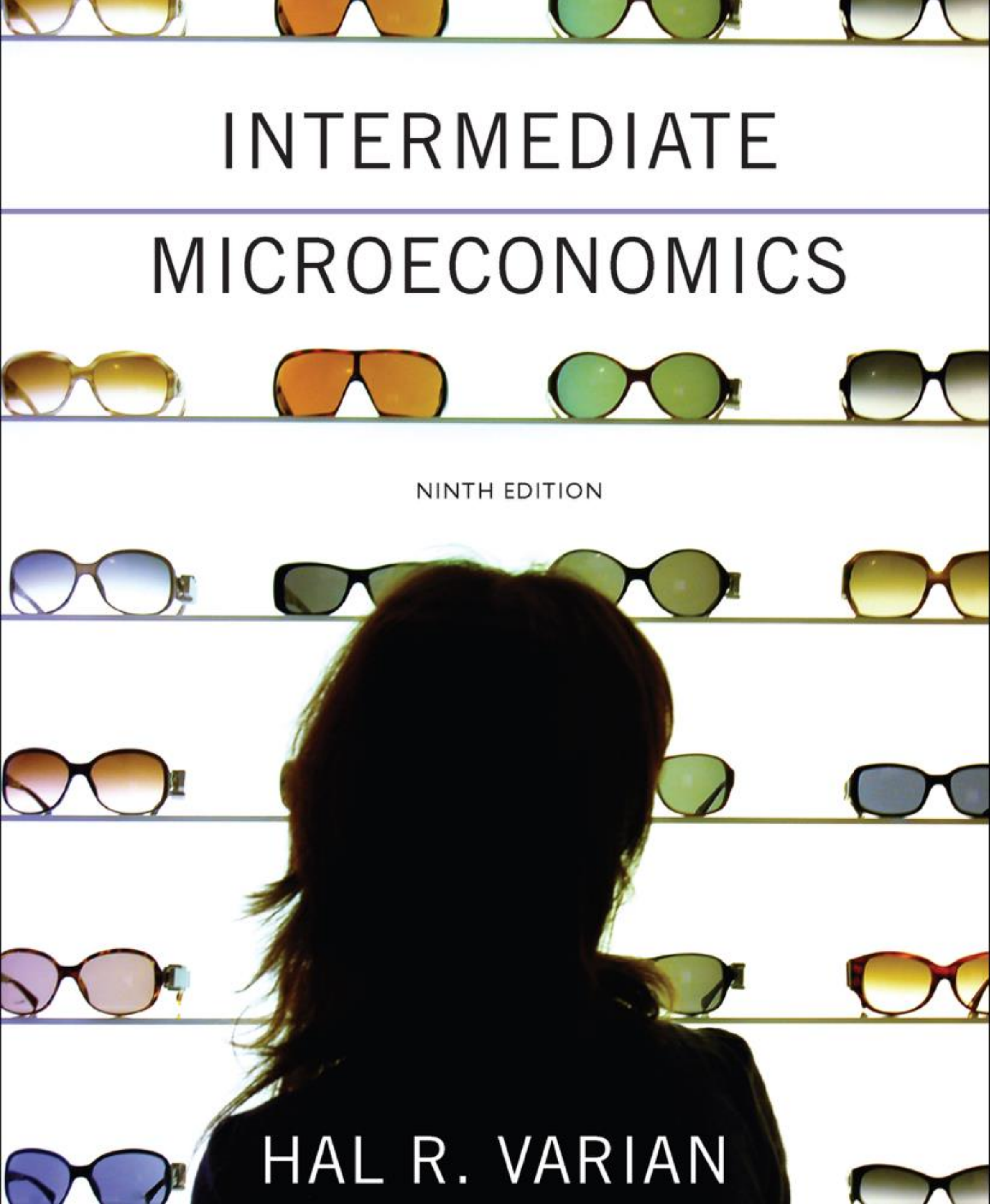



# CHAPTER 29

## Game Theory

# Game Theory

Game theory helps to model **strategic interaction** by agents who understand that their actions affect the actions of other agents.

**Some applications** of game theory:

- the study of oligopolies
    - In Chapter 28 we already used game theoretical concepts…
- the study of cartels
- the study of military strategies
- bargaining and negotiations

# What is a Game?

A **game** consists of

- a set of **players**
- a set of **strategies** for each player
- the **payoffs** to each player for every possible combination of strategies by all players

# Two-Player Games

A game with just two players is a **two-player game**.

We will study games in which there are only two players, each of whom can choose between only two actions.

# A First Example of a Two-Player Game

The players are called A and B.

Player A has two actions, called top ($T$) and bottom ($B$).

Player B has two actions, called left ($L$) and right ($R$).

A table shows the payoffs to both players for each of the four possible action combinations. This is the game's **payoff matrix**.

# A First Example of a Two-Player Game

Player B

| | | *Left* | *Right* |
|---|---|---|---|
| Player A | *Top* | 3, 9 | 2, 8 |
| | *Bottom* | 0, 2 | 1, 1 |

This is the game's payoff matrix.

Player A's payoff is shown first.
Player B's payoff is shown second.

# A First Example of a Two-Player Game

Player B

| | | *L* | *R* |
|---|---|---|---|
| Player A | *T* | 3, 9 | 2, 8 |
| | *B* | 0, 2 | 1, 1 |

This is the game's payoff matrix.

For example, if A plays up and B plays right then A's payoff is 2 and B's payoff is 8.

# A First Example of a Two-Player Game

Player B

| Player A | | $L$ | $R$ |
|---|---|---|---|
| | $T$ | 3, 9 | 2, 8 |
| | $B$ | 0, 2 | 1, 1 |

What do we think would happen if this game were played?

## Dominant Strategy

This game provides an example of a **dominant strategy**: a strategy for a player that is best no matter what the other player does.

In this example, top $T$ is a dominant strategy for A.

Similarly, left $L$ is a dominant strategy for B.

# A First Example of a Two-Player Game

| | | Player B | |
|---|---|---|---|
| | | ***L*** | ***R*** |
| Player A | ***T*** | **3, 9** | **2, 8** |
| | ***B*** | **0, 2** | **1, 1** |

By eliminating for consideration strategies that have been dominated, we can find our equilibrium—in this example, strategies $(T, L)$ with payoffs $(3, 9)$.

# A Second Example of a Two-Player Game

Player B

| | | $L$ | $R$ |
|---|---|---|---|
| Player A | $T$ | 3, 9 | 1, 8 |
| | $B$ | 0, 0 | 2, 1 |

What do we think would happen if this game were played?

# A Second Example of a Two-Player Game

Player B

| | | $L$ | $R$ |
|---|---|---|---|
| Player A | $T$ | 3, 9 | 1, 8 |
| | $B$ | 0, 0 | 2, 1 |

Is $(T, R)$ a likely play?

If B plays right then A's best reply is down since this improves A's payoff from 1 to 2. So, $(T, R)$ is not a likely play.

# A Second Example of a Two-Player Game

Player B

Is $(B, L)$ a likely play?

| Player A | | $L$ | $R$ |
|---|---|---|---|
| | $T$ | $3, 9$ | $1, 8$ |
| | $B$ | $0, 0$ | $2, 1$ |

If A plays down then B's best reply is right. So, $(B, L)$ is not a likely play.

# A Second Example of a Two-Player Game

Player B

| | | $L$ | $R$ |
|---|---|---|---|
| Player A | $T$ | 3, 9 | 1, 8 |
| | $B$ | 0, 0 | 2, 1 |

Is $(B, R)$ a likely play?

If B plays right then A's best reply is down and if A plays down, B's best reply is right. So, $(B, R)$ *is* a likely play.

# A Second Example of a Two-Player Game

Player B

| | | $L$ | $R$ |
|---|---|---|---|
| Player A | $T$ | 3, 9 | 1, 8 |
| | $B$ | 0, 0 | 2, 1 |

Is $(T, L)$ a likely play?

If B plays left then A's best reply is up and if A plays up, B's best reply is left. So, $(T, L)$ *is* also a likely play.

# Nash equilibrium

A **Nash equilibrium** is a pair of strategies $(p^*, q^*)$ such that $p^*$ is a best reply of player A to $q^*$ and $q^*$ is a best reply of player B to $p^*$.

Our second game had two Nash equilibria: $(T, L)$ and $(B, R)$.

Note that the Cournot equilibrium is a Nash equilibrium: $y_1^*$ is the best response of firm 1 to $y_2^*$ and $y_2^*$ is the best response of firm 2 to $y_1^*$ .

# Nash equilibrium

In general, how to find a Nash equilibrium in a game that can be described by a pay-off matrix as shown previously?

For each player, **underline the payoff of the best response** given each strategy of the other player. If there is a cell with all payoffs underlined, then that must be a Nash equilibrium.

# Pure Strategies

| | | Player B | |
|---|---|---|---|
| | | ***L*** | ***R*** |
| Player A | ***T*** | **3, 9** | **1, 8** |
| | ***B*** | **0, 0** | **2, 1** |

This game has two Nash equilibria: $(T, L)$ and $(B, R)$. Player A has been thought of as choosing to play either $T$ or $B$, but no combination of both. In other words, A is playing **purely** $T$ or $B$.

$T$ and $B$ are player A's **pure strategies**. Similarly, $L$ and $R$ are player B's pure strategies.

# Pure Strategies

Player B

| | | $L$ | $R$ |
|---|---|---|---|
| Player A | $T$ | 3, 9 | 1, 8 |
| | $B$ | 0, 0 | 2, 1 |

Consequently, $(T, L)$ and $(B, R)$ are **pure strategy Nash equilibria**.
Must every game have at least one pure strategy Nash equilibrium?

# Pure Strategies – New Game

Player B

| | | *L* | *R* |
|---|---|---|---|
| Player A | *T* | 1, 2 | 0, 4 |
| | *B* | 0, 5 | 3, 2 |

Notice that there are NO pure strategy Nash equilibria in this game. However, it can be shown that there is a **mixed strategy Nash equilibrium**.

# Mixed Strategies

Instead of playing purely up or down, player A may select a probability distribution $(\pi_T, 1-\pi_T)$, meaning that with probability $\pi_T$ player A will play top $T$ and with probability $1-\pi_T$ will play bottom $B$.

Player A is **mixing** over the pure strategies top and bottom.

The probability distribution $(\pi_T, 1-\pi_T)$ is a **mixed strategy** for player A.

# Mixed Strategies

Similarly, player B may select a probability distribution ($\pi_L$ , 1−$\pi_L$), meaning that with probability $\pi_L$ player B will play left $L$ and with probability 1−$\pi_L$ will play right $R$.

Player B is **mixing** over the pure strategies left and right.

The probability distribution ($\pi_L$, 1−$\pi_L$) is a **mixed strategy** for player B.

# Mixed Strategies

| | | Player B | |
|---|---|---|---|
| | | $L, \pi_L$ | $R, 1-\pi_L$ |
| Player A | $T, \pi_T$ | 1, 2 | 0, 4 |
| | $B, 1-\pi_T$ | 0, 5 | 3, 2 |

**Nash equilibrium:**

(i) Player B chooses $\pi_L^*$ to make player A indifferent between T and B.

(ii) Player A chooses $\pi_T^*$ to make player B indifferent between L and R.

In this case, (i) gaurantees that $\pi_T^*$ by A is a best response to $\pi_L^*$, and (ii) gaurantees that $\pi_L^*$ by B is a best response to $\pi_T^*$.

# Mixed Strategies – Condition (i)

| | | Player B | |
|---|---|---|---|
| | | $L, \pi_L$ | $R, 1-\pi_R$ |
| Player A | $T, \pi_T$ | 1, 2 | 0, 4 |
| | $B, 1-\pi_T$ | 0, 5 | 3, 2 |

A's expected value of choosing up is $\pi_L$.
A's expected value of choosing down is $3(1 - \pi_L)$.
If $\pi_L > 3(1 - \pi_L)$, A will choose only up. If $\pi_L < 3(1 - \pi_L)$, A will choose only down.

# Mixed Strategies – Condition (i)

Player B

| | | $L, \pi_L$ | $R, 1-\pi_R$ |
|---|---|---|---|
| Player A | $T, \pi_T$ | 1, 2 | 0, 4 |
| | $B, 1-\pi_T$ | 0, 5 | 3, 2 |

If there is a NE, necessarily $\pi_L = 3(1 - \pi_L) \Rightarrow \pi_L = 3/4$.

# Mixed Strategies – Condition (ii)

| | | Player B | |
|---|---|---|---|
| | | $L, \frac{3}{4}$ | $R, \frac{1}{4}$ |
| Player A | $T, \pi_T$ | 1, 2 | 0, 4 |
| | $B, 1-\pi_T$ | 0, 5 | 3, 2 |

B's expected value of choosing left is $2\pi_T + 5(1 - \pi_T)$.
B's expected value of choosing right is $4\pi_T + 2(1 - \pi_T)$.
If $2\pi_T + 5(1 - \pi_T) > 4\pi_T + 2(1 - \pi_T)$, B will choose only left.
If $2\pi_T + 5(1 - \pi_T) < 4\pi_T + 2(1 - \pi_T)$, B will choose only right.

# Mixed Strategies – Condition (ii)

Player B

| Player A | | $L, \frac{3}{4}$ | $R, \frac{1}{4}$ |
|---|---|---|---|
| | $T, \frac{3}{5}$ | 1, 2 | 0, 4 |
| | $B, \frac{2}{5}$ | 0, 5 | 3, 2 |

If there is a NE, necessarily $2\pi_T + 5(1 - \pi_T) = 4\pi_T + 2(1 - \pi_T) \Rightarrow \pi_T = 3/5$.

# Mixed Strategies – Nash Equilibrium

Player B

| | | $L, \frac{3}{4}$ | $R, \frac{1}{4}$ |
|---|---|---|---|
| Player A | $T, \frac{3}{5}$ | **1, 2** | **0, 4** |
| | $B, \frac{2}{5}$ | **0, 5** | **3, 2** |

The game's only Nash equilibrium consists of A playing the mixed strategy (3/5, 2/5) and B playing the mixed strategy (3/4, 1/4).

# Mixed Strategies - Payoffs

| | | Player B | |
|---|---|---|---|
| | | $L, \frac{3}{4}$ | $R, \frac{1}{4}$ |
| Player A | $T, \frac{3}{5}$ | $1, 2$ $\frac{9}{20}$ | $0, 4$ $\frac{3}{20}$ |
| | $B, \frac{2}{5}$ | $0, 5$ $\frac{6}{20}$ | $3, 2$ $\frac{2}{20}$ |

A's NE expected payoff is $(1 \times 9/20) + (0 \times 3/20) + (0 \times 6/20) + (3 \times 2/20) = 3/4$.

B's NE expected payoff is $(2 \times 9/20) + (4 \times 3/20) + (5 \times 6/20) + (2 \times 2/20) = 16/5$.

# How Many Nash Equilibria are there?

It can be shown that a game with a finite number of players, each with a finite number of pure strategies, has at least one Nash equilibrium.

So, if the game has no pure strategy Nash equilibrium, then it must have at least one mixed strategy Nash equilibrium.

# The Prisoner's Dilemma

To see if Pareto-efficient outcomes are (always) the Nash equilibrium of a game, consider the famous game called the **prisoner's dilemma**.

Bonnie and Clyde are caught by the police.

There are two strategies: stay silent ($S$) or confess ($C$).

Their sentence depends upon their own strategy, but also upon the strategy of the other.

# The Prisoner's Dilemma

Clyde

| | | *S* | *C* |
|---|---|---|---|
| Bonnie | *S* | -5, -5 | -30, -1 |
| | *C* | -1, -30 | -10, -10 |

What plays are we likely to see for this game?

# The Prisoner's Dilemma

| | | Clyde: *S* | Clyde: *C* |
|---|---|---|---|
| Bonnie | ***S*** | **-5, -5** | **-30, -1** |
| | ***C*** | **-1, -30** | **-10, -10** |

If Bonnie stays silent, then Clyde's best response is to confess (-1 > -5).
If Bonnie confesses, Clyde's best response is still to confess (-10 > -20).

# The Prisoner's Dilemma

| | | Clyde *S* | Clyde *C* |
|---|---|---|---|
| Bonnie | *S* | -5, -5 | -30, -1 |
| Bonnie | *C* | -1, -30 | -10, -10 |

The same is true for Bonnie. So, for both players, confess is the dominant strategy.

# The Prisoner's Dilemma

| | | Clyde | |
|---|---|---|---|
| | | *S* | *C* |
| Bonnie | *S* | -5, -5 | -30, -1 |
| | *C* | -1, -30 | -10, -10 |

Notice that that (C,C) is the Nash equilibrium even though (S,S) would yield better outcome for both players. In fact, (S,S) is **Pareto efficient** – there is no other strategy that makes both players better off – whereas (C,C) is not.

# Sequential Games

Unto so far, we assumed that the players chose their strategies simultaneously. Such games are **simultaneous games**.

But there are other games in which one player plays before another player. Such games are **sequential games**.

The player who plays first is the **leader**. The player who plays second is the **follower**.

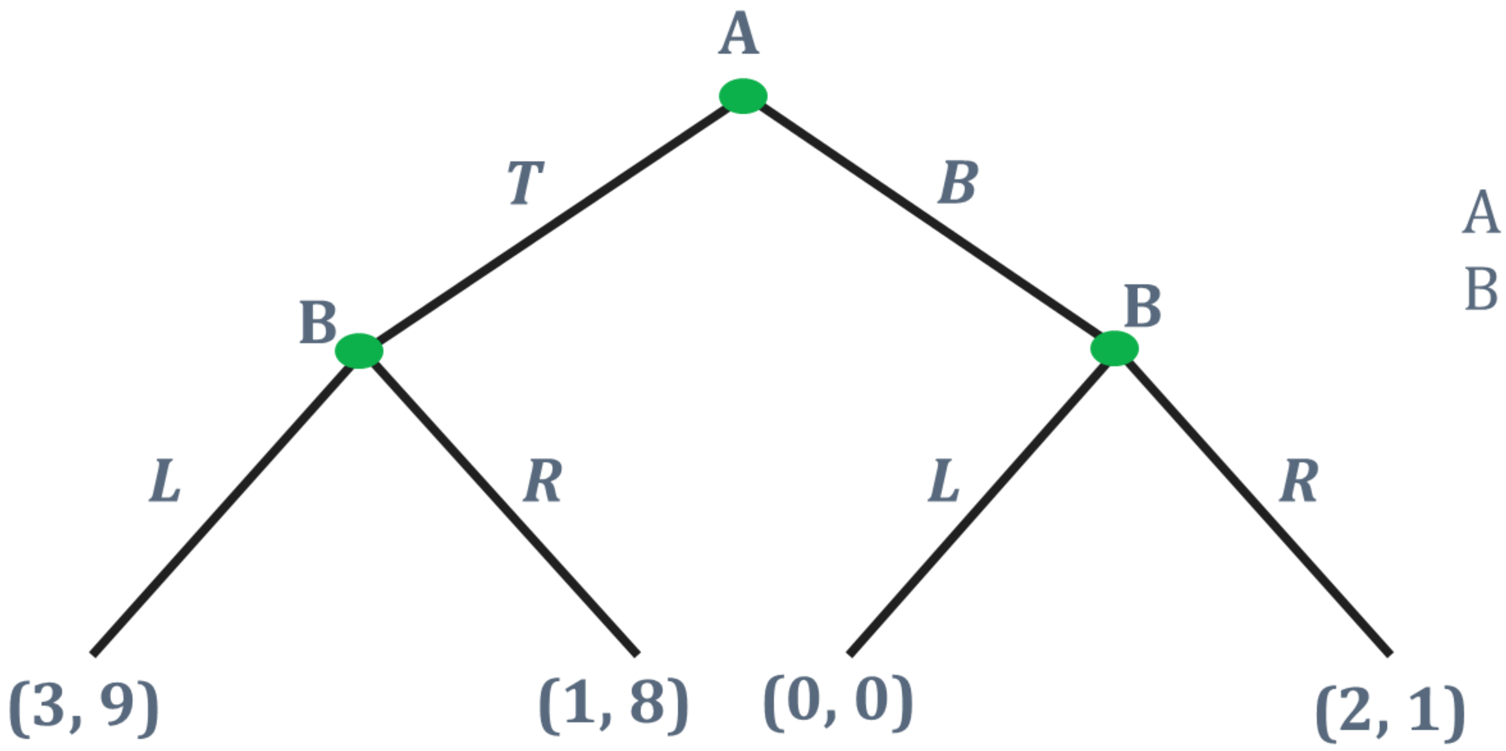
# A Sequential Game Example

| | | Player B | |
|---|---|---|---|
| | | *Left* | *Right* |
| Player A | *Top* | 3, 9 | 1, 8 |
| | *Bottom* | 0, 0 | 2, 1 |

Consider the game we analyzed before, but now it is played **sequentially**, with A leading and B following.

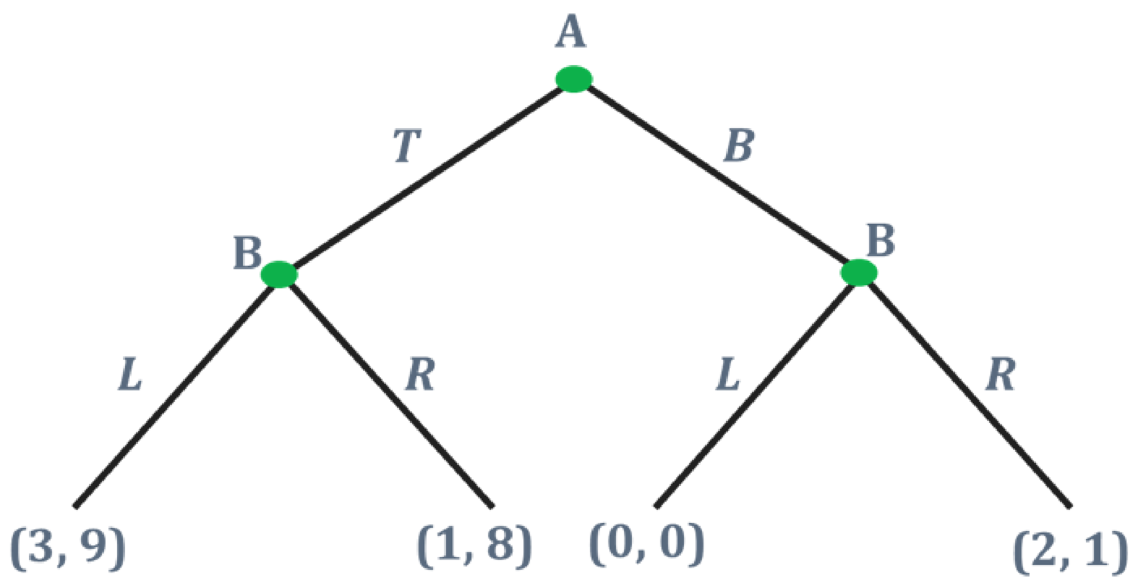
We can rewrite the game in **extensive form**: a game tree that represents the time pattern of the choices.

# A Game in Extensive Form




A plays first.
B plays second.

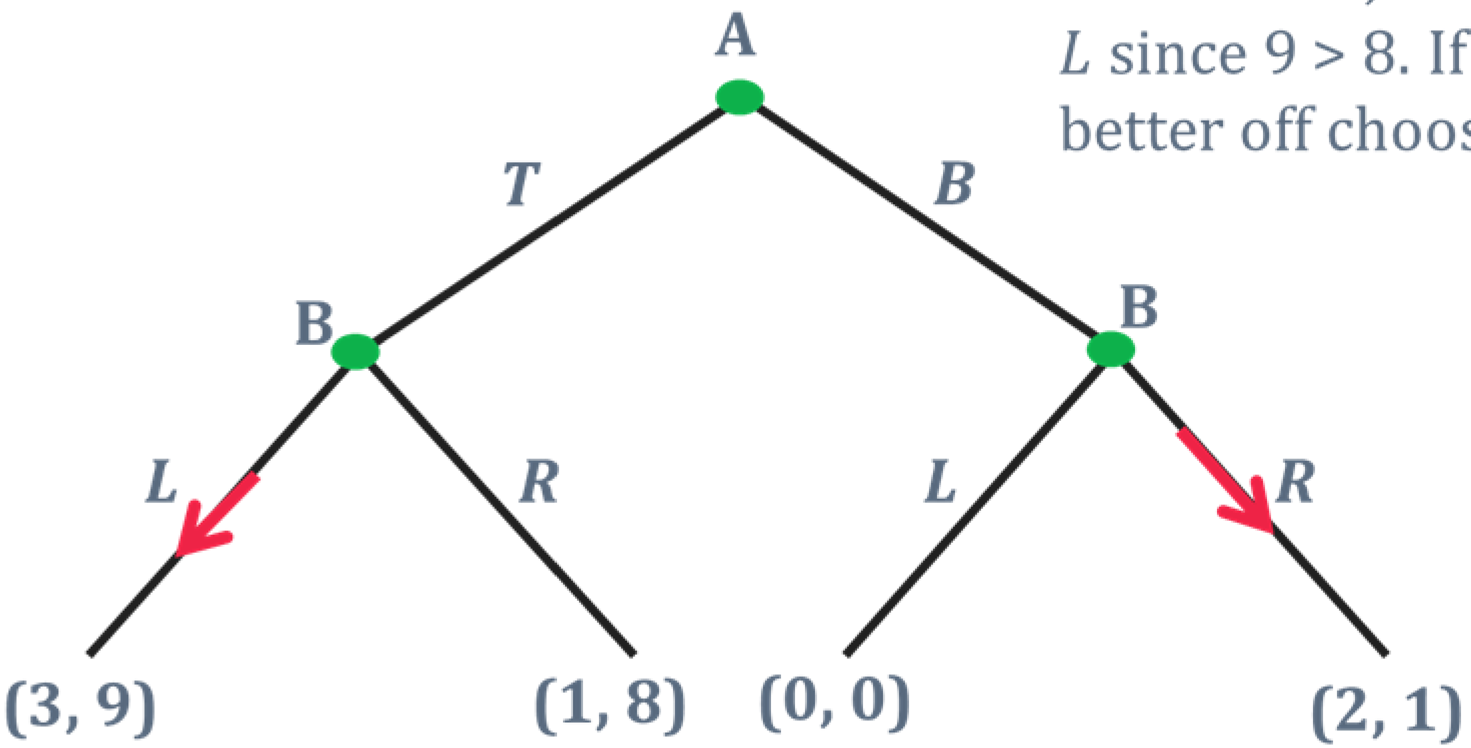
# Backward Induction




**Backward induction:** solve the game from back to front by crossing out all inferior options.

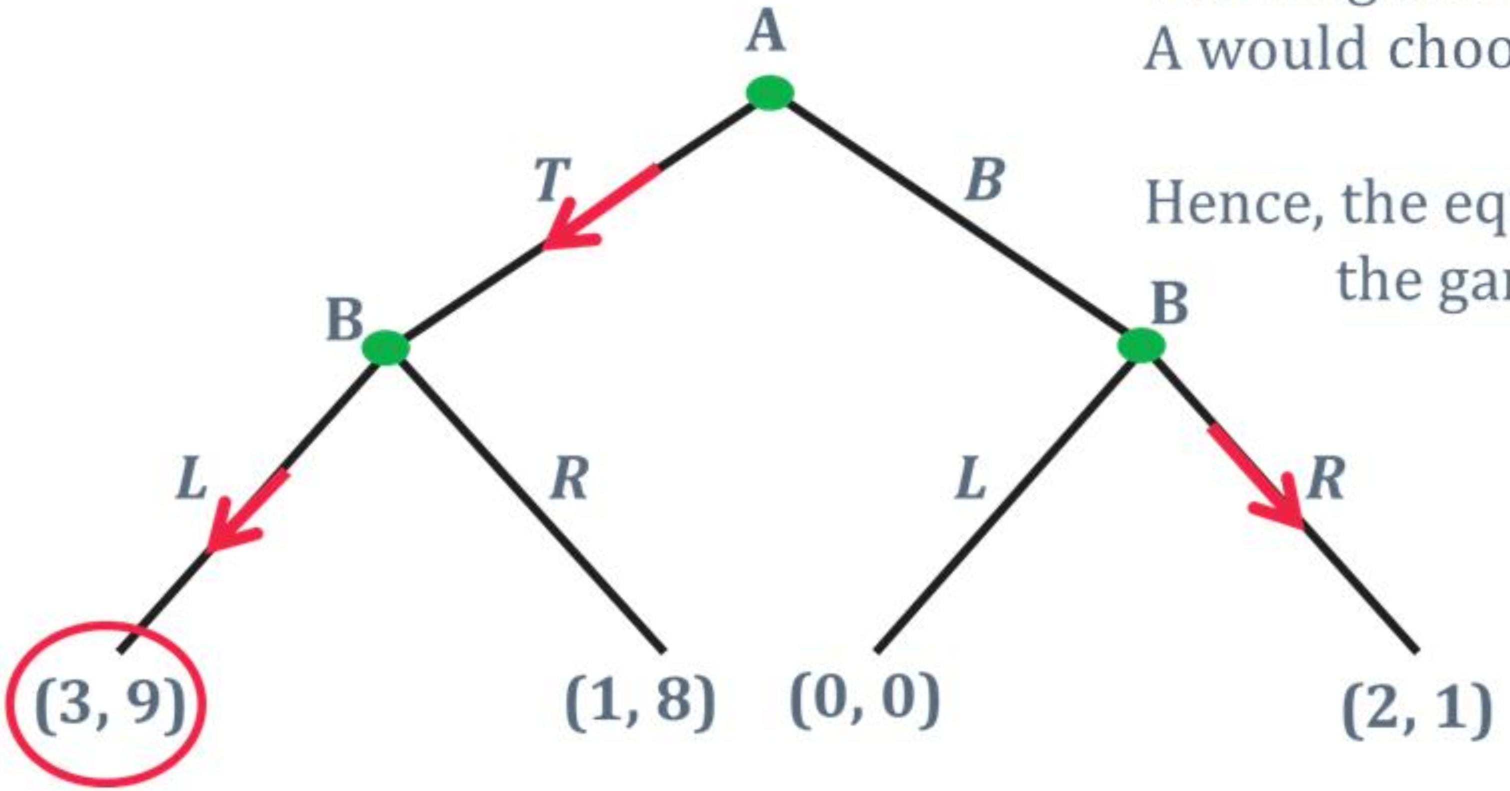
The method of backward induction (BI) always delivers a Nash equilibrium in pure strategies, since neither player has an incentive to deviate after BI.

# A Sequential Game Example




If A chose $T$, B would be better off choosing $L$ since $9 > 8$. If A chose $B$, B would be better off choosing $R$ since $1 > 0$.

# A Sequential Game Example




Working backward, since A knows this, A would choose *T* since 3 > 2.

Hence, the equilibrium in this version of the game is *(T, L)*.

# Repeated Games

A game is a **repeated game** if there are several periods, and the same game is played once in each period.

What strategies are sensible in repeated games depends greatly on whether the game

- is repeated over only a **finite** number of periods, or
- is repeated over an **infinite** number of periods.

Repeated games are often used to study and explain how a cartel can be enforced.

# The Prisoner's Dilemma: Repeated Infinitely

Clyde

| | | *S* | *C* |
|---|---|---|---|
| Bonnie | *S* | -5, -5 | -30, -1 |
| | *C* | -1, -30 | -10, -10 |

If the game is repeated for an infinite number of periods then the game has a huge number of credible NE.

# The Prisoner's Dilemma: Repeated Infinitely

$(S, S)$ forever is one such NE because a player can punish the other for choosing to confess $C$. One example of a strategy that could deliver (S,S) forever as NE is the tit-for-tat strategy.

**Tit-for-tat strategy**: in the first round you stay silent $S$. On every round thereafter, you follow the strategy of your opponent in the previous round: if she chose $S$ you choose $S$, if she chose $C$ you choose $C$.

Intuition: The threat of punishment via a confession $C$ – and the lower payoffs that may follow – may sustain $(S, S)$ as an equilibrium if both players care enough about the future.

However, $(C, C)$ forever is also a NE…

# The Prisoner's Dilemma: Repeated Finitely

Clyde

| | | *S* | *C* |
|---|---|---|---|
| Bonnie | *S* | **-5, -5** | **-30, -1** |
| | *C* | **-1, -30** | **-10, -10** |

Suppose that this game will be played in each of only 3 periods; $t = 1, 2, 3$.
What is the likely outcome?

# The Prisoner's Dilemma: Repeated Finitely

- Use **backward induction** and start at period $t = 3$. Since there is no future, this is a one-shot game. Both should choose confess $C$ at $t = 3$.
- Then let's go to $t = 2$. Clyde and Bonnie know each will choose confess $C$ in $t = 3$. So, if Clyde chooses S in $t = 2$, Bonnie will choose C for sure. And vice versa. Both should choose confess $C$ in $t = 2$ as well.
- In $t = 1$ Clyde and Bonnie both expect that each will choose confess $C$ in $t = 2$ and $t = 3$. Both should choose confess $C$ in $t = 1$.
- The only credible (subgame perfect) NE for this game is where both Clyde and Bonnie choose confess in all three periods $(C,C)$. This is true even if the game is repeated for a large, but still finite, number of periods. This is due to backward induction: **there is never a threat of punishment** via a confession C in the future, since C in the future is certain.