1.1. (1 point)

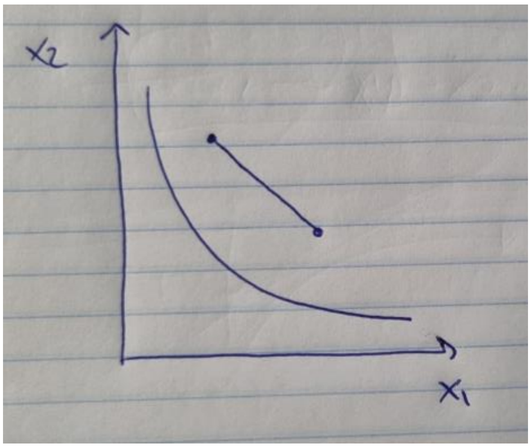
$$TRS = -\frac{\frac{df}{dx_1}}{\frac{df}{dx_2}} = -\frac{MP_1}{MP_2} - \frac{\alpha\theta x_1^{\alpha-1} x_2^{\beta}}{\beta\theta x_1^{\alpha} x_2^{\beta-1}} = -\frac{\alpha x_2}{\beta x_1}$$

1.2. (2 points)

When $\alpha \to 0$ then $TRS \to 0$.

Economic intuition: TRS tells us if x_1 increases, how much x_2 can decrease while keeping output constant. When $\alpha \to 0$, then $\frac{df}{dx_1} = MP_1 \to 0$. Hence, if x_1 increases then output does not increase as the marginal productivity of x_1 is near zero, and so x_2 cannot decrease at all. Hence, TRS should be near zero. In other words, $TRS = \frac{MP_1}{MP_2} = 0$ when $MP_1 = 0$.

1.3. (1 point)



1.4. (1 point)

The input requirement set are all points (x_1, x_2) to the north-east side of the isoquant, as these points allow the firm to produce at least $\bar{y}$. This set is convex since one can connect any two points within that set by a straight line, and that straight line also lies entirely within the set.

2.1. (1 point)

WAPM is that

$$p^t y^t - \boldsymbol{w}^t \boldsymbol{x}^t \geq p^t y^s - \boldsymbol{w}^t \boldsymbol{x}^s \; \forall \; t \; and \; s \neq t$$

So, we can test WAPM using t=1 and s=2, and using t=2 and s=1:

With t=1 and s=2

$$p^1 y^1 - \boldsymbol{w}^1 \boldsymbol{x}^1 \geq p^1 y^2 - \boldsymbol{w}^1 \boldsymbol{x}^2$$

$$40 - 14 > 40 - 16$$

With t=2 and s=1

$$p^2 y^2 - \boldsymbol{w}^2 \boldsymbol{x}^2 \geq p^2 y^1 - \boldsymbol{w}^2 \boldsymbol{x}^1$$

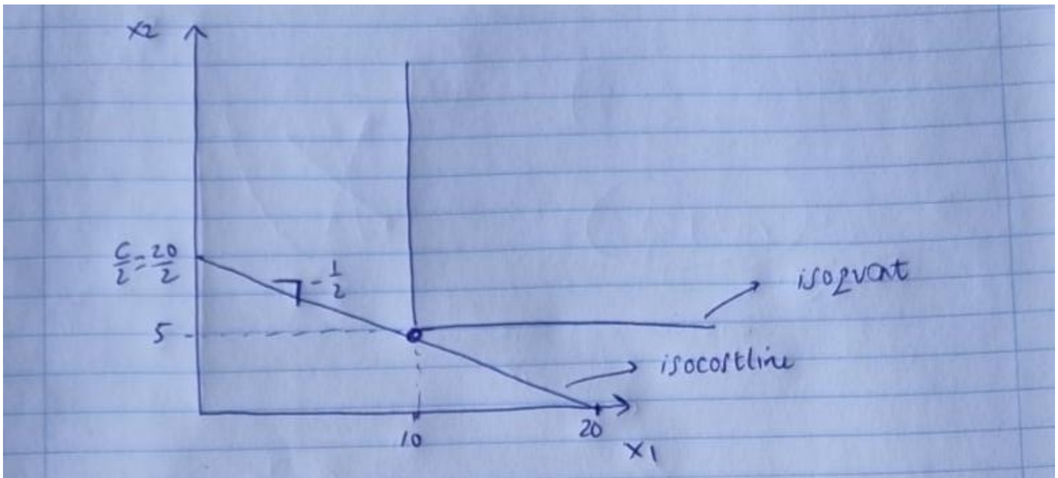
$$30 - 12 > 30 - 13$$

So WAPM cannot be rejected.

2.2. (2 points)

A necessary *and sufficient* condition for profit maximization is that the inequality above holds for *all feasible production plans*. However, we do not observe all feasible plans, but only those selected by the firm in months 1 and 2. As a result, WAPM provides only a sufficient condition for profit maximization. If WAPM is rejected, we can conclusively say that the firm did not maximize profits. However, if WAPM is not rejected, it does not necessarily mean that the firm maximized profits, as there may be a feasible plan that would have led to higher profits, but the firm did not select it in months 1 and 2.

2.3. (2 points)



isoquant: $\min(1x_1, 2x_2) = 10$ if $x_1 = 10$ & $x_2 \geq 5$,
or $x_1 \geq 10$ & $x_2 = 5$

isocostline: $1x_1 + 2x_2 = c$

$x_2 = \frac{c}{2} - \frac{1}{2}x_1$,

if $c = 20$, then $(x_1, x_2) = (10, 5)$ is on the isocostline and costs are minimized as it just touches the isoquant.

$x_1 = 10, x_2 = 5$

$c = 20$

2.4. (1 point)

$x_1 = 10, x_2 = 5$

$c = 30$

3.1. (3 points)

Set up the Lagrange

$$L = 2x_1 + 1x_2 - \lambda(x_1^{\frac{2}{3}}x_2^{\frac{1}{3}} - 10)$$

Take FOCs

$$\frac{\partial L}{\partial x_1} = 2 - \lambda\frac{2}{3}x_1^{-\frac{1}{3}}x_2^{\frac{1}{3}} = 0$$

$$\frac{\partial L}{\partial x_2} = 1 - \lambda\frac{1}{3}x_1^{\frac{2}{3}}x_2^{-\frac{2}{3}} = 0$$

$$\frac{\partial L}{\partial \lambda} = x_1^{\frac{2}{3}}x_2^{\frac{1}{3}} - 10 = 0$$

Divide the first two FOCs by each other to get

$$2 = 2\frac{x_2}{x_1}$$

$$x_1 = x_2$$

Plug this into the third FOC

$$x_1^{\frac{2}{3}}x_1^{\frac{1}{3}} - 10 = 0$$

$$x_2^{\frac{2}{3}}x_2^{\frac{1}{3}} - 10 = 0$$

And solve for x_i to obtain the factor demands:

$$x_1^* = 10$$

$$x_2^* = 10$$

Finally, we can find the minimum costs by plugging the factor demands into the cost function:

$$c = 2x_1^* + 1x_2^*$$

$$c = 2 * 10 + 1 * 10$$

$$= 30$$

3.2. (2 points)

The FOCs are the first derivatives of the Lagrange towards the choice variables.

The SOC condition is that the production function $f(x_1, x_2)$ needs to be concave.

The intuition behind the FOCs can be gotten by dividing the first two FOCs by each other. This division gives us that the ratio of the input prices, which is the slope of isocost line, equals the ratio of the marginal productivities, which is the slope of isoquant. Hence, at the optimum the isocost line just touches the isoquant.

The intuition behind the SOC is that a concave production function implies that the input requirement set is convex. This implies that the isoquant always lies above the isocost line (except where they touch, which is the FOC). Hence, if we move away from the optimum by keeping output constant (a move along the isoquant), then we are going to increase costs (an isocost line with a higher intercept).

4.1. (2 points)

Let

$$f(\boldsymbol{x}) = y$$

DRTS implies for t>1

$$f(t\boldsymbol{x}) < ty$$

We can write DRTS as

$$f(t\boldsymbol{x}) = \tilde{t}y \;\; with \;\; 1 < \tilde{t} < t$$

Now consider that the costs of producing $y = f(\boldsymbol{x})$ are

$$\boldsymbol{wx} = c$$

So that the costs of producing $\tilde{t}y = f(t\boldsymbol{x})$ are

$$\boldsymbol{w}t\boldsymbol{x} = t\boldsymbol{wx} = tc$$

Then we can write the average costs of producing y as

$$\frac{c}{y}$$

And the average cost of producing $\tilde{t}y$ as

$$\frac{tc}{\tilde{t}y}$$

Note that we have:

$$\frac{tc}{\tilde{t}y} > \frac{c}{y}$$

Indeed, $\tilde{t} < t$ implies $\frac{t}{\tilde{t}} > 1$. Hence average cost at $\tilde{t}y$ are larger than at y.

4.2. (2points)

The short-run total costs equal the long-run total costs when production occurs at a point where the fixed factor is fixed at a level that coincides with the long-run optimal factor demand, i.e., when that factor is no longer fixed.