3. Affine models for European term structures of interest rates

3.1. Introduction

The identification of the factors that determine the time series and cross section behaviour of the term structure of interest rates is a relevant issue, namely for assessing the impact of economic policy measures or for hedging purposes (see, for instance, Fleming and Remolona (1999) and Bliss (1997)).

Most papers have focused on the U.S. term structure. The pronounced hump-shape of the US yield curve and the empirical work pioneered by Litterman and Scheinkman (1991), as underlined in the previous chapter, have led to the conclusion that three factors are required to explain the movements of the whole term structure of interest rates.

Moreover, given the stochastic properties of interest rates volatility, Gaussian models are often rejected, as they are based on the assumption of constant volatility. Therefore, several recent papers have used 3-factor models with stochastic volatility in order to model the term structure of interest rates (see, for instance, Balduzzi *et al.* (1996) and Gong and Remolona (1997a)).

However, it was also referred that stochastic volatility models pose admissibility problems. Besides, some term structures may have properties identifiable with less complex models. For instance, according to Buhler *et al.* (1999), principal component analysis reveal that two factors explain more than 95 percent of the variation in the German term structure of interest rates consistently during 1970 to 1999. Therefore, the initial step in modelling the term structure of interest rates will consist in assessing the adequacy of Gaussian models to the German term structure of interest rates. These models overcome the empirical problems posed by stochastic volatility models and are capable of reproducing a wide variety of shapes of the yield curve, though they face some shortcomings regarding the limiting properties of the instantaneous forward rate.[1]

As the data revealed supportive of the constant volatility assumption, a two-latent factor Gaussian model is fitted. The model specification implies that the short-term interest rate is the sum of a constant with the two latent factors. This hypothesis is consistent with the idea that the Bundesbank followed a type of Taylor rule in setting official interest rates as recently documented by Clarida *et al*. (1998) and Clarida and Gertler (1997). In accordance with such a rule the short-term interest rate was adjusted in response to the deviation of inflation and output from their targets.

The model is also consistent with the idea that nominal interest rates can be (approximately) decomposed into two components: the expected rate of inflation and the expected real interest rate (Fisher hypothesis) or, in the C-CAPM model, the expected growth rate of consumption. A three-latent factor Gaussian model is also estimated in order to capture other potential influences in the evolution of the German yield curve, like for example, external factors.

Both two and three-factor models fit quite well the yield and the volatility curve, also providing reasonable estimates for the one-period forward and term premium curves. However, two periods of poorer model performance are identified, both related to world wide gyrations in bond markets - Spring 1994 and 1998 - which were characterised by sharp changes in long-term interest rates while short-term rates remained stable. Thus, it seems that the third factor fails to capture the potential external influences on German interest rates. Given that both models are nested, a Chi-square test is performed and the two-factor model is not rejected.

[1] See, e.g., Campbell *et al*. (1997), pp. 433, on the limitations of a one-factor homoskedastic model.

As Backus *et al.* (1997) mention, the major outstanding issue is the economic interpretation of the interest rate behaviour approximated with affine models in terms of its monetary and real economic factors.[2] In this chapter, the subject is assessed in three different ways: the first two are based on the explicit identification of one of the factors with the inflation rate and relating the other factor to the *ex-ante* real interest rate, in line with Zin (1997). The third way chosen to analyse the identification issue was based on econometric evidence about the leading indicator properties of one factor for inflation developments in Germany.

Within this framework, a first exercise consisted in identifying one factor with the inflation rate process, modelled as an AR(1) process, following Fung *et al.* (1999).[3] The second factor is left unconstrained and should reflect the “real” determinants of the term structure of interest rates, such as the output gap or the real interest rate.

A second exercise performed is based on the assumption that the inflation is given by two factors, being one of them a common factor with the term structure.[4] As it can be shown, this assumption provides the general case of a joint model developed in Fung *et al.* (1999), where the factor loadings are not restricted to the values of the parameters of the interest rate loadings.

The second factor of the term structure is left unconstrained and should reflect the “real” determinants of the term structure of interest rates, such as the output gap or the real interest rate, while the second factor of the inflation shall reflect the non-core inflation moves.

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[2] This is also the major theoretical drawback in arbitrage pricing theory (APT) developed by Ross (1976).

[3] In Remolona et al. (1998), inflation-indexed bonds issued by the UK Government are used for this purpose. However, these securities do not exist in Germany. Possible alternatives include the specification of the inflation as an observable factor or regression analysis between the latent factors and the observable variables to which they are supposedly related.

[4] This joint factor can be taken as a proxy for the core inflation.

The behaviour of the two-factor model is also good when the inflation rate is explicitly included, though the estimates for the term premium are higher than in the non-identified models. The two-factor model is also applied to a longer database but comprising only interest rates from one-year to ten-year maturity. The results obtained are similar to the previous ones.

The conclusions obtained are in accordance with previous empirical findings on the relationship between the term structure, on one side, and future inflation and real interest rates, on the other side.

A joint 2-factor Gaussian model for the German and French term structures of interest rates is subsequently presented and developed. This model differs from the one presented in Fung *et al*. (1999), as the latter considers stochastic volatilities for the US and Canada term structures. The results obtained suggest that the assumption of a common factor for the whole period considered and/or the assumption of constant volatility of the French term structure is not adequate.

Assuming only latent factors, the Kalman Filter was used to estimate them and a maximum likelihood procedure to estimate the time-constant parameters, following the pioneer work by Chen and Scott (1993a and 1993b).

The chapter is structured as follows. In the next section the two-factor model to be estimated is fully specified. The third section focuses on the identification of the factors. The fourth section concerns the joint model for the German and French term structures. The econometric methodology is presented in the fifth section. The data and the estimation results are included in the sixth section. The main conclusions are stated at the end.

3.2. The two-factor Gaussian model

The two-factor Gaussian or constant volatility model is a generalisation of the Vasicek (1977) one-factor model and a particular case of the DK model. Following equation (2.19), the pricing kernel in a two- or a three-factor Gaussian model may be written as:[5]

$$(3.1)\quad -m_{t+1} = \delta + \sum_{i=1}^{k}\left(\frac{\lambda_i^2}{2}\sigma_i^2 + z_{it} + \lambda_i \sigma_i \varepsilon_{t+1}\right)$$

with k = 2 or 3. The factors are assumed to follow a first-order autoregressive process, with zero mean:[6]

$$(3.2)\quad z_{i,t+1} = \varphi_i z_{it} + \sigma_i \varepsilon_{i,t+1}$$

In this model, from (2.21), the logs of bond prices are written as:

$$(3.3)\quad -p_{n,t} = A_n + B_{1,n} z_{1t} + B_{2,n} z_{2t}$$

Within the DK framework, this model is thus characterised as in Table 3.1.

Table 3.1 – DK characterisation of the two- or three-factor Gaussian model

K	θ_i	Φ	α_i	β_i	δ	γ_i
2 or 3	0	$\begin{bmatrix}\varphi_1 & & \\ & \ddots & \\ & & \varphi_k\end{bmatrix}$	σ_i^2	0	$\delta + \sum_{i=1}^{3}\frac{\lambda_i^2}{2}\sigma_i^2$	1

[5] As it will be seen later, this specification was chosen in order to write the short-term interest rate as the sum of a constant (δ) with the factors.

[6] This corresponds to considering the differences between the "true" factors and their means.

According to (2.22), (2.23) and (3.1), the recursive restrictions are:

(3.4) $$A_n = A_{n-1} + \delta + \frac{1}{2}\sum_{i=1}^{k}\left[\lambda_i^2\sigma_i^2 - \left(\lambda_i\sigma_i + B_{i,n-1}\sigma_i\right)^2\right]$$

(3.5) $$B_{i,n} = \left(1 + B_{i,n-1}\varphi_i\right)$$

As it is seen from equation (3.1), in a homoskedastic model, the element on the RHS of (2.19) related to the risk is zero. Thus, there are no interactions between the risk and the factors influencing the term structure.

Therefore, yields are defined as:

(3.6) $$y_{n,t} = \frac{1}{n}\left(A_n + B_{1,n}z_{1t} + B_{2,n}z_{2t}\right)$$

Given the common normalisation $p_{ot} = 0$, $A_0 = B_{10} = B_{20} = 0$, the short term interest rate is:

(3.7) $$y_{1,t} = \delta + \sum_{i=1}^{k} z_{it}$$

As referred in Campbell *et al.* (1997) in a one-factor homoskedastic model setting, the $B_{i,n}$ coefficients measure the sensitivity of the log of bond prices to changes in short-term interest rate.[7] This model has the appealing feature of the short-term being the sum of two factors plus a constant. Our conjecture is that one factor is

[7] This is different from duration, as it does not correspond to the impact on bond prices of changes in the respective yields, but instead in the short rate.

related to inflation expectations and that the other factor reflects the *ex-ante* real interest rate.[8]

In this model, the one-period forward rate in (2.30), (2.37) or (2.39) is given by:

$$(3.8)\qquad f_{n,t} = \delta + \frac{1}{2}\sum_{i=1}^{2}\left[\lambda_i^2\sigma_i^2 - \left(\lambda_i\sigma_i + \frac{1-\varphi_i^n}{1-\varphi_i}\sigma_i\right)^2\right] + \sum_{i=1}^{2}\left[\varphi_i^n z_{it}\right]$$

The specification of the forward-rate curve in this model accommodates very different shapes. However, as referred in the previous chaper, the limiting forward rate cannot be simultaneously finite and time-varying. In fact, if φ_i<1, the limiting value will not depend on the factors, corresponding to the following expression:

$$(3.9)\qquad \lim_{n\to\infty} f_{n,t} = \delta + \sum_{i=1}^{k}\left[-\frac{\lambda_i\sigma_i^{\,2}}{(1-\varphi_i)} - \frac{\sigma_i^{\,2}}{2(1-\varphi_i)^2}\right]$$

If $\varphi_i = 1$, the limiting value of the instantaneous forward becomes time-varying but assumes infinite values. Effectively, according to (2.38), $\frac{1-\varphi_i^n}{1-\varphi_i} = n$ in this case. Thus, the expression for the instantaneous forward will be:

$$(3.10)\qquad f_{n,t} = \delta + \sum_{i=1}^{k}\left[-n\lambda_i\sigma_i^2 - \frac{1}{2}n^2\sigma_i^2\right] + \sum_{i=1}^{k} z_{it}$$

Accordingly, if λ_i <0, the forward may start by increasing, but in longer terms it will decrease infinitely. The volatility curve in (2.28) corresponds in this model to:

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[8] This conjecture is in line with the assertion in Zin (1997).

(3.11) $Var_t\left(y_{n,t+1}\right)=\dfrac{1}{n^2}\sum_{i=1}^{k}\left(B_{i,n}^2\sigma_i^2\right)$

and the conditional volatility of the one-period forward rate from equation (3.8):

(3.12) $Var_t\left(f_{n,t+1}\right)=\sum_{i=1}^{k}\left(\varphi_i^{2n}\sigma_i^2\right)$

Notice that, as the factors have constant volatility, given by $Var_t\left(z_{i,t+1}\right)=\sigma_i^2$, the volatility of yields depends neither on the level of the factors, nor on the level of the short-rate.

The term premium equation (2.31) resulting from this model is equal to:

(3.13)
$$\begin{aligned}\Lambda_{n,t} &= E_t p_{n,t+1}-p_{n+1,t}-y_{1,t} = \frac{1}{2}\sum_{i=1}^{k}\left[\lambda_i^2\sigma_i^2-\left(\lambda_i\sigma_i+\frac{1-\varphi_i^n}{1-\varphi_i}\sigma_i\right)^2\right]\\ &=\sum_{i=1}^{k}\left[-\lambda_i\sigma_i^{\,2}B_{i,n}-\frac{B_{i,n}^{\;2}\sigma_i^{\,2}}{2}\right]\end{aligned}$$

The term premium may also be written as in (2.35). In a Gaussian model, the conditional variances in this equation are constant and correspond to $B_{i,n}^{\;2}Var_t\left(z_{i,t+1}\right)$. Consequently, equation (2.35) is equivalent to:

(3.14) $\Lambda_{n,t}=\sum_{i=1}^{k}\left[B_{i,n}COV_t\left(z_{i,t+1},m_{t+1}\right)-B_{i,n}^{\;2}Var_t\left(z_{i,t+1}\right)/2\right]$

According to equations (3.1) and (3.2), the first term on the right hand side of (3.14) is $\sum_{i=1}^{k} -\lambda_i \sigma_i^{\,2} B_{i,n}$, while the second term is $-\frac{\sum_{i=1}^{k} B_{i,n}^{\ 2} \sigma_i^{\,2}}{2}$. Thus, equation (3.14) corresponds to (3.13).[9]

In line with the description in the previous chapter for the general DK model, the first component of (3.13) is a pure risk premium, being λ_i the market prices of risks, and the second component is a Jensen's inequality term. In this model, following equations (3.8) and (3.13), the instantaneous forward corresponds to the sum of the term premium with a constant and with the factors weighted by the auto-regressive parameters of the factors.

Once again, the limiting cases are worth noting. When $\varphi_i < 1$, the limiting value of the risk premium differs from the forward only in δ. Then, within this constant volatility model and in this particular case, the expected short-term rate for a very distant settlement date is δ, i.e., the average short-term interest rate. When $\varphi_i = 1$, the instantaneous forward differs from the risk premium in the sum of δ with the factors, but on average the difference is still δ.

In order to assess the adequacy of the model, forward regressions as in (2.42) will be estimated. It can be shown that, in the two-factor Gaussian model under analysis, the forward regression coefficient in (2.47) must be equal to one in all maturities.

[9] This can be easily demonstrated by comparing the terms in the last brackets of the numerator and the denominator of (2.47):

$$B_1 - \Phi^T (B_n - B_{n-1}) = B_1 - \Phi^T B_n + \Phi^T B_{n-1} + B_1 - B_1 = \left(B_1 + \Phi^T B_{n-1}\right) - \Phi^T B_n = B_n - \Phi^T B_n = \left(I - \Phi^T\right) B_n$$

$$B_1 + B_n - B_{n+1} = B_1 + B_n - \left(B_1 + B_n^T \Phi\right)^T = (I - \Phi) B_n$$

3.3. Identification of the model

Contrary to the pioneer interest rate models, such as Cox *et al.* (1985a), where the short-term interest influenced the whole term structure, the latent factor models do not use explicit determinants of the yield curve. Therefore, as referred in Backus *et al.* (1997), the major outstanding issue in this context is the economic interpretation of the interest rate behaviour approximated with affine models, in terms of its monetary and real economic factors. Our conjecture is that two factors seem to drive the German term structure of interest rates: one factor related to the *ex-ante* real interest rate and a second factor linked to inflation expectations.

As previously stated, the identification will thus be assessed in three different ways, being the first based on the model developed in Fung *et al*. (1999). As it is supposed that one of the latent factors is related to inflation, one factor is identified with that variable, which is assumed to be an AR(1) process. The inflation, denoted by π_t, will thus be defined as:[10]

$$(3.15)\quad (\pi_{t+1} - \overline{\pi}) = \rho(\pi_t - \overline{\pi}) + u_{t+1}$$

where $\overline{\pi}$ is the unconditional mean of the inflation rate and ρ is a parameter that measures the rate of mean-reversion. Considering that the short-term interest rate in (3.7) is a risk-free rate, as there are no expectation revisions in a one period investment, the hypothesis is that it is the sum of the short-term real interest rate ($r_{1,t}$) with the one-period inflation rate expectation:

[10] Inflation is measured as the annual change in the log CPI. This measure is preferred because the quarter-on-quarter annualised change in the log CPI is extremely volatile with often negative readings. Our measure should be interpreted as capturing the underlying or smoothed one-period inflation rate. Inflation is measured as year-on-year changes in the CPI in deviations from its mean, due to the definition of the factors as having zero mean.

(3.16) $y_{1,t} - \delta = (r_{1,t} - \overline{r}) + E_t(\pi_{t+1} - \overline{\pi})$

where the long-run mean of the short-term interest rate is equal to the sum of the unconditional means of the real interest rate and inflation rate ($\delta = \overline{r} + \overline{\pi}$). The component of the short-term rate related to the second factor may be considered as the one-period inflation expectation:

(3.17) $z_{2,t} = E_t(\pi_{t+1} - \overline{\pi}) = \rho(\pi_t - \overline{\pi}) = \rho\tilde{\pi}_t$

where $\tilde{\pi}_t$ denotes deviation of inflation from the steady-state value.

From (3.15) and (3.17) the value of the second factor in t+1 is:

(3.18) $z_{2,t+1} = E_{t+1}(\tilde{\pi}_{t+2}) = \rho\tilde{\pi}_{t+1} = \rho(\rho\tilde{\pi}_t + u_{t+1}) = \rho z_{2,t} + \rho u_{t+1}$

Comparing equations (3.2) and (3.18), we have the following identification between the parameters of the second factor and the inflation processes:

(3.19) $$\begin{aligned} \rho &= \varphi_2 \\ \rho u_{t+1} &= \sigma_2 \varepsilon_{2,t+1} \end{aligned}$$

Using equations (3.17) to (3.19), the following relationship between the inflation and the second factor is obtained:

(3.20) $\tilde{\pi}_t = \dfrac{1}{\varphi_2} z_{2,t}$

Using this identification method of the term structure, *ex-ante* real interest rates can be derived. Following Remolona *et al*. (1998), one can extract the expected average one-period inflation from t to $t+n$ as the difference between the nominal and the real yields on zero-coupon bonds with term to maturity equal to n, less the inflation risk premium.

In fact, from (3.16) one can define the expected average one-period inflation from t to $t+n$ as the difference between the expected average short-term nominal interest rate and the expected average short-term real interest rate:

$$(3.21)\quad \begin{aligned} &\frac{1}{n}E_t\left(\sum_{i=0}^{n-1}\tilde{y}_{1,t+i}\right)=\frac{1}{n}E_t\left(\sum_{i=0}^{n-1}\tilde{r}_{1,t+i}\right)+\frac{1}{n}\sum_{i=0}^{n-1}E_t(\tilde{\pi}_{t+i})\Leftrightarrow \\ &\frac{1}{n}\sum_{i=0}^{n-1}E_t(\tilde{\pi}_{t+i})=\frac{1}{n}E_t\left(\sum_{i=0}^{n-1}\tilde{y}_{1,t+i}\right)-\frac{1}{n}E_t\left(\sum_{i=0}^{n-1}\tilde{r}_{1,t+i}\right)\end{aligned}$$

In Remolona *et al*. (1998), nominal and indexed-bonds are used in order to estimate the expected inflation rate. In our case, as there are no indexed bonds, the exercise is done in the opposite way, i.e., the inflation is estimated by the model, using only information on nominal yield and inflation itself. Afterwards, estimates for the *ex-ante* real interest rates will be obtained from inflation and nominal interest rate estimates.

The left-hand side of (3.21) is given by:

$$(3.22)\quad \frac{1}{n}\sum_{i=0}^{n-1}E_t(\tilde{\pi}_{t+i})=\frac{1}{n}\sum_{i=0}^{n-1}\rho^i\tilde{\pi}_t=\frac{1}{n}\frac{(1-\rho^n)}{(1-\rho)}\tilde{\pi}_t$$

The elements on the RHS of (3.21) are given by the identity between the n-term nominal and real interest rates, on one hand, and the sum of the expected average nominal and real short-term interest rates with the respective one-period term premiums, on the other hand:

(3.23) $\tilde{r}_{n,t} = \frac{1}{n} E_t \left(\sum_{i=0}^{n-1} \tilde{r}_{1,t+i} \right) + \frac{1}{n} E_t \left(\sum_{i=0}^{n-1} \Lambda_{1,t+i}^{R} \right)$

(3.24) $\tilde{y}_{n,t} = \frac{1}{n} E_t \left(\sum_{i=0}^{n-1} \tilde{y}_{1,t+i} \right) + \frac{1}{n} E_t \left(\sum_{i=0}^{n-1} \Lambda_{1,t+i}^{N} \right)$

where $\Lambda_{1,t}^{R}$ and $\Lambda_{1,t}^{N}$ are respectively the one-period real and nominal term premiums with one period to maturity. The difference between these premiums corresponds to the one-period inflation risk premium:

(3.25) $\Lambda_{1,t}^{\pi} = \Lambda_{1,t}^{N} - \Lambda_{1,t}^{R}$

Inserting equations (3.23) to (3.25) in (3.21), we obtain the following expression for real interest rates:

$$\begin{aligned} \tilde{r}_{n,t} &= \tilde{y}_{n,t} - \frac{1}{n} E_t \sum_{i=0}^{n-1} \tilde{\pi}_{t+i} - \frac{1}{n} E_t \sum_{i=0}^{n-1} \Lambda_{1,t+i}^{\pi} = \\ &= \tilde{y}_{n,t} - \frac{1}{n} \frac{(1-\rho^n)}{(1-\rho)} \tilde{\pi}_t - \frac{1}{n} E_t \sum_{i=0}^{n-1} \Lambda_{1,t+i}^{\pi} = \\ &= \tilde{y}_{n,t} - \frac{1}{n} \frac{(1-\rho^n)}{(1-\rho)} \tilde{\pi}_t - \Lambda_{1,t}^{\pi} \end{aligned} \qquad (3.26)$$

In Remolona *et al*. (1998), the inflation-risk premium in (3.26) is estimated as the difference between the nominal and the real term premiums implicit in nominal and indexed bonds. In our case the real interest rate is calculated from (3.26) were the inflation risk premium is identified with the contribution of the second factor to the term premium derived from (3.13). Thus

(3.27) $r_{n,t} = y_{n,t} - \bar{\pi} - \frac{1}{n} \frac{(1-\rho^n)}{(1-\rho)} \tilde{\pi}_t - \Lambda_{1,t}^{\pi}$

The major drawback of this technique is that it implies the second factor to explain simultaneously the inflation as well as the long term rates, which in some periods may evidence significantly different volatilities. Consequently, during the periods of higher volatility of the long-term rates, the estimated inflation tends to present a more irregular behaviour than the true inflation.

Secondly, the procedure sketched above is based on the assumption in (3.15), which is not necessarily the optimal model for forecasting inflation. In fact, this model is too simple concerning its lag structure and also does not allow for the inclusion of other macro-economic information that market participants may use to form their expectations of inflation.[11] For example, information about developments in monetary aggregates, commodity prices, exchange rates, wages and unit labour costs, etc, may be used by market participants to forecast inflation and, thus, may be reflected in the bond pricing process. However, a more complex model would certainly not allow a simple identification of the factor.

One way to overcome these problems is by using a joint model for the term structure and the inflation, where the latter still shares a common factor with the interest rates but is also determined by a second specific factor:[12]

Inflation is modelled as a function of two factors:

(3.28) $$\pi_t = \frac{1}{n}\left(A_\pi + B_\pi{}^T z_{\pi t}\right)$$

where $z_{\pi t} = \begin{bmatrix} z_{2t} \\ z_{1\pi,t} \end{bmatrix}$ and

[11] A more general ARMA model would perhaps be necessary to model the dynamics of inflation. Nevertheless, given that φ_2 is close to 1, equation (3.20) implies a loss of leading indicator properties of the second factor regarding inflation.

(3.29) $z_{1\pi,t+1} = \varphi_{1\pi} z_{1\pi,t} + \sigma_{1\pi} \varepsilon_{1\pi,t+1}$

Comparing to the model developed in Fung *et al.* (1999), this model offers the advantage of not restricting the factor loadings to the values of the parameters of the term structure loadings, as well as to provide an additional factor to explain the inflation moves. Therefore, the second term structure factor does not have to be liable simultaneously for the long-term interest rates and the inflation, avoiding more volatile estimates for the inflation in periods of higher volatility of the long-term rates.

In order to overcome the problems detected in the first identification model, the leading indicator properties of z_{2t} for inflation were also analysed, by setting up a VAR model with p lags (see Hamilton (1994a), chapter 11):

(3.30) $x_t = A_1 x_{t-1} + \ldots + A_p x_{t-p} + \mu + u_t$

where x_t is (2 x 1) and each of the A_i is a (2 x 2) matrix of parameters with generic element denoted $\left[a_{kj}^{i}\right]$ and $u_t \sim IN(0, \Sigma)$. The vector x_t is defined as $x_t = \begin{bmatrix} \pi_t \\ z_{2t} \end{bmatrix}$.

Testing whether z_{2t} has leading indicator properties for inflation corresponds to testing the hypothesis $H_0 : a_{12}^{1} = \ldots = a_{12}^{p} = 0$. This is a test for Granger causality, i.e., a test of whether past values of the factor along with past values of inflation better "explain" inflation than past values of inflation alone. This of course does not imply that bond yields cause inflation. Instead it means that z_{2t} is possibly reflecting bond market's expectations as to where inflation might be headed.

[12] In this case, the loading of the second factor in the inflation equation is independent from the loading in term structure equations, as it is not assumed any explicit relationship between the yields and the inflation and, consequently, there are no recursive restrictions between the yields and the inflation.

In assessing the leading indicator properties of z_{2t}, the Granger causality test can be supplemented with an impulse-response analysis. The vector $MA(\infty)$ representation of the VAR is given by:

(3.31) $x_t = \mu + u_t + \Psi_1 u_{t-1} + \Psi_2 u_{t-2} + ...$

The matrix Ψ_s will then correspond to:

(3.32) $\frac{\partial x_{t+s}}{\partial u_t'} = \Psi_s;$

that is, the row i, column j element of Ψ_s identifies the consequences of a one-unit increase in the jth variable's innovation at date t (u_{jt}) for the value of the i-th variable at time $t+s$ ($x_{i,t+s}$), holding all other innovations at all dates constant. A plot of row i, column j element of $\Psi_s(\frac{\partial x_{i,t+s}}{\partial u_{jt}})$, as a function of s is the impulse-response function. It describes the response of ($x_{i,t+s}$) to a one-time impulse in $x_{j,t,}$ with all other variables dated t or earlier held constant. Supposing that the date t value of the first variable in the autoregression z_{2t} is higher than expected, so that $u_{1,t}$ is positive, then we have:[13]

(3.33) $\frac{\partial \hat{E}(x_{i,t+s} \mid z_{2t}, x_{t-1}', x_{t-2}', ..., x_{t-p}')}{\partial z_{2t}} = \frac{\partial x_{i,t+s}}{\partial u_{1t}}$

[13] We use a Cholesky decomposition of the variance-covariance of the innovations to identify the shocks. We order inflation first in the VAR to reflect the idea that whereas inflation does not respond, contemporaneously, to shocks to expectations, these may be affected by contemporaneous information on inflation.

Thus if z_{2t} is a leading indicator of inflation, a revision in market expectations of inflation $\partial\hat{E}(x_{i,t+s} \mid z_{2t}, x'_{t-1},...,x'_{t-p})$ should be captured by the marginal impact of a shock to the innovation process in the equation for z_{2t}.

3.4. A two-countries Gaussian model

The prospects about the European Monetary Union closed the gaps between the short-term interest rates of the participating countries. However, the long-term interest rates were not equalised, due to different fiscal, liquidity and rating features. Consequently, it is important to explain how these spreads are formed, namely between the core countries of the Euro area. In this section, the focus will be on the spreads between Germany and France.

A first attempt to model jointly the term structures of two countries is found in Fung *et al.* (1999), where a two-factor stochastic volatility model is used to estimate simultaneously the U.S. and the Canadian term structures. In this case, it was assumed that both countries share a common factor related to the real interest rate, following the close trade relationship between those countries. As each country pursued its own monetary policy, it was assumed that the U.S. and the Canadian term structures also depended on a specific factor, related to the inflation expectations and, accordingly, to the monetary policy.

In the Euro area, the opposite happens, i.e., there is a common monetary policy and real interest rates differ among the eleven countries. It is also arguable that, before the Euro launch, ERM-EMS participants followed German monetary policy. Consequently, the model used in this section assumes a common factor related to the inflation expectations and a specific factor that is supposed to be related to the real interest rate. Additionally, it was assumed that the common inflation factor corresponded to the German factor, due to the anchor role performed by the German economy and the Bundesbank among national European central banks, before the Euro launch.

Considering the results obtained in the previous section and the similarities between the term structures of Germany and France, both factors were assumed to have constant volatility. Thus, the German interest rates will be modelled as in the previous section, while the French interest rates were assumed to be a function of a specific real factor and a common nominal factor. Accordingly, the sdf for France is written as:

$$(3.34)\quad -m_{t+1}^{*} = \delta^{*} + \frac{\lambda_1^{*2}}{2}\sigma_i^{*2} + z_{it}^{*} + \lambda_1^{*}\sigma_1^{*}\varepsilon_{t+1}^{*} + \frac{\lambda_2^{2}}{2}\sigma_2^{2} + z_{2t} + \lambda_2\sigma_2\varepsilon_{t+1}$$

The specific factor of the French term structure is assumed to follow a first-order autoregressive order, with zero mean, as previously stated for Germany. Correspondingly, the logs of bond prices will be:

$$(3.35)\quad -p_{n,t}^{*} = A_n^{*} + B_{1,n}^{*} z_{1t}^{*} + B_{2,n} z_{2t}$$

where the specific factor follows

$$(3.36)\quad z_{1,t+1}^{*} = \varphi_1^{*} z_{1t}^{*} + \sigma_1^{*}\varepsilon_{1,t+1}^{*}$$

and the yield curve will correspond to:

$$(3.37)\quad y_{n,t}^{*} = \frac{1}{n}\left(A_n^{*} + B_{1,n}^{*} z_{1t}^{*} + B_{2,n} z_{2t}\right)$$

3.5. Econometric methodology

As the factors that determine the dynamics of the yield curve are non-observable and the parameters are unknown, a Kalman filter and a maximum likelihood procedure were chosen for the estimation of the model.[14] In a brief description, the Kalman Filter is an algorithm that computes the optimal estimate for the state variables at a moment t using the information available up to t-1.[15] The starting point for the derivation of the Kalman filter is to write the model in state-space form, which includes an observation or measurement equation and a state or transition equation, respectively:

(3.38) $$\underset{(r\times 1)}{Y_t} = \underset{(r\times n)}{A} \cdot \underset{(n\times 1)}{X_t} + \underset{(r\times k)}{H} \cdot \underset{(k\times 1)}{Z_t} + \underset{(r\times 1)}{w_t}$$

(3.38a) $$\underset{(k\times 1)}{Z_t} = \underset{(k\times 1)}{C} + \underset{(k\times k)}{F} \cdot \underset{(k\times 1)}{Z_{t-1}} + G\ \underset{(k\times 1)}{v_t}$$

where r is the number of variables to estimate, n is the number of observable exogenous variables and k is the number of non-observable or latent exogenous variables (the factors). w_t and v_t are i.i.d. residuals, distributed as $w_t \sim N(0,R)$ and $v_t \sim N(0,Q)$.

The variance matrices are written as:

(3.39) $$\underset{(r\times r)}{R} = E(w_t w_t')$$

(3.40) $$\underset{(k\times k)}{Q} = E(v_{t+1} v_{t+1}')$$

[14] The maximum likelihood procedure is usually adopted when the parameters are unknown.
[15] For further details see, for instance, Hamilton (1994a), chapter 13, Hamilton (1994b) or Harvey (1990).

According to equation (3.6), the measurement or observation equation in our one-country non-identified model may be written as:

$$(3.41)\quad \begin{bmatrix} y_{1,t} \\ \vdots \\ y_{l,t} \end{bmatrix} = \begin{bmatrix} a_{1,t} \\ \vdots \\ a_{l,t} \end{bmatrix} + \begin{bmatrix} b_{1,1} & b_{2,1} \\ \vdots & \vdots \\ b_{1,l} & b_{2,l} \end{bmatrix} \begin{bmatrix} z_{1,t} \\ z_{2,t} \end{bmatrix} + \begin{bmatrix} w_{1,t} \\ \vdots \\ w_{l,t} \end{bmatrix}$$

where $y_{1,t}$, … , $y_{l,t}$ are the l zero-coupon yields at time t with maturities j = 1, … , u periods and $w_{1,t}$, … ,$w_{l,t}$ are the normally distributed i.i.d. errors, with null mean and standard-deviation equal to e_j^2, of the measurement equation for each interest rate considered, $a_j = A_j/j$, $b_{1,j} = B_{1,j}/j$, $b_{2,j} = B_{2,j}/j$.

Following equation (3.2), the transition or state equation is:

$$(3.42)\quad \begin{bmatrix} z_{1,t+1} \\ z_{2,t+1} \end{bmatrix} = \begin{bmatrix} \varphi_1 & 0 \\ 0 & \varphi_2 \end{bmatrix} \begin{bmatrix} z_{1,t} \\ z_{2,t} \end{bmatrix} + \begin{bmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \end{bmatrix} \begin{bmatrix} v_{1,t+1} \\ v_{2,t+1} \end{bmatrix}$$

being $v_{1,t+1}$ and $v_{2,t+1}$ orthogonal shocks with null mean and variances equal to σ_1^2 and σ_2^2, respectively. As in the estimation of the models the information on the homoskedasticity of yields is exploited, in line with equation (3.11), the observation equation may be rewritten as follows:

$$(3.43)\quad \begin{bmatrix} y_{1,t} \\ \vdots \\ y_{l,t} \\ Var_t\left(y_{1,t+1}\right) \\ \vdots \\ Var_t\left(y_{l,t+1}\right) \end{bmatrix} = \begin{bmatrix} a_{1,t} \\ \vdots \\ a_{l,t} \\ a_{l+1,t} \\ \vdots \\ a_{2l} \end{bmatrix} + \begin{bmatrix} b_{1,1} & b_{2,1} \\ \vdots & \vdots \\ b_{1,l} & b_{2,l} \\ 0 & 0 \\ \vdots & \vdots \\ 0 & 0 \end{bmatrix} \begin{bmatrix} z_{1,t} \\ z_{2,t} \end{bmatrix} + \begin{bmatrix} \upsilon_{1,t} \\ \vdots \\ \upsilon_{l,t} \\ \upsilon_{l+1,t} \\ \vdots \\ \upsilon_{2l,t} \end{bmatrix}$$

where $a_{l+j,t} = \frac{1}{n^2}\left(B_{1,j}^2\sigma_1^2 + B_{2,j}^2\sigma_2^2\right)$ and 2l is the number of variables to estimate (r in (3.38)). In this way we adjust simultaneously the yield curve and the volatility curve, avoiding implausible estimates for the latter.

In our model A is a column vector with elements $a_{j,t}$ for the first l rows and $\frac{1}{n^2}\left(B_{1,j}^2\sigma_1^2 + B_{2,j}^2\sigma_2^2\right)$ for the next l rows; X_t is a 2l-dimension column vector of one's ($n = 1$), C is a column vector of zeros and F is a $k \times k$ diagonal matrix, with typical element $F_{ii} = \varphi_i$ ($k = 2$).

When the link between the second factor and the inflation is considered in line with (3.20), the observation equation becomes:

$$\text{(3.44)} \quad \begin{bmatrix} y_{1,t} \\ \vdots \\ y_{l,t} \\ Var_t(y_{1,t+1}) \\ \vdots \\ Var_t(y_{l,t+1}) \\ \tilde{\pi}_t \end{bmatrix} = \begin{bmatrix} a_{1,t} \\ \vdots \\ a_{l,t} \\ a_{l+1,t} \\ \vdots \\ a_{2l} \\ 0 \end{bmatrix} + \begin{bmatrix} b_{1,1} & b_{2,1} \\ \vdots & \vdots \\ b_{1,l} & b_{2,l} \\ 0 & 0 \\ \vdots & \vdots \\ 0 & 0 \\ 0 & b_\pi \end{bmatrix} \begin{bmatrix} z_{1,t} \\ z_{2,t} \end{bmatrix} + \begin{bmatrix} \upsilon_{1,t} \\ \vdots \\ \upsilon_{l,t} \\ \upsilon_{l+1,t} \\ \vdots \\ \upsilon_{2l,t} \\ \upsilon_\pi \end{bmatrix}$$

Conversely, when the inflation is modelled as motivated by two factors, as in (3.28), the observation equations is:

$$\text{(3.45)} \quad \begin{bmatrix} y_{1,t} \\ \vdots \\ y_{l,t} \\ Var_t(y_{1,t+1}) \\ \vdots \\ Var_t(y_{l,t+1}) \\ \tilde{\pi}_t \end{bmatrix} = \begin{bmatrix} a_{1,t} \\ \vdots \\ a_{l,t} \\ a_{l+1,t} \\ \vdots \\ a_{2l} \\ 0 \end{bmatrix} + \begin{bmatrix} b_{1,1} & b_{2,1} & 0 \\ \vdots & \vdots & \vdots \\ b_{1,l} & b_{2,l} & 0 \\ 0 & 0 & 0 \\ \vdots & \vdots & \vdots \\ 0 & 0 & 0 \\ 0 & b_{2\pi} & b_{1\pi} \end{bmatrix} \begin{bmatrix} z_{1,t} \\ z_{2,t} \\ z_{1\pi,t} \end{bmatrix} + \begin{bmatrix} \upsilon_{1,t} \\ \vdots \\ \upsilon_{l,t} \\ \upsilon_{l+1,t} \\ \vdots \\ \upsilon_{2l,t} \\ \upsilon_\pi \end{bmatrix}$$

In this case, the transition equations become:

$$(3.46)\quad \begin{bmatrix} z_{1,t+1} \\ z_{2,t+1} \\ z_{1\pi,t+1} \end{bmatrix} = \begin{bmatrix} \varphi_1 & 0 & 0 \\ 0 & \varphi_2 & 0 \\ 0 & 0 & \varphi_{1\pi} \end{bmatrix} \begin{bmatrix} z_{1,t} \\ z_{2,t} \\ z_{1\pi,t} \end{bmatrix} + \begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_{1\pi} \end{bmatrix} \begin{bmatrix} \varepsilon_{1,t+1} \\ \varepsilon_{2,t+1} \\ \varepsilon_{1\pi;t+1} \end{bmatrix}$$

Finally, for the two-country model the observation equation is written as:

$$(3.47)\quad \begin{bmatrix} y_{1,t} \\ \vdots \\ y_{l,t} \\ y^*_{1,t} \\ \vdots \\ y^*_{l,t} \\ Var_t\left(y_{1,t+1}\right) \\ \vdots \\ Var_t\left(y_{l,t+1}\right) \\ Var_t\left(y^*_{1,t+1}\right) \\ \vdots \\ Var_t\left(y^*_{l,t+1}\right) \\ \tilde{\pi}_t \end{bmatrix} = \begin{bmatrix} a_{1,t} \\ \vdots \\ a_{l,t} \\ a^*_{1,t} \\ \vdots \\ a^*_{l,t} \\ a_{l+1,t} \\ \vdots \\ a_{2l} \\ a^*_{l+1,t} \\ \vdots \\ a^*_{2l} \\ 0 \end{bmatrix} + \begin{bmatrix} b_{1,1} & b_{2,1} & 0 \\ \vdots & \vdots & \vdots \\ b_{1,l} & b_{2,l} & 0 \\ 0 & b_{2,1} & b^*_{1,1} \\ \vdots & \vdots & \vdots \\ 0 & b_{2,l} & b^*_{1,l} \\ 0 & 0 & 0 \\ \vdots & \vdots & \vdots \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ \vdots & \vdots & \vdots \\ 0 & 0 & 0 \\ 0 & b_\pi & 0 \end{bmatrix} \begin{bmatrix} z_{1,t} \\ z_{2,t} \\ z^*_{1,t} \end{bmatrix} + \begin{bmatrix} \upsilon_{1,t} \\ \vdots \\ \upsilon_{l,t} \\ \upsilon^*_{1,t} \\ \vdots \\ \upsilon^*_{l,t} \\ \upsilon_{l+1,t} \\ \vdots \\ \upsilon_{2l,t} \\ \upsilon^*_{l+1,t} \\ \vdots \\ \upsilon^*_{2l,t} \\ \upsilon_\pi \end{bmatrix}$$

The estimation departs from assuming that the starting value of the state vector S is obtained from a normal distribution with mean $\bar{Z}_o$ and variance P_0. $\hat{Z}_o$ can be seen as a guess concerning the value of Z using all information available up to and including t = 0. As the residuals are orthogonal to the state variables, $\bar{Z}_o$ cannot be obtained using the data and the model. P_0 is the uncertainty about the prior on the values of the state variables.

Using $\bar{Z}_o$ and P_0 and following (3.16), the optimal estimator for Z_1 will be given by:

(3.48) $\hat{Z}_{1|0} = C + F\hat{Z}_0$

Consequently, the variance matrix of the estimation error of the state vector will correspond to:

(3.49)
$$\begin{aligned}
P_{1|0} &= E\left[\left(Z_1 - \hat{Z}_{1|0}\right)\left(S_1 - \hat{Z}_{1|0}\right)'\right] \\
&= E\left[(C + FZ_0 + Gv_1 - C - FZ_0)(C + FZ_0 + Gv_1 - C - FZ_0)'\right] \\
&= E\left[(Fv_0 + Gv_1)(v_0'F' + v_1'G')\right] \\
&= E(Fv_0v_0'F') + E(Gv_1v_1'G') \\
&= FP_{1|0}F' + GQ_1G'
\end{aligned}$$

Given that $vec(ABC) = (C' \otimes A) \cdot vec(B)$, $P_{1|0}$ may be obtained from:

(3.50)
$$\begin{aligned}
vec\left(P_{1|0}\right) &= vec\left(FP_{1|0}F'\right) + vec(GQ_1G') \\
&= (F \otimes F) \cdot vec\left(P_{1|0}\right) + (G \otimes G) \cdot vec(Q_1) \\
&= \left[\underset{(n^2 \times n^2)}{I} - (F \otimes F)\right]^{-1} \left[(G \otimes G) \cdot vec(Q_1)\right]
\end{aligned}$$

Generalising equations (3.48) and (3.49), we have the following prediction equations:

(3.51) $\hat{Z}_{t|t-1} = C + F\hat{Z}_{t-1}$

(3.52) $P_{t|t-1} = FP_{t|t-1}F' + GQ_tG'$

As w_t is independent from X_t and from all the prior information on y and x (denoted by ζ_{t-1}), we can obtain the forecast of y_t conditional on X_t and ζ_{t-1} directly from (3.38):

(3.53) $E(y_t | X_t, \zeta_{t-1}) = AX_t + H\hat{Z}_{t|t-1}$

Therefore, from (3.38) and (3.53), we have the following expression for the forecasting error:

(3.54) $Y_t - E(Y_t | X_t, \zeta_{t-1}) = (AX_t + HZ_t + w_t) - \left(AX_t + H\hat{Z}_{t|t-1}\right) = H\left(Z_t - \hat{Z}_{t|t-1}\right) + w_t$

Given that w_t is also independent from Z_t and $\hat{Z}_{t|t-1}$ and considering (3.54), the conditional variance-covariance matrix of the estimation error of the observation vector will be:

(3.55)

$$\begin{aligned}E\left\{\left[Y_t - E(Y_t | X_t, \zeta_{t-1})\right]\left[Y_t - E(Y_t | X_t, \zeta_{t-1})\right]'\right\} &= E\left\{\left[H\left(Z_t - \hat{Z}_{t|t-1}\right) + w_t\right]\left[H\left(Z_t - \hat{Z}_{t|t-1}\right) + w_t\right]'\right\} \\ &= HE\left[\left(Z_t - \hat{Z}_{t|t-1}\right)\left(Z_t - \hat{Z}_{t|t-1}\right)'\right]H' + E(w_t w_t') \\ &= HP_{t|t-1}H' + R\end{aligned}$$

In order to characterise the distribution of the observation and state vectors, it is also required to compute the conditional covariance between both forecasting errors. From (3.54) we get:

(3.56)

$$\begin{aligned}E\left\{\left[Y_t - E(Y_t | X_t, \zeta_{t-1})\right]\left[Z_t - E(Z_t | X_t, \zeta_{t-1})\right]'\right\} &= E\left\{\left[H\left(Z_t - \hat{Z}_{t|t-1}\right) + w_t\right]\left[Z_t - \hat{Z}_{t|t-1}\right]'\right\} \\ &= HE\left[\left(Z_t - \hat{Z}_{t|t-1}\right)\left(Z_t - \hat{Z}_{t|t-1}\right)'\right] \\ &= HP_{t|t-1}\end{aligned}$$

Therefore, using (3.53), (3.55) and (3.56), the conditional distribution of the vector (Y_t, Z_t) is:

(3.57) $$\begin{bmatrix} Y_t | X_t, \zeta_{t-1} \\ Z_t | X_t, \zeta_{t-1} \end{bmatrix} \sim N\left(\begin{bmatrix} AX_t + H_{t|t-1} \\ \hat{Z}_{t|t-1} \end{bmatrix}, \begin{bmatrix} HP_{t|t-1}H'+R & HP_{t|t-1} \\ P_{t|t-1}H' & P_{t|t-1} \end{bmatrix}\right)$$

According to a well-known result,[16] if X_1 and X_2 have a joint normal conditional distribution characterised by

(3.58) $$\begin{bmatrix} X_1 \\ X_2 \end{bmatrix} \sim N\left(\begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix}, \begin{bmatrix} \Omega_{11} & \Omega_{12} \\ \Omega_{21} & \Omega_{22} \end{bmatrix}\right)$$

then the distribution of $X_2 | X_1$ is $N(m, \Sigma)$, being

(3.59) $$m = \mu_2 + \Omega_{21}\Omega_{11}^{-1}(X_1 - \mu_1)$$

(3.60) $$\Sigma = \Omega_{22} - \Omega_{21}\Omega_{11}^{-1}\Omega_{12}$$

Equations (3.59) and (3.60) correspond to the optimal forecast of X_2 given X_1 and to the mean square error of this forecast respectively. Consequently, following (3.57), the distribution of Z_t given Y_t, X_t and ζ_{t-1} is $N\left(\hat{Z}_{t|t}, P_{t|t}\right)$, where $\hat{Z}_{t|t}$ and $P_{t|t}$ are respectively the optimal forecast of Z_t given $P_{t|t}$ and the mean square error of this forecast, corresponding (using (3.59) and (3.60)) to the following updating equations of the Kalman Filter:

(3.61) $$\hat{Z}_{t|t} = \hat{Z}_{t|t-1} + P_{t|t-1}H'\left(HP_{t|t-1}H'+R\right)^{-1}\left[Y_t - \left(AX_t + H_{t|t-1}\right)\right]$$

(3.62) $$P_{t|t} = P_{t|t-1} - P_{t|t-1}H'\left(HP_{t|t-1}H'+R\right)^{-1}HP_{t|t-1}$$

[16] See, e.g., Mood *et al.* (1986), pp. 167-168.

After estimating $\hat{S}_{t|t}$ and $P_{t|t}$, we can proceed with the estimation of $\hat{S}_{t+1|t}$ and $P_{t+1|t}$. Considering (3.51) and (3.52), we obtain:

$$\begin{aligned}\hat{Z}_{t+1|t} &= C + F\hat{Z}_{t|t} = C + F\left\{\hat{Z}_{t|t-1} + P_{t|t-1}H'\left(HP_{t|t-1}H' + R\right)^{-1}\left[Y_t - \left(AX_t + H_{t|t-1}\right)\right]\right\} \\ &= C + F\hat{Z}_{t|t-1} + FP_{t|t-1}H'\left(HP_{t|t-1}H' + R\right)^{-1}\left[Y_t - \left(AX_t + H_{t|t-1}\right)\right]\end{aligned} \tag{3.63}$$

$$\begin{aligned}P_{t+1|t} &= FP_{t|t}F' + GQ_tG' \\ &= F\left[P_{t|t-1} - P_{t|t-1}H'\left(HP_{t|t-1}H' + R\right)^{-1}HP_{t|t-1}\right]F' + GQ_tG' \\ &= FP_{t|t-1}F' - FP_{t|t-1}H'\left(HP_{t|t-1}H' + R\right)^{-1}HP_{t|t-1}F' + GQ_tG'\end{aligned} \tag{3.64}$$

The matrix $FP_{t|t-1}H'\left(HP_{t|t-1}H' + R\right)^{-1}$ is usually known as the gain matrix, since it determines the update in $\hat{Z}_{t+1|t}$ due to the estimation error of Y_t. The equation (3.64) is known as a Ricatti equation. Concluding, the Kalman Filter may be applied after specifying starting values for $\hat{Z}_{1|0}$ and $P_{1|0}$ using equations (3.53), (3.55), (3.61), (3.62) and iterating on equations (3.63) and (3.64).

The parameters are estimated using a maximum likelihood procedure. The log-likelihood function corresponds to:

$$\log L(Y_T) = \sum_{t=1}^{T} \log f(Y_t | I_{t-1}) \tag{3.65}$$

being

$$f\left(Y_t | I_{t-1}\right) = (2\pi)^{-1/2} \left|HP_{t|t-1}H' + R\right|^{-1/2} \cdot \exp\left[-\frac{1}{2}\left(Y_t - A - H\hat{Z}_{t|t-1}\right)'\left(HP_{t|t-1}H' + R\right)^{-1}\left(y_t - A - H\hat{Z}_{t|t-1}\right)\right] \quad (3.66)$$

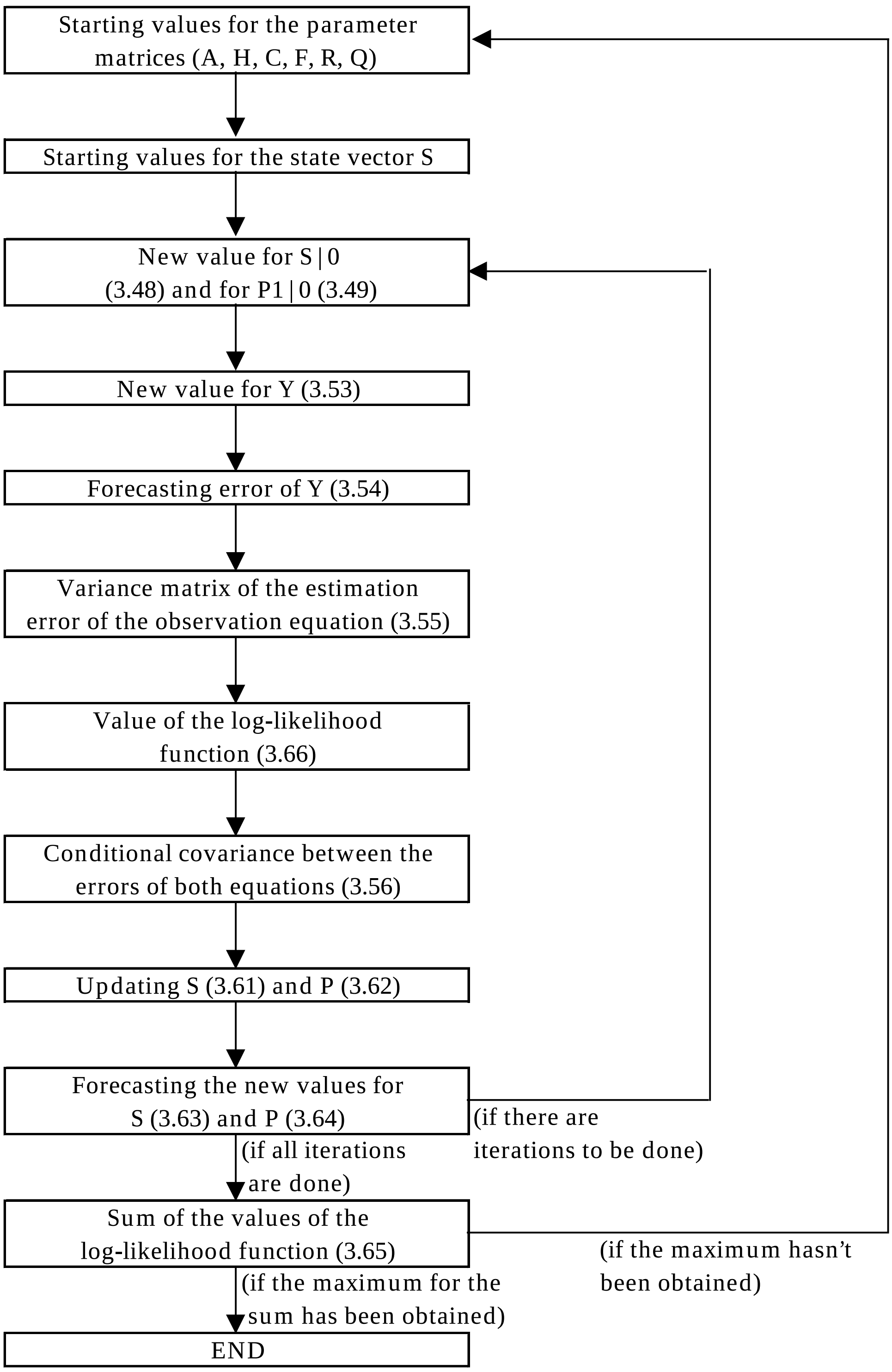
for $t = 1, \ldots, T$.

In order to make inference about the statistical significance of the variables, the standard-errors of the parameters were computed using the method described in Hamilton (1994a), pp. 389.

Concluding, the estimation procedure may be resumed as in figure 3.1:[17]

[17] This figure is based on the flowchart in Kim and Nelson (1999), pp.26.

Figure 3.1 - Kalman filter algorithm with unknown parameters



3.6. Data and estimation results

The data used for the German term structure consist in two databases. The first comprises monthly averages of nine daily spot rates for maturities of 1 and 3 months and 1, 2, 3, 4, 5, 7 and 10 years, between January 1986 and December 1998.[18] These spot rates were estimated from euro-mark short-term interest rates, obtained from Datastream, and par yields of German government bonds, obtained from J.P. Morgan, using the Nelson and Siegel (1987) and Svensson (1994) smoothing techniques. One-period forward rates were calculated for this sample, which allowed the forward regressions to be performed.[19]

The second database for Germany covers a longer period, between September 1972 and December 1998.[20] However this sample includes only spot rates for annual maturities between 1 and 10 years, excluding the 9-year maturity. The inflation rate was computed as the difference to the mean of the yearly inflation rate, obtained from Datastream.[21] The data regarding the French term structure consists in spot rates computed from par yields supplied by J.P. Morgan, for 1, 3, 5 and 10-year maturities.

The Kalman filtering and maximum likelihood estimation was carried out using a Matlab code.[22] Besides the recursive restrictions, the parameters were estimated subject to the usual signal restrictions.

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[18] Therefore, the parameter l in the state-space model is equal to 9 and there are 18 equations in the measurement equation.

[19] The research assistance of Fátima Silva, from the Research Department of the Banco de Portugal, on this issue is acknowledged.

[20] We are grateful to Manfred Kremer, from the Research Department of the Bundesbank, for providing the data.

[21] Given that this database does not include information on the money market rates, the 3-factor model was not estimated.

[22] The matlab codes were written upon codes made available by Mike Wickens and Eli Remolona.

Before the estimation results, it is worthwhile to present the most significant stylised facts concerning the data. The properties of the German yield curve are summarised in Tables 3.2 and 3.3. A number of features are worth noting. Firstly, between 1986 and 1998, the term structure is negatively slopped at the short end, in contrast with the more familiar concave appearance observed for the USA market (see, Backus *et al.* (1997)). Secondly, yields are very persistent, with monthly autocorrelations above 0.98 for all maturities.[23] Thirdly, yields are highly correlated along the curve, but correlation is not equal to one, with lower correlations obtained for the ends of the yield curve. This suggests that non-parallel shifts of the yield curve are important. Therefore, one-factor models seem to be insufficient to explain the German term structure of interest rates. Lastly, as expected, the volatility curve of the yields is downward sloping.

Table 3.2 - Properties of German Government Bond Yields in 1986-1988

Maturity	1	3	12	24	36	48	60	84	120
Mean	5.732	5.715	5.681	5.823	6.020	6.203	6.359	6.587	6.793
St.Dev.	2.267	2.224	2.087	1.882	1.686	1.524	1.398	1.228	1.095
Skew.	0.450	0.449	0.490	0.522	0.519	0.487	0.432	0.282	0.049
Kurt.	-1.258	-1.267	-1.139	-1.051	-0.975	-0.871	-0.737	-0.426	0.001
Autocorrel.	0.991	0.992	0.991	0.990	0.989	0.989	0.988	0.987	0.983

Normality Tests

Maturity	1	3	12	24	36	48	60	84	120
c2(2)	15.5558	15.6717	14.6832	14.2669	13.1898	11.1065	8.3923	3.2515	0.0627
P(X>x)	0.0004	0.0004	0.0006	0.0008	0.0014	0.0039	0.0151	0.1968	0.9691

Correlation matrix

y1	1.000	0.998	0.969	0.936	0.913	0.891	0.867	0.816	0.743
y3		1.000	0.982	0.955	0.934	0.912	0.889	0.839	0.765
y12			1.000	0.993	0.980	0.964	0.944	0.898	0.829
y24				1.000	0.996	0.986	0.972	0.935	0.875
y36					1.000	0.997	0.988	0.961	0.911
y48						1.000	0.997	0.980	0.940
y60							1.000	0.992	0.962
y84								1.000	0.989
y120									1.000

[23] Interest rates are close to being non-stationary.

Table 3.3 - Properties of German Government Bond Yields in 1972-1988

Maturity	12	24	36	48	60	72	84	96	120
Mean	6.348	6.609	6.856	7.051	7.202	7.321	7.414	7.487	7.594
St.Dev.	2.290	2.053	1.891	1.770	1.674	1.598	1.535	1.483	1.401
Skew.	0.545	0.349	0.224	0.151	0.108	0.084	0.075	0.077	0.112
Kurt.	-0.513	-0.671	-0.677	-0.636	-0.575	-0.495	-0.397	-0.281	-0.006
Autocorrel.	0.981	0.985	0.985	0.985	0.985	0.985	0.985	0.984	0.981
Normality Tests									
Maturity	12	24	36	48	60	72	84	96	120
$\chi 2(2)$	19.28	12.44	8.76	6.59	5.01	3.64	2.39	1.37	0.67
P(X>x)	0.000	0.002	0.013	0.037	0.082	0.162	0.302	0.505	0.715
Correlation matrix									
y12	1.000	0.986	0.964	0.942	0.919	0.896	0.874	0.853	0.812
y24		1.000	0.993	0.979	0.962	0.943	0.924	0.906	0.870
y36			1.000	0.996	0.986	0.973	0.958	0.943	0.912
y48				1.000	0.997	0.990	0.980	0.968	0.943
Y60					1.000	0.998	0.992	0.984	0.964
Y72						1.000	0.998	0.994	0.979
Y84							1.000	0.999	0.989
Y96								1.000	0.995
y120									1.000

The first estimations to be performed were those concerning the forward regressions. Charts 3.1 (Simple test) and Chart 3.2 (Forward regressions) show the results of the tests of the expectations theory of the term structure, in line with equations (2.40) and (2.42). In table 3.4, the parameter estimates are exhibited.

The pure expectations theory is easily rejected: as shown in Chart 3.1 and in the table below, average one-period short-term forward rates vary with maturity, which contradicts equation (2.40) with $\Pi_n = 0$.[24] However, as shown in Chart 3.2, forward regressions generate slope coefficients close to one for all maturities, with relatively small standard errors,[25] suggesting that the assumption of constant term premiums might be a reasonable approximation.

[24] In the steady state, it is expected that short-term interest rates remain constant. Thus the one-period forward curve should be flat under the absence of risk premium.
[25] Newey-West standard errors.

Chart 3.1
Simple test of the pure expectations theory of interest rates
Forward rate average curve in 1986-1998

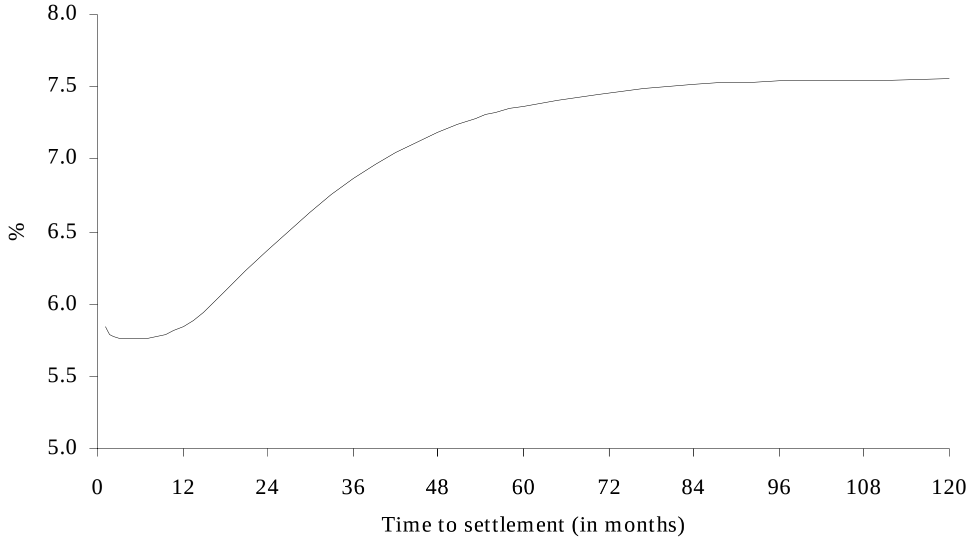


Chart 3.2
Test of the expectations theory of interest rates
Forward regression coefficient in 1986-1998

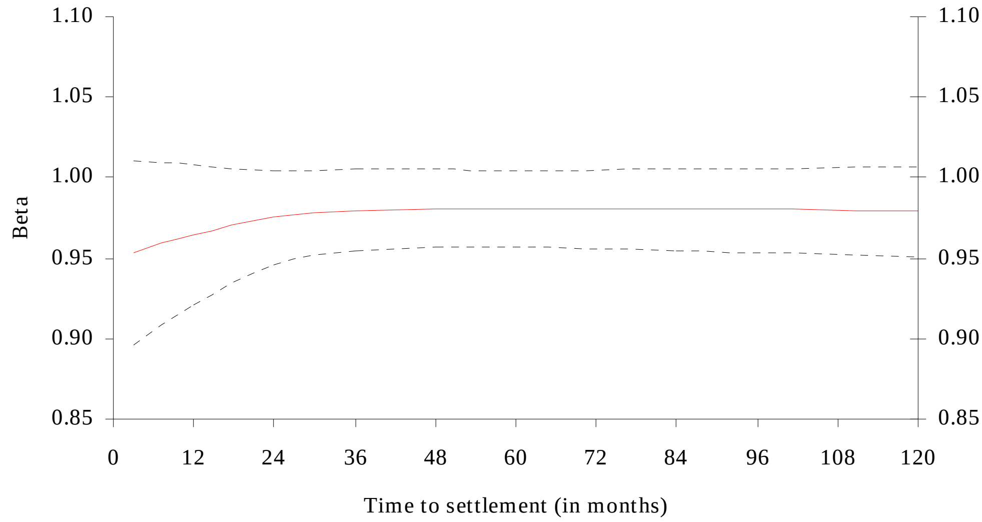


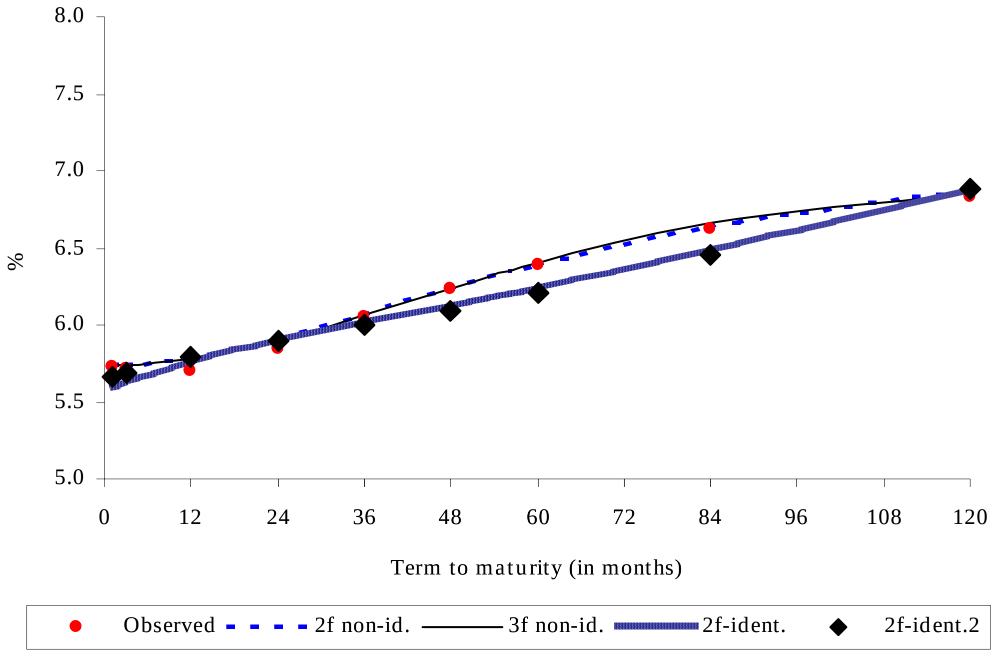
Table 3.4 – Parameter estimates in the forward regressions

m	3	12	24	36	48	60	84	120
β	0.953	0.965	0.975	0.980	0.981	0.981	0.980	0.979
$\sigma(\beta)$	0.029	0.022	0.015	0.013	0.012	0.012	0.013	0.014
α	-0.003	0.005	0.020	0.029	0.034	0.038	0.043	0.046
$\sigma(\alpha)$	0.007	0.019	0.025	0.030	0.033	0.036	0.040	0.043

The results of the term structure estimation are shown in Charts 3.3a and 3.3b (average yield curves), 3.4a and 3.4b (Volatility curves), 3.5a and 3.5b (Term premium curves), 3.6a and 3.6b (One-period forward curves), 3,7a and 3.7b (Expected short-term interest rate curves), 3.8a and 3.8b (Time-series yields), 3.9a and 3.9b (Time-series inflation), 3.10a and 3.10b (Factor loadings) and 3.11a, 3.11b, 3.11c, 3.11d, 3.11e and 3.11f (Time-series factors). The parameters and their standard errors are reported in Tables 3.5 and 3.6.

Chart 3.3a

Average Yield Curves 1986-1998



According to Charts 3.3a, 3.3b, 3.4a and 3.4b, all estimates reproduce very closely both the average yield and the volatility curves, though worse results are obtained for 1986-1998 when the second factor is identified with inflation as in (3.44) or in (3.45).[26] This result must be associated to the link between the behaviour of the second factor and the inflation, which implies a worse fitting for the long-term interest rates.

Chart 3.3b
Average Yield Curve 1972-1998

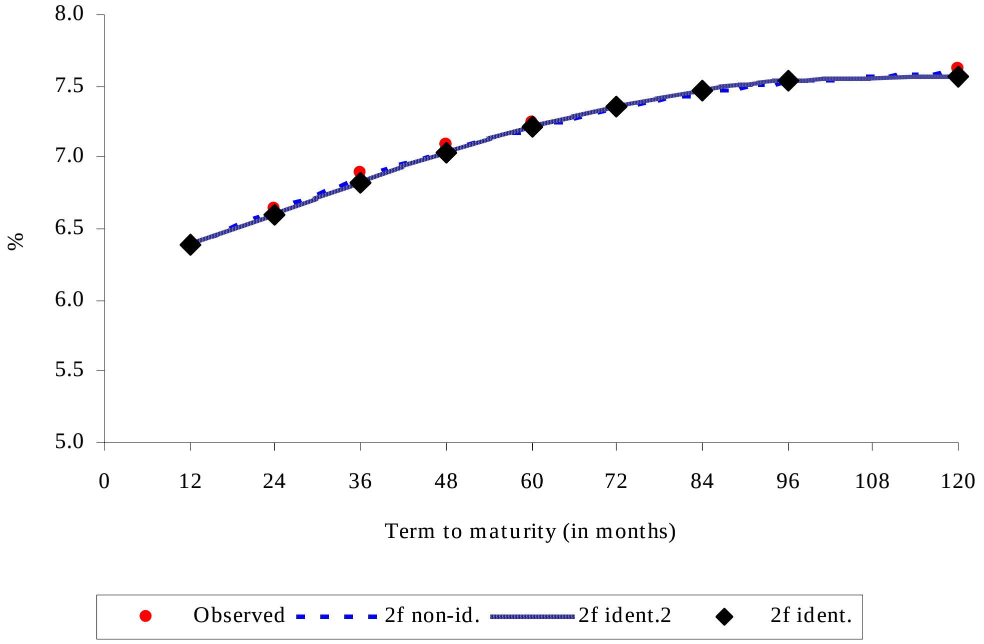
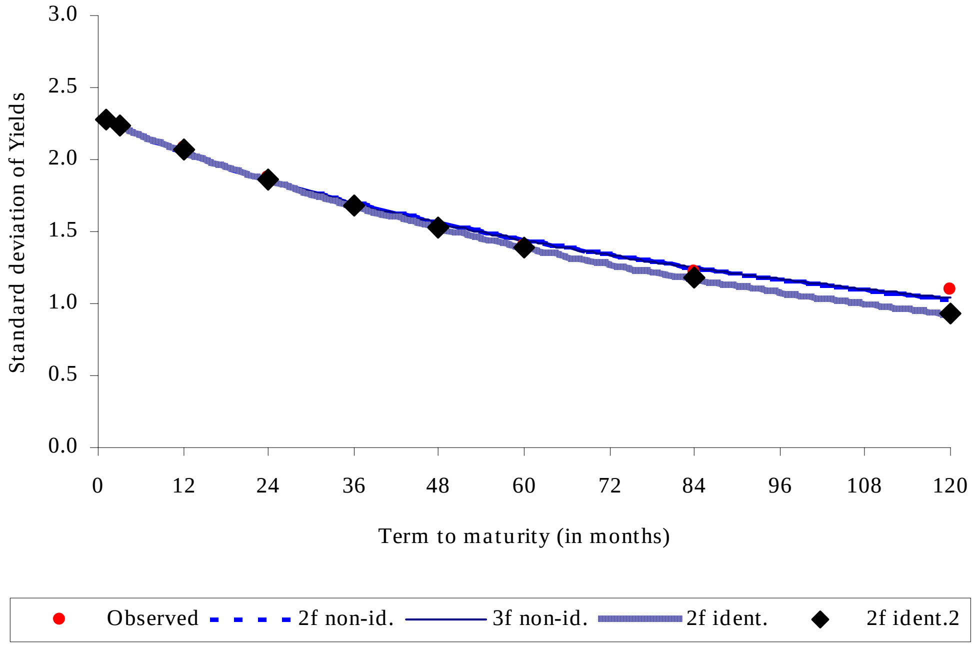
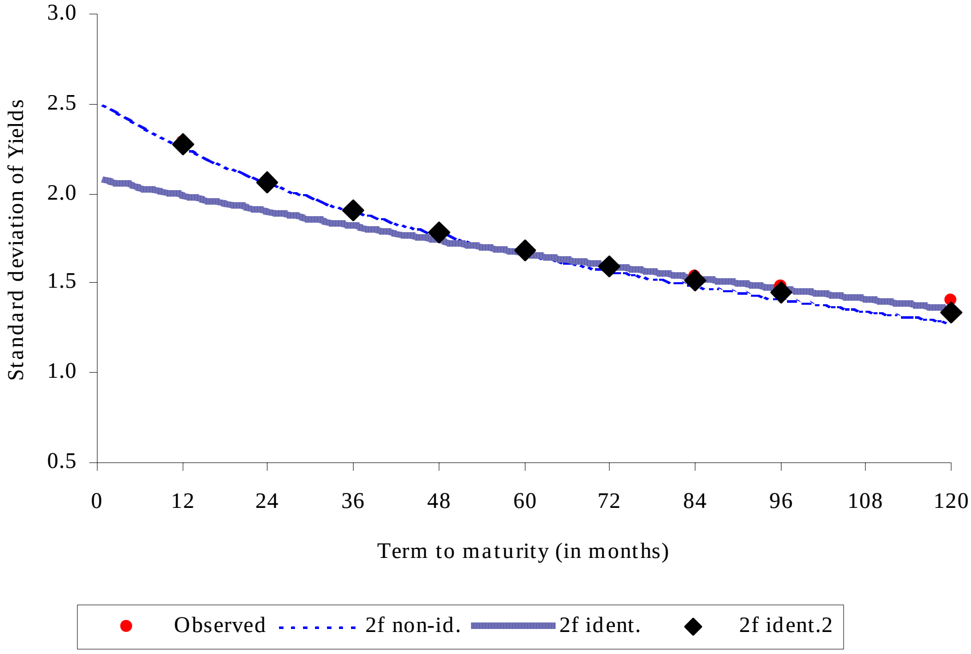


Chart 3. 4a
Volatility Curve 1986-1998



[26] The results with the alternative identified model (in (3.45)) are labelled “2f-ident.2”.

Chart 3.4b
Volatility Curve 1972-1998



Concerning the term premium and the one-period forward, charts 3.5a, 3.5b, 3.6a and 3.6b show again that the results obtained differ in the shorter sample, depending on using identified or non-identified models. The estimates obtained with the identified model in (3.44) point to lower risk premium in the maturities between 2 and 7 years, while the risk premium is higher at both ends of the curve. This model still presents an estimate for the 10-year risk premium around 1.7 in both samples, though it converges monotonically to a positive value, around 4% in maturities higher than 20 years.[27] In the non-identified models, the term premium rises rapidly up to 1.5 in the 5-year maturity, converging to around 1.6 and 1.25 in the 10-year maturity, respectively using the two- and the three-factor model.[28]

---

[27] Notice that the identified model in (3.45) provides lower estimates for the risk premium as the market price of risk of the first factor is higher, giving a lower contribution for the risk premium.

[28] The term premium estimate for the 10-year maturity using the two-factor non-identified model is line with other estimates obtained for different term structures. See, e.g., de Jong (1997), where it is presented an estimate of 1.65 for the 10-year term premium in the US term structure.

Chart 3.5a
Term Premium Curve 1986-1998

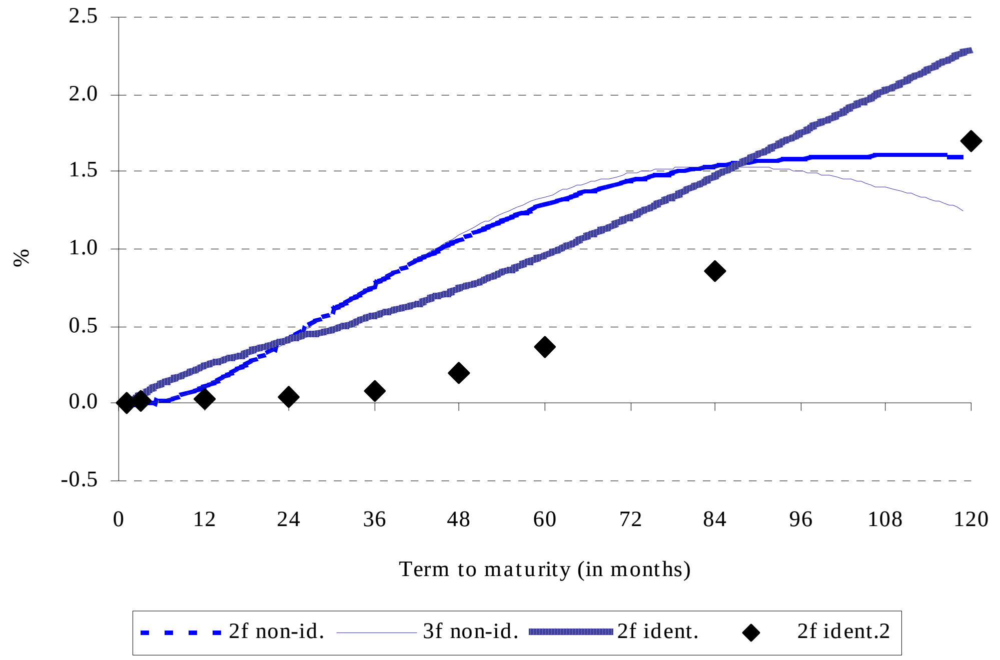


Chart 3.5b
Term Premium Curve 1972-1998

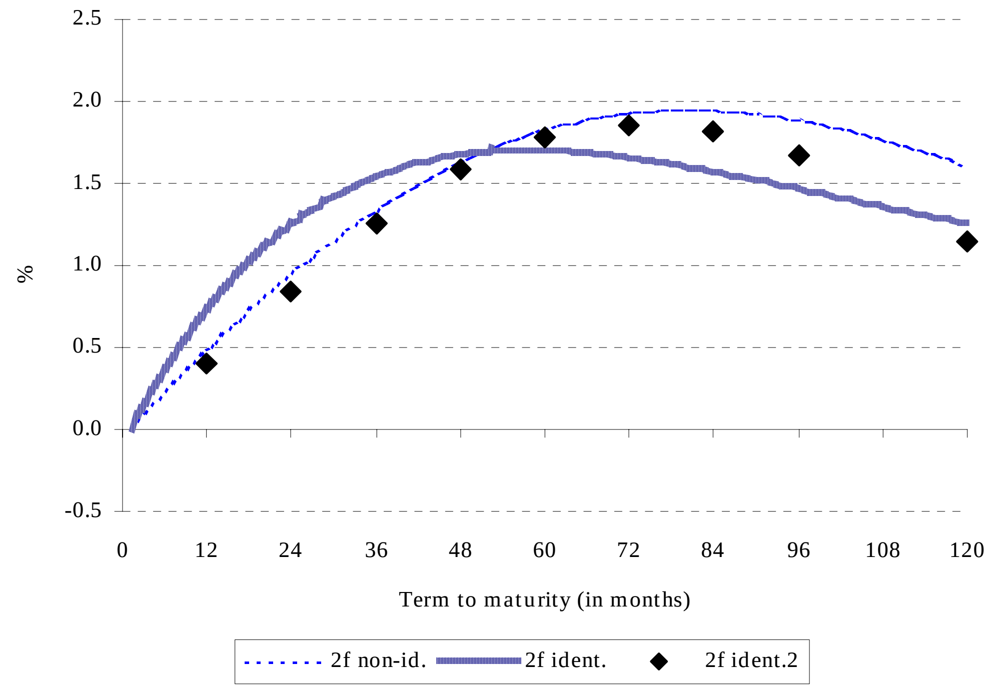


Chart 3.6a
Average Forward Curve 1986-1998

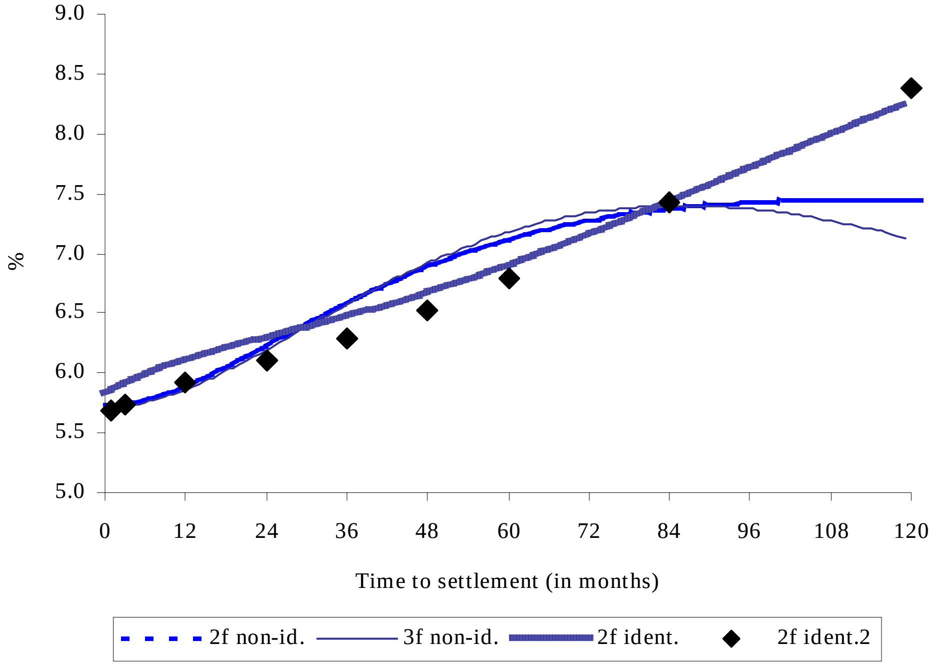
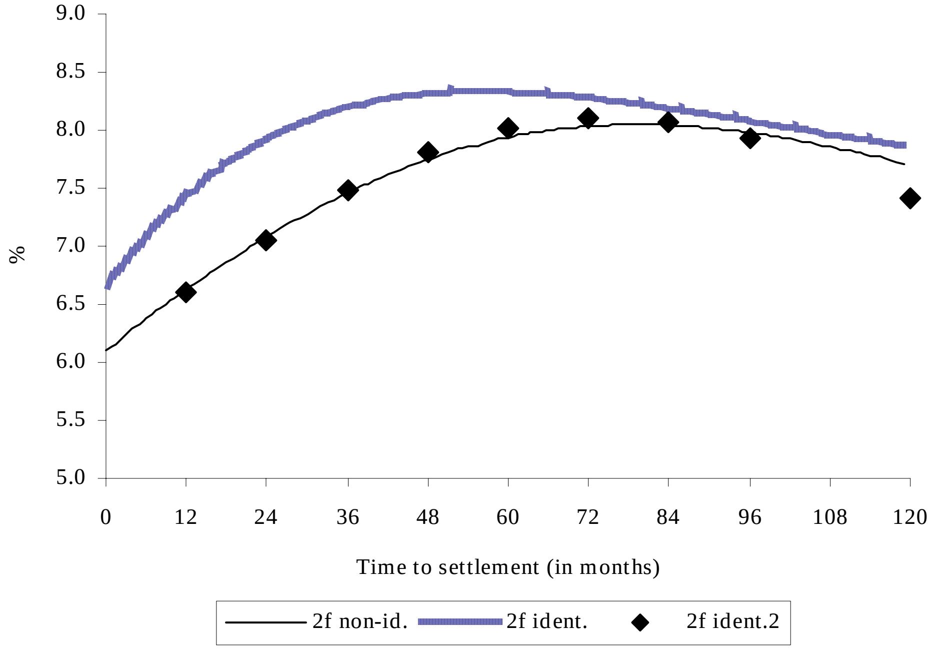


Chart 3.6b
Average Forward Curve 1972-1998



Accordingly, the expected short-term interest rates exhibit differences between the identified and the non-identified models, as well as between the samples. The most remarkable feature is the positive slope in the identified models and using the shorter sample, namely with the model in (3.45). In the longer sample, the short-term interest rate is roughly constant in all models, as the shape of the one-period forward and the term premium curves are similar.

Chart 3.7a
Average Expected Short-term Interest Rate 1986-1998

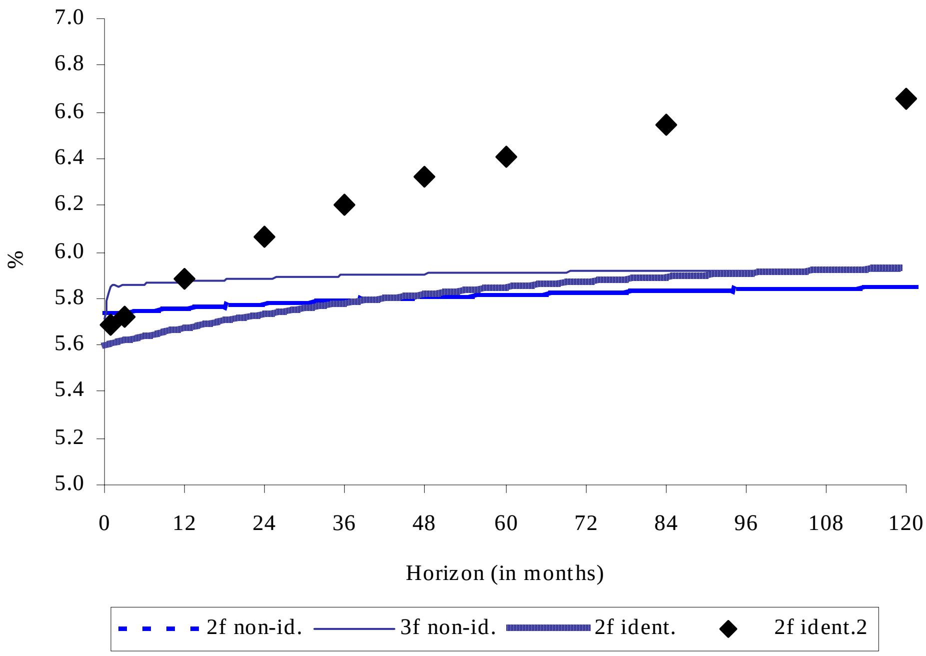
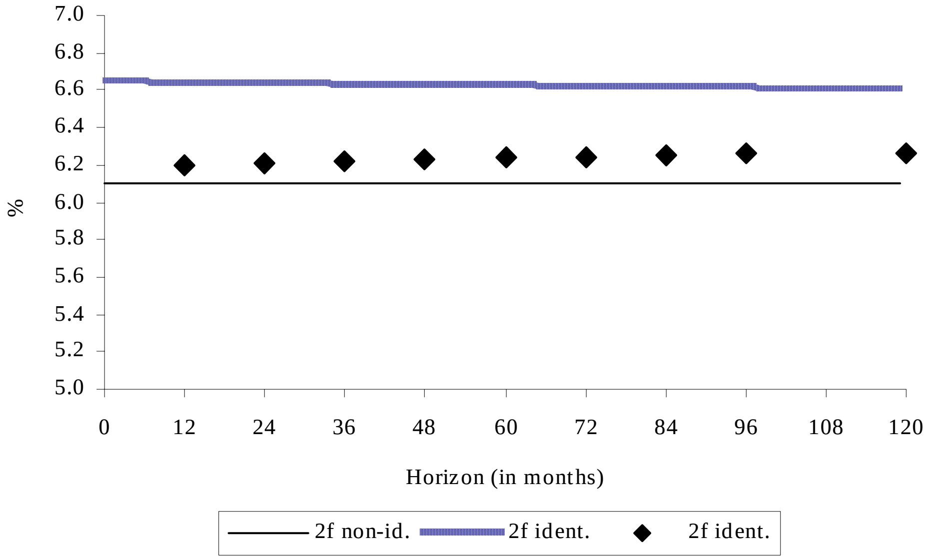


Chart 3.7b
Average Expected Short-term Interest Rate 1972-1998



Regarding the time-series results, charts 3.8a and 3.8b show the observed and the estimated yields for several maturities considered, while in charts 3.9a and 3.9b the observed and the estimated inflation (according to the models in (3.44) and (3.45)). It can be concluded that all estimates reproduce very closely both the yields across the maturity spectrum and the inflation.[29] However, the fit for the yields is poorer in two sub-periods. The first one concerns to Spring 1994, where predicted short-term rates are above actual rates and predicted long term rates are below actual rates.[30] The second episode has occurred in 1998, when long bond yields fell as a consequence of the Russian and Asian crisis, whilst short-rates remained stable. Therefore, the estimated long-term interest rates remained higher than the actual rates.

---

[29] The quality of the fit for the yields contrasts sharply with the results found for the USA by Gong and Remolona (1997c). In fact, they concluded that at least two different two-factor heteroskedastic models are needed to model the whole USA term structures of yields and volatilities: one to fit the medium/short-end of the curves and another to fit the medium/long-terms. The lowest time-series correlation coefficients between the observed and the estimated figures is 0.87, for the 10-year maturity, and the cross-section correlation coefficients are, in most days of the sample, above 0.9.
[30] The surprising behaviour of bond yields during 1994 is discussed in detail in Campbell (1995) with reference to the US market.

The identified model in (3.45) provides more accurate estimates for the inflation, as it uses a specific factor for this variable, instead of the inflation being explained only by the second factor of the term structure.

Chart 3.8a - Time-series yield estimation results - 1986-1998

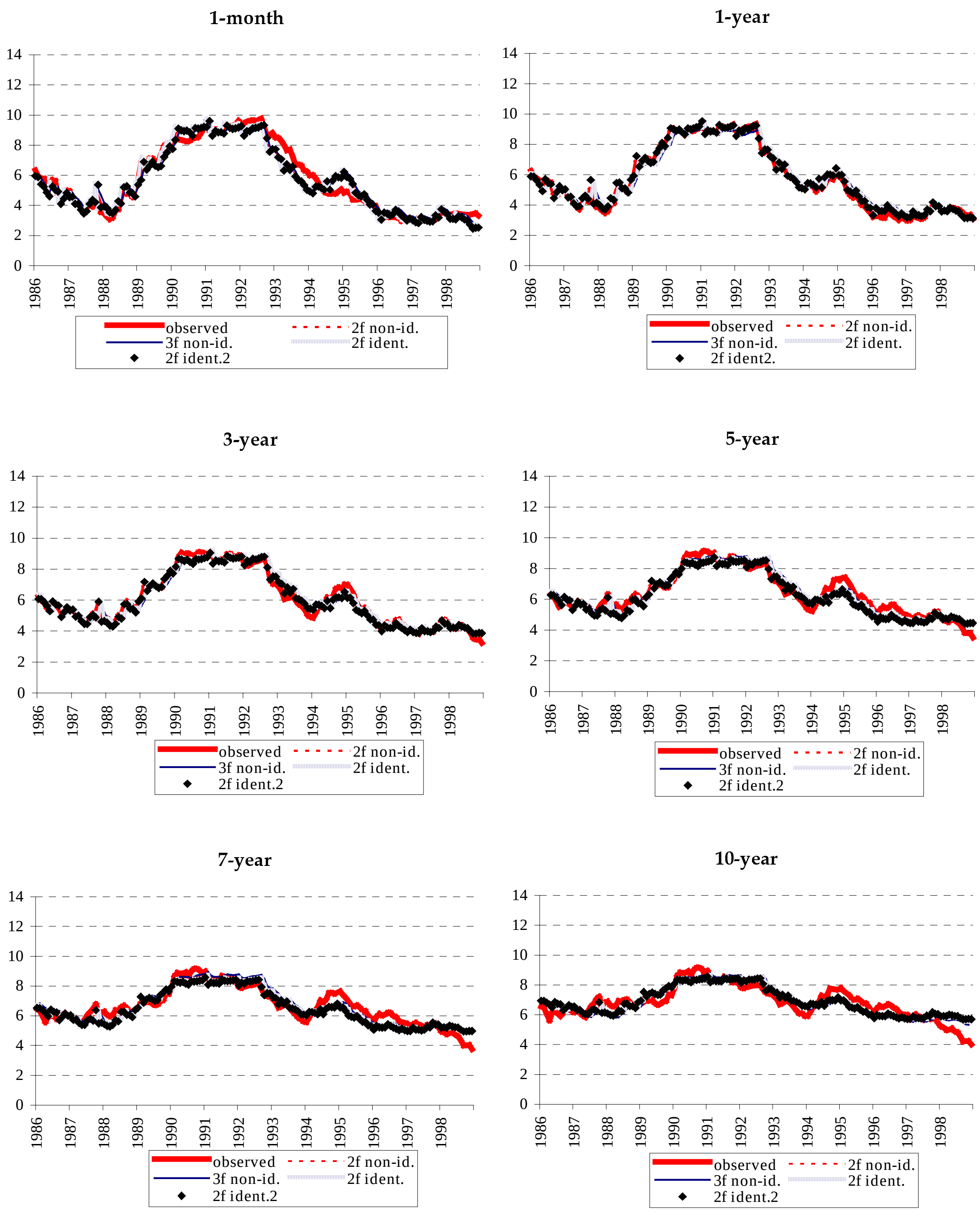


Chart 3.8b - Time-series yield estimation results - 1972-1998

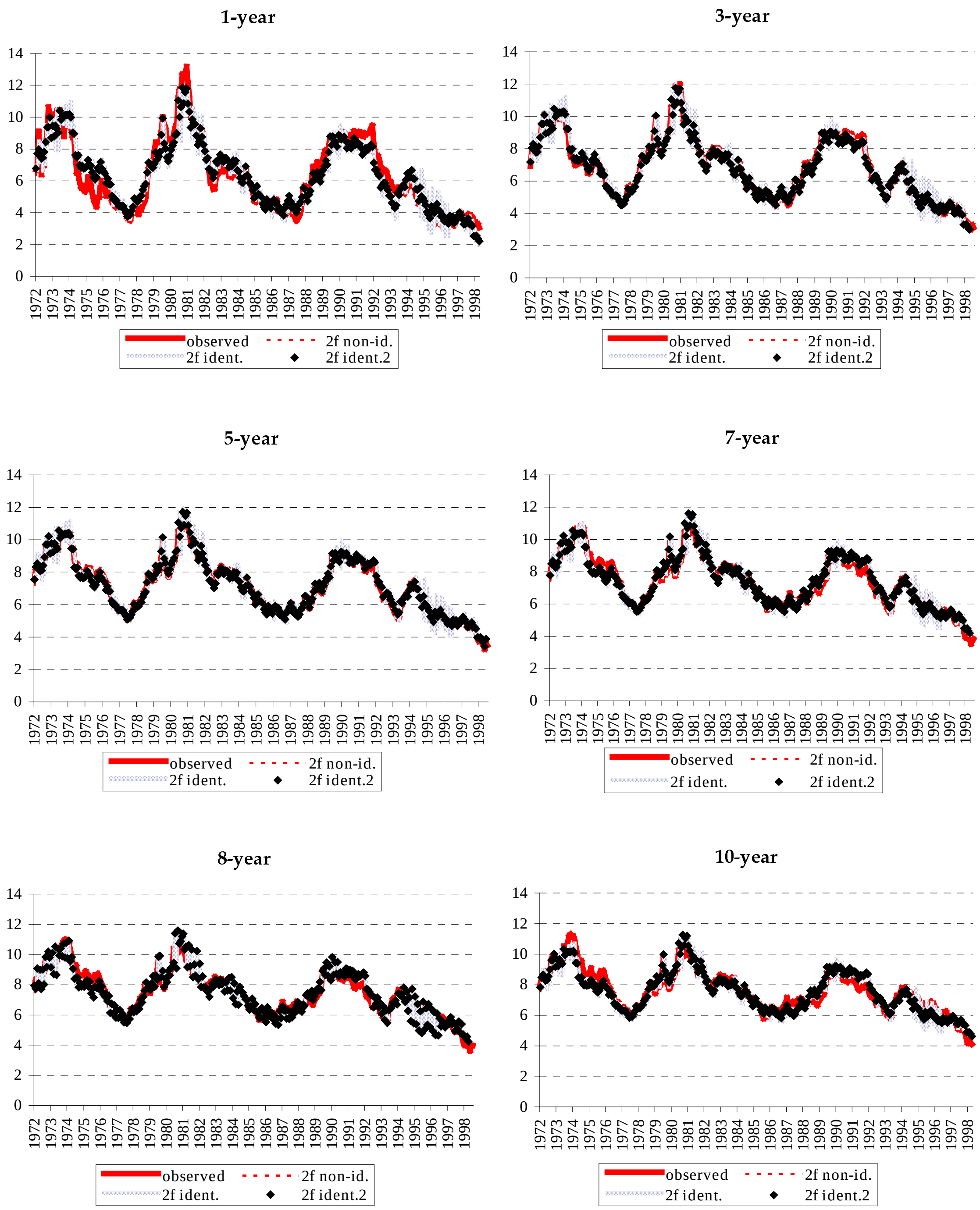


Chart 3.9a
Observed and Estimated Yearly Inflation 1986-1998

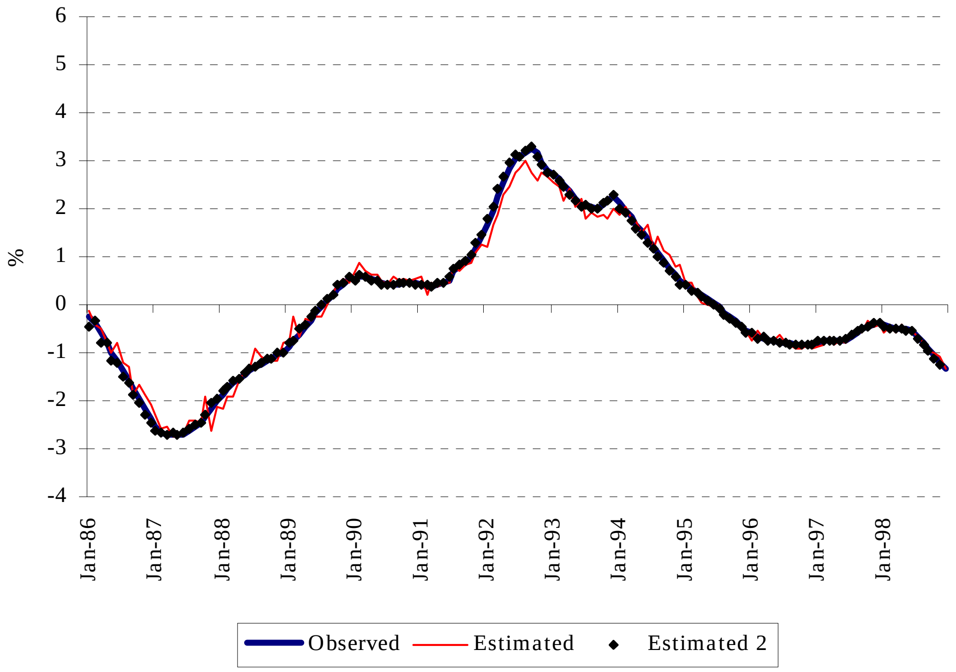


Chart 3.9b
Observed and Estimated Yearly Inflation 1972-1998

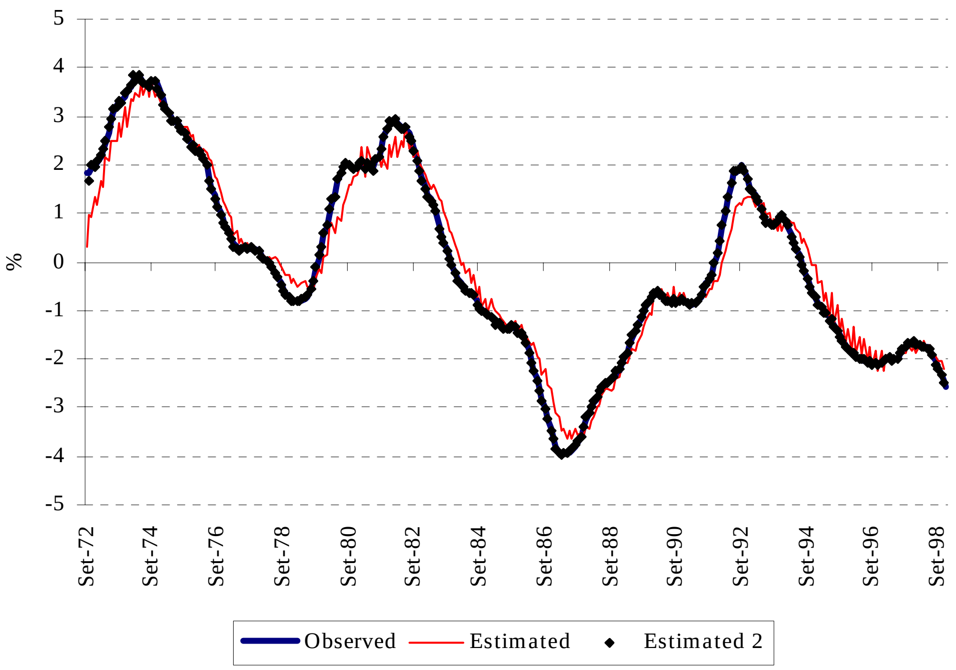


Table 3.5 - Parameter estimates

1986-1998							
	δ	σ_1	φ_1	$\lambda_1\sigma_1$	σ_2	φ_2	$\lambda_2\sigma_2$
2f non-id.	0.00489	0.00078	0.95076	0.22192	0.00173	0.98610	-0.09978
3f non-id.	0.00494	0.00114	0.97101	1.13983	0.00076	0.97543	-1.71811
2f ident.	0.00498	0.00156	0.98163	3.09798	0.00109	0.98216	-4.44754
2f ident.2	0.00566	0.00154	0.98169	3.20068	0.00111	0.98225	-4.45554
1972-1998							
2f non-id.	0.00508	0.00127	0.96283	0.10340	0.00165	0.99145	-0.10560
2f ident.	0.00547	0.00159	0.99225	2.30797	0.00070	0.99259	-5.29858
2f ident.2	0.00526	0.00141	0.95572	0.09198	0.00160	0.99298	-0.09869

	$\sigma_{1\pi}$	$\varphi_{1\pi}$	$\lambda_{1\pi}\sigma_{1\pi}$	$\lambda_{2\pi}\sigma_{1\pi}$	$B_{1\pi}$	$B_{2\pi}$	σ_3	φ_3	$\lambda_3\sigma_3$
1986-1998									
2f non-id.	-	-	-	-	-	-	-	-	-
3f non-id.	-	-	-	-	-	-	0.001317	0.989882	0.006502
2f ident.	-	-	-	-	-	-	-	-	-
2f ident.2	6.89340	0.99773	1.72529	1.55172	0.00004	-0.05438	-	-	-
1972-1998									
2f non-id.	-	-	-	-	-	-	-	-	-
2f ident.	-	-	-	-	-	-	-	-	-
2f ident.2	0.00931	0.99748	0.08405	-2.65631	0.25683	0.06685	-	-	-

Focusing on the parameter estimates, the factors are very persistent, as the estimated φ's are close to one and exhibit low volatility.[31] It is the second factor that contributes positively to the term premium ($\sigma_2\lambda_2$) and exhibits higher persistency. The market price of risk of this factor becomes higher when it is identified with inflation. Conversely, its volatility decreases. All variables are statistically significant, as the

[31] Nevertheless, given that the standard-deviations are low, the unit root hypothesis is rejected and, consequently, the factors are stationary. We acknowledge the comments of Jerome Henry on this issue.

parameters evidence very low standard-deviations. This result confirms the general assertion about the high sensitivity of the results to the parameter estimates.[32]

Table 3.6 - Standard deviation estimates

	δ	σ_1	φ_1	$\lambda_1\sigma_1$	σ_2	φ_2	$\lambda_2\sigma_2$
1986-1998							
2f non-id.	0.00009	0.00001	0.00142	0.00648	0.00003	0.00100	0.00084
2f ident.	0.00001	0.00001	0.00000	0.00000	0.00071	0.00100	0.00005
2f ident.2	0.00001	0.00001	0.00000	0.00000	0.00071	0.00100	0.00005
1972-1998							
2f non-id.	0.00001	0.00001	0.00000	0.00000	0.00249	0.00084	0.00013
2f ident.	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000
2f ident.2	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000

	$\sigma_{1\pi}$	$\varphi_{1\pi}$	$\lambda_{1\pi}\sigma_{1\pi}$	$\lambda_{2\pi}\sigma_{1\pi}$	$B_{1\pi}$	$B_{2\pi}$	σ_3	φ_3	$\lambda_3\sigma_3$
1986-1998									
2f non-id.	-	-	-	-	-	-	-	-	-
2f ident.	-	-	-	-	-	-	-	-	-
2f ident.2	0.12343	0.00154	0.00653	0.00000	0.00000	0.00004	-	-	-
1972-1998									
2f non-id.	-	-	-	-	-	-	-	-	-
2f ident.	-	-	-	-	-	-	-	-	-
2f ident.2	0.00000	0.00000	0.00000	0.00000	0.00864	0.00025	-	-	-

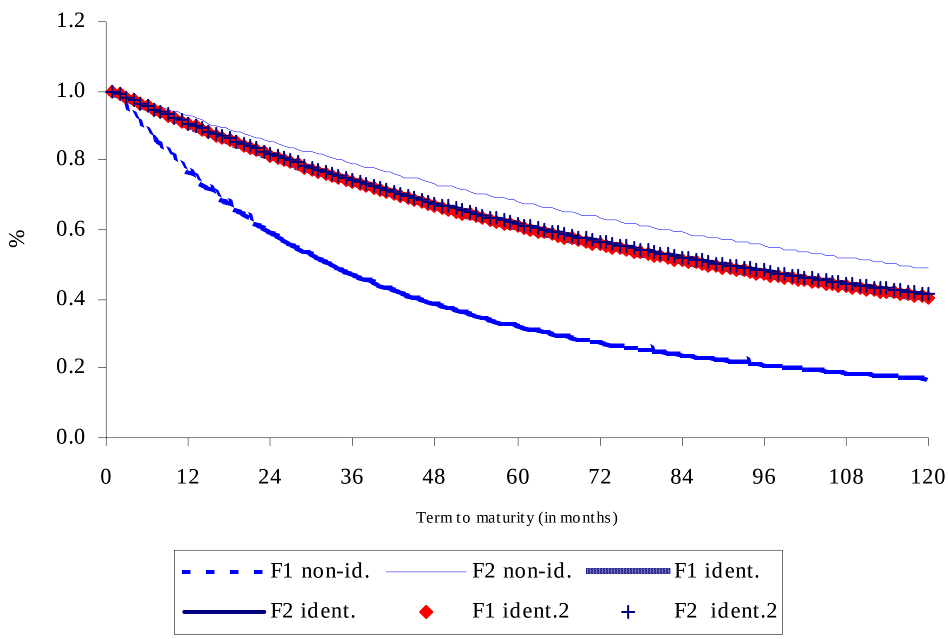
As it was seen in the previous charts, the results obtained with the 3-factor model and using the shorter sample exhibit some differences to those obtained with the two-factor models. In order to opt between the two- and the three-factor model, a decision must be made based on the estimation accuracy of the models. Consequently, a hypothesis test is performed, considering that both models are nested and the two-factor model corresponds to the three-factor version imposing $\varphi_3 = \sigma_3 = 0$.

[32] Very low standard-deviations for the parameters have been usually obtained in former Kalman filter estimates of term structure models, as in Babbs and Nowman (1998), Gong and Remolona (1997a), Remolona *et al.* (1998) and Geyer and Pichler (1996).

This test consists in computing the usual likelihood ratio test $l = -2 \cdot (\ln v - \ln v^*) \sim \chi^2(q)$, being v and v* the sum of the likelihood functions respectively of the two- and the three-factor models and q the number of restrictions (in this case two restrictions are imposed). The values for ln v and ln v* were –306.6472 and –304.5599, implying that $l = 4.1746$. As $\chi^2(2)_{0.95} = 5.991$, the null hypothesis is not rejected and the two-factor model is chosen. This conclusion is also suggested by the results already presented, due to the fact that the three-factor model does not seem to increase significantly the fitting quality. Consequently, the factor analysis and the identification is focused on the two-factor models.

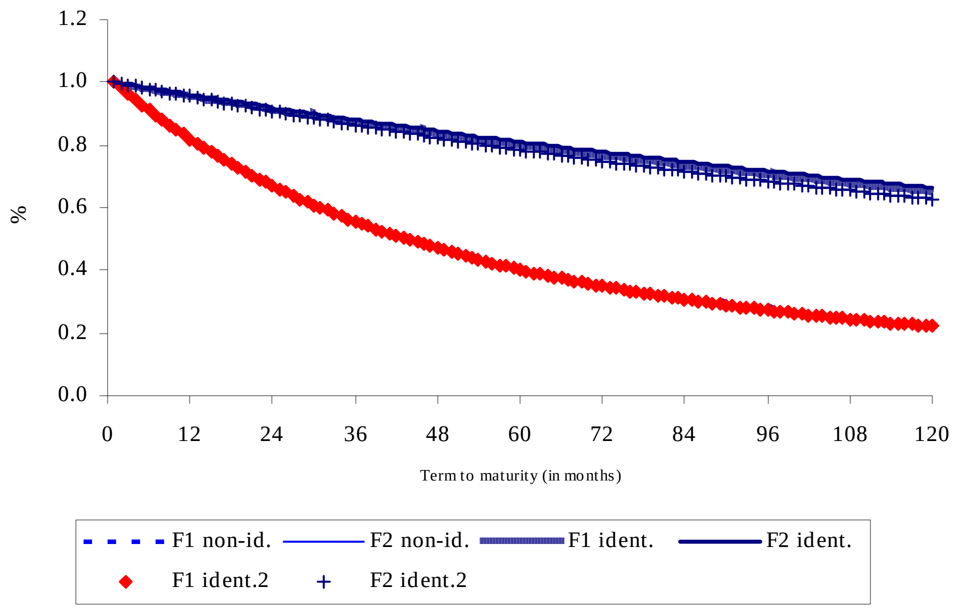
In what concerns to the factor loadings (Charts 3.10a and 3.10b), the first factor – less persistent and volatile - is relatively more important for the short end of the curve, while the second assumes that role in the long end of the curve.[33]

Chart 3.10a
Factor loadings in the two-factor models 1986-1998



[33] Excepting the identified 2-factor models for the shorter sample.

Chart 3.10b
Factor loadings in the two-factor models 1972-1998



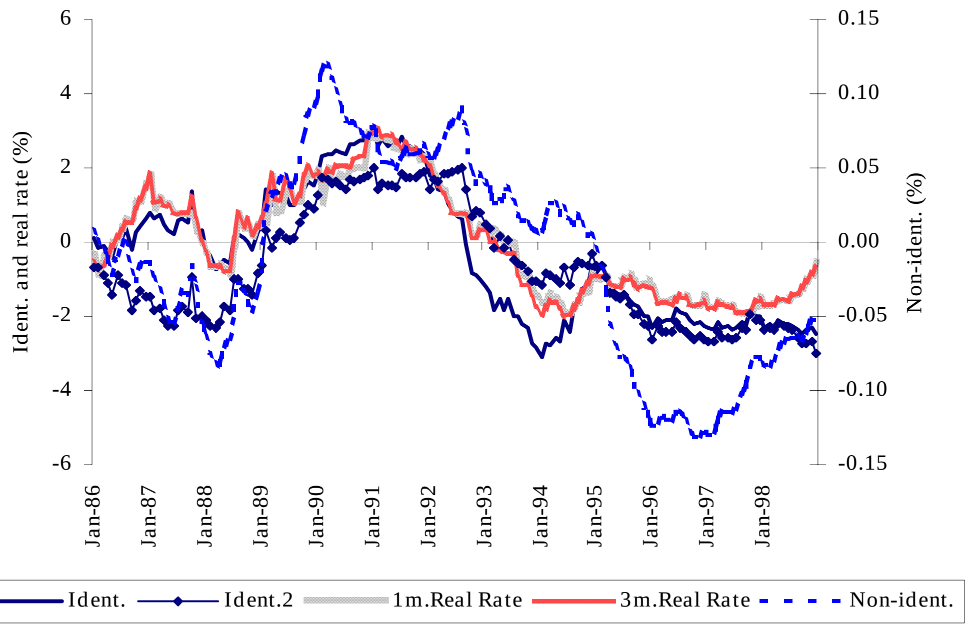
The results already presented illustrate two ways of assessing the identification problem. A third one, as previously referred, is to analyse the correlation between the factors and the variables with which they are supposed to be related – the real interest rate and the expected inflation - including the analysis of the leading indicator properties of the second factor concerning the inflation rate.

The comparison between the unobservable factors, on one hand, and a proxy for the real interest rate and the inflation rate, on the other hand, shows (in Charts 3.11a, 3.11b, 3.11c, and 3.11d) that the correlation between the factors and the corresponding economic variables is high. In fact, using *ex-post* real interest rates as proxies for the *ex-ante* real rates, the correlation coefficients between one-month and three-month real rates, on one side, and the first factor, on the other side, are around

0.75 and 0.6 respectively in the shorter and in the longer sample.[34] Using the identified models, that correlation coefficient decreases to 0.6 and 0.7 in the shorter sample, respectively using the models in (3.44) and in (3.45), while in the larger sample it increases to 0.75 using the model in (3.44) and slightly decreases to 0.56 with the model in (3.45).

The correlation coefficients between the second factor and the inflation are close to 0.6 and 0.7 in the same samples. When the identified model in (3.44) is used, the second factor basically reproduces the inflation behaviour in the shorter sample, with the correlation coefficient achieving figures above 0.98 in both samples. As the model in (3.45) includes a specific factor the inflation, the correlation between the second factor and that variable decreases to around 0.5 in the shorter sample and remains close to 0.7 in the larger.

Chart 3.11a
Time-series evolution of the 1st. factor 1986-1998



[34] This finding contrasts with the results in Gerlach (1995) for the Germany, between January 1967 and January 1995. However, it is line with the conclusion in Mishkin (1990b) that U.S. short rates have information content regarding real interest rates.

Chart 3.11b
Time-series evolution of the 1st. factor 1972-1998

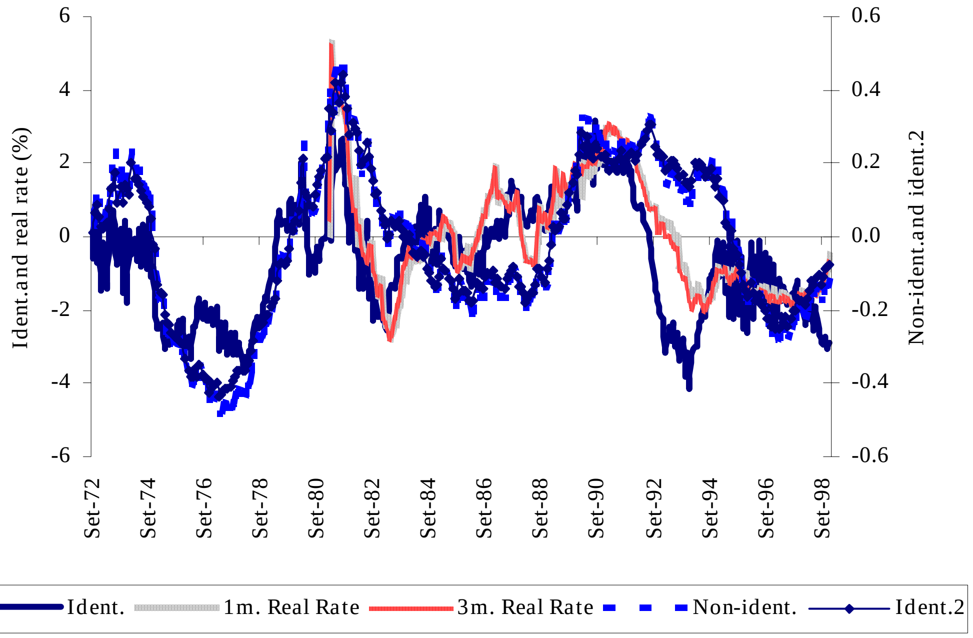


Chart 3.11c
Time-series evolution of the 2nd. factor 1986-1998

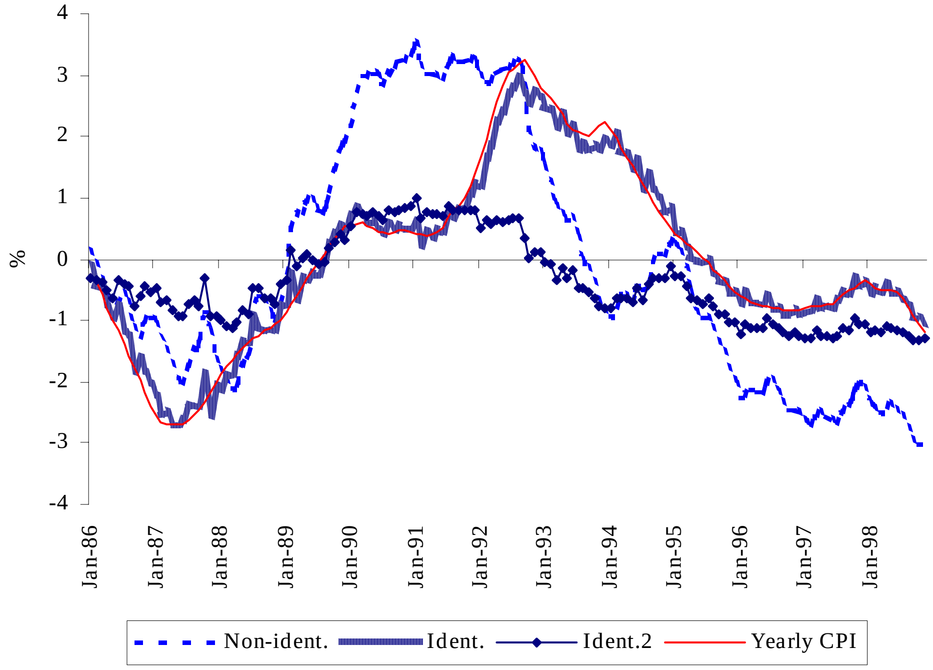
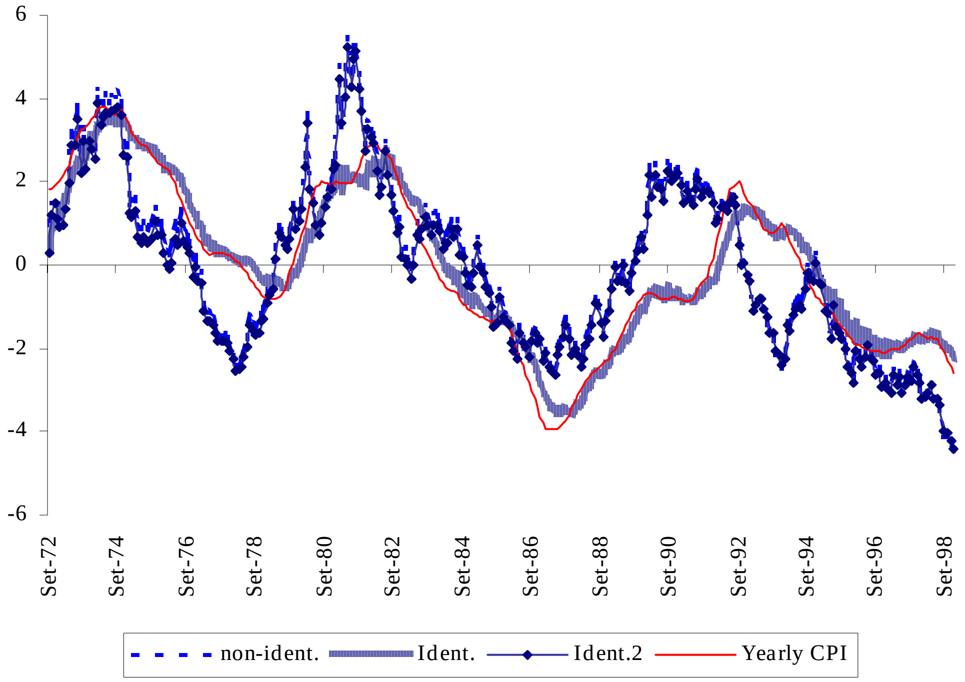


Chart 3.11d
Time-series evolution of the 2nd. factor 1972-1998



In charts 3.11e and 3.11f, the behaviour of the specific inflation factor vis-à-vis the observed inflation is presented. This factor in both samples is highly correlated to the inflation, which implies that the correlation between the second factor and the inflation rate becomes lower.

One of the most striking results obtained is that the second factor appears to be a leading indicator of inflation, according to charts 3.11c and 3.11d. This is in line with the pioneer findings of Fama (1975) regarding the forecasting ability of the term structure on future inflation. It is also consistent with the idea that inflation expectations drive long-term interest rates, given that the second factor is relatively more important for the dynamics of interest rates at the long end of the curve, as previously stated.

If z_{2t} is a leading indicator for π_t, then the highest (positive) correlation should occur between lead values of π_t and z_{2t}. Table 3.7 shows cross-correlation between π_t and z_{2t} in the larger sample. The shaded figures are the correlation coefficients between z_{2t} and lags of π_t. The first figure in each row is k and the next corresponds to the correlation between π_{t-k} and z_{2t} (negative k means lead). The next to the right is the correlation between π_{t-k-1} and z_{2t}, etc.

According to table 3.7, the highest correlation (in bold) is at the eighth lead of inflation. Additionally, that correlation increases with leads of inflation up to eight and steadily declines for lags of inflation.

Chart 3.11e
Time-series evolution of the inflation factor 1986-1998

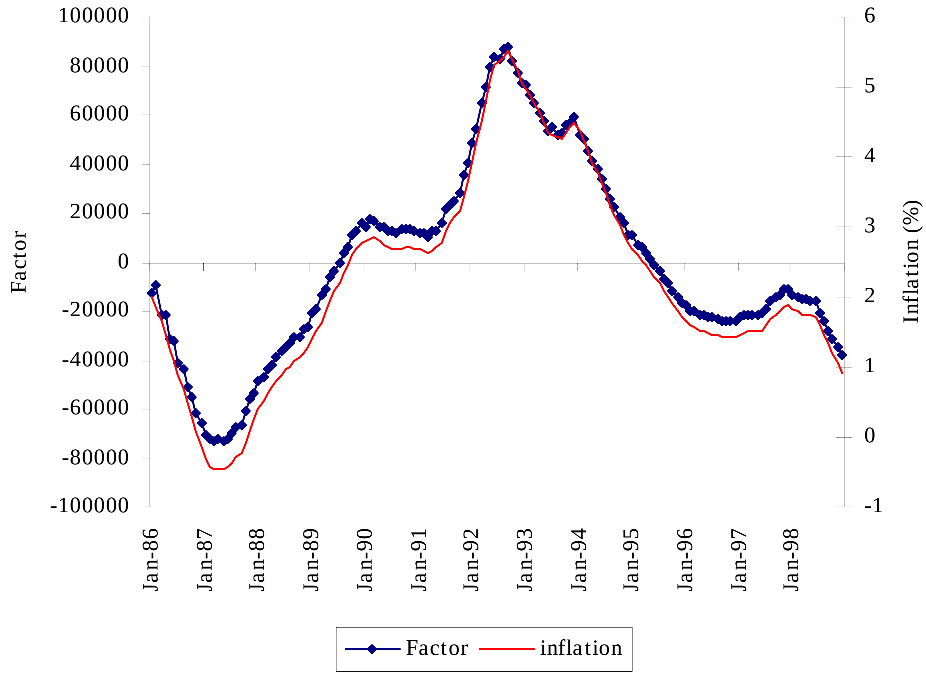


Chart 3.11f
Time-series evolution of the inflation factor 1972-1998

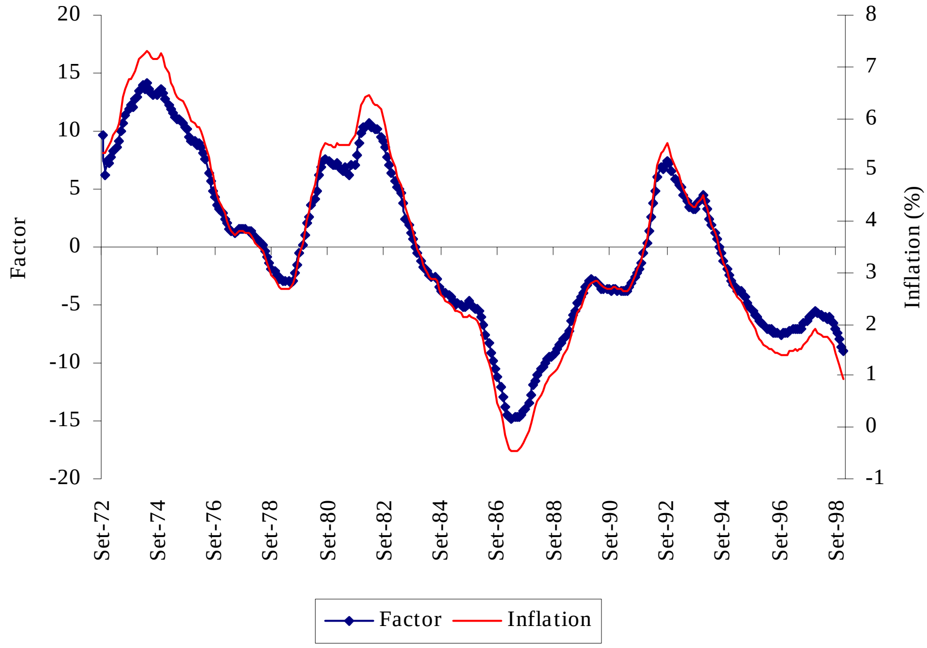


Table 3.8 presents the Granger causality test and Chart 3.12 the impulse-response functions. The results strongly support the conjecture that z_{2t} has leading indicator properties for inflation. At the 5% level of confidence one can reject that z_{2t} does not Granger cause inflation. This is confirmed by the impulse-response analysis.

A positive shock to the innovation process of z_{2t} is followed by a statistically significant increase in inflation, as it is illustrated by the first panel in chart 3.12, where the confidence interval of the impulse-response function does not include zero. However a positive shock to the innovation process of inflation does not seem to be followed by a statistically significant increase in z_{2t}, according to the last panel in chart 3.12, where the confidence interval includes zero. These results suggest that an innovation to the inflation process does not contain "news" for the process of expectation formation. This is in line with the forward-looking interpretation of shocks to z_{2t} as reflecting "news" about the future course of inflation.

Table 3.7 – Cross-correlations of series π_t and z_{2t} (Monthly data from 1972:09 to 1998:12)

-25:	0.3974759	0.4274928	0.4570631	0.4861307	0.5129488	0.5394065
-19:	0.5639012	0.5882278	0.6106181	0.6310282	0.6501370	0.6670820
-13:	0.6829368	0.6979626	0.7119833	0.7235875	0.7324913	0.7385423
-7:	0.7432024	0.7453345	0.7457082	**0.7458129**	0.7443789	0.7411556
-1:	0.7368274	0.7311008	0.7144727	0.6954511	0.6761649	0.6561413
5:	0.6360691	0.6159540	0.5949472	0.5726003	0.5478967	0.5229193
11:	0.4960254	0.4706518	0.4427914	0.4156929	0.3870690	0.3590592
17:	0.3311699	0.3011714	0.2720773	0.2425888	0.2126804	0.1829918
23:	0.1527293	0.1226024	0.0927202	0.0636053	0.0362558	0.0085952
29:	-0.0167756	-0.0421135	–0.0673895	–0.0911283	-0.1139021	-0.1356261

Note: Correlation between π_{t-k} and z_{2t} (i.e. negative k means lead).

Table 3.8 - Granger causality tests

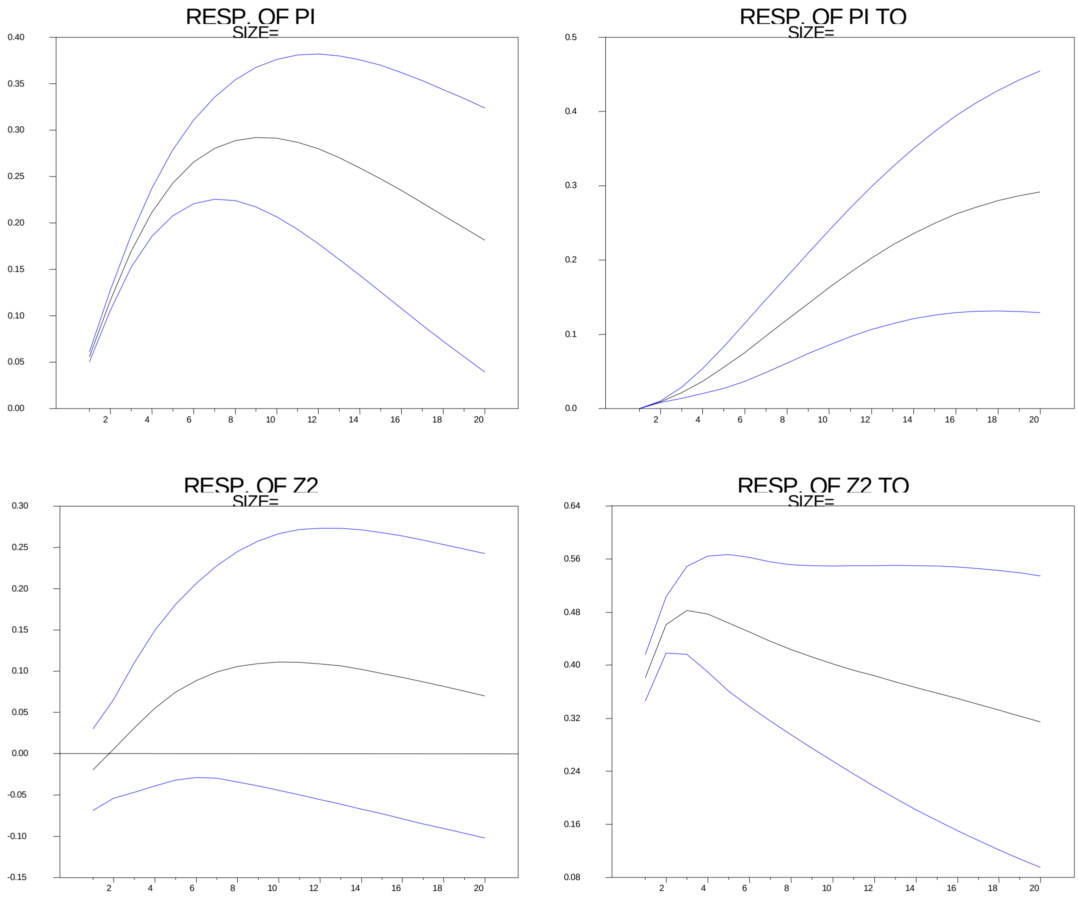
Variable	F-Statistic	Significance
Inflation does not Granger cause Z_2		
Variable	F-Statistic	Significance
π_t	5.8	0.03 *
Z_2 does not Granger cause inflation		
Variable	F-Statistic	Significance
Z_2	6.12	0.03 *

* Means rejection at 5% level

Overall, considering the results for the factor loadings and the relationship between the latent factors and the economic variables, the short-term interest rates are mostly driven by the real interest rates. In parallel, the inflation expectations exert the most important influence on the long-term rates, as it is usually assumed. Similar results concerning the information content of the German term structure, namely in the longer terms, regarding future changes in inflation rate were obtained in several previous papers, namely Schich (1996), Gerlach (1995) and Mishkin (1991), using different samples and testing procedures.[35]

[35] Mishkin (1991) and Jorion and Mishkin (1991) results on Germany are contradictory as, according to Mishkin (1991), the short-end of the term structure does not contain information on future inflation for all OECD countries studied, except for France, United Kingdom and Germany. Conversely, Jorion and

Chart 3.12 - Impulse-response functions and two standard error bands



Fleming and Remolona (1999) also found that macroeconomic announcements of the CPI and the PPI affect mostly the long-end of the term structure of interest rates, using high frequency data. Nevertheless, it is also important to have in mind

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Mishkin (1991) conclude that the predictive power of the shorter rates about future inflation is low in the U.S., Germany and Switzerland.
Fama (1990) and Mishkin (1990a and 1990b) present identical conclusions concerning the information content of U.S. term structure regarding future inflation and state that the U.S. dollar short rates have information content regarding future real interest rates and the longer rates contain information on inflation expectations. Mishkin (1990b) also concludes that for several countries the information on inflation expectations is weaker than for the United States. Mehra (1997) presents evidence of a cointegration relation between the nominal yield on 10-year Treasury Bond and the actual U.S. inflation rate. Koedijk and Kool (1995), Mishkin (1991) and Jorion and Mishkin (1991) supply some evidence on the information content of the term structure concerning inflation rate in several countries.

that, according to Schich (1999), the information content of the term structure on future inflation is time-varying and depends on the country considered.

The relationship between the factors and the referred variables is consistent with the time-series properties of the factors and those variables. Actually, as observed in Clarida *et al*. (1998), the $I(1)$ hypothesis is rejected in Dickey-Fuller tests for the German inflation rate and short-term interest rate. Besides, as previously stated, the factors are stationary.

Using the result in (3.27), estimates for the average real yield curves are obtained, pointing to an average 10-year real yield close to 5.5 in both samples.[36] The resulting average inflation expectations are in line with the estimates obtained from the inflation-indexed OAT in France.[37]

The results obtained with the joint model for the German and French term structures (equation 3.47) are shown next. From charts 3.13a and 3.13b it is visible that the model has very weak fitting properties regarding the short end of the French curve and the long end of the German curve. Concerning the volatility and the term premium curves (respectively, charts 3.14a, 3.14b and 3.15), poorer results are obtained for the French data.[38]

[36] The gap between the nominal and the real curves tightens with the maturity, as the auto-regressive specification adopted for the inflation implies that the mean expected inflation decreases with the maturity. The mean expected inflation for a 10-year forecasting horizon was 1.4 and 2.3, respectively for the shorter and the larger samples.

[37] The indexed OAT started to be issued in 1998, the ending year of our sample. The spread between these bond yields and nominal bond yields with similar maturity may be considered as a proxy for the average expected inflation in France. As the spreads between French and German bonds are tight, the comparison between this market measure of inflation expectations and the results from our estimations may be done. In December 1998, the average expected inflation from the indexed bonds was 0.85, while our model provides a figure of 0.83.

[38] Several alternatives were considered to model simultaneously these two term structures, namely splitting the database in two sub-periods or eliminating the link between the common factor and the German inflation rate. Unfortunately, the results did not improve significantly.

Chart 3.13a
Germany Average Yield Curves 1986-1998:
Joint model

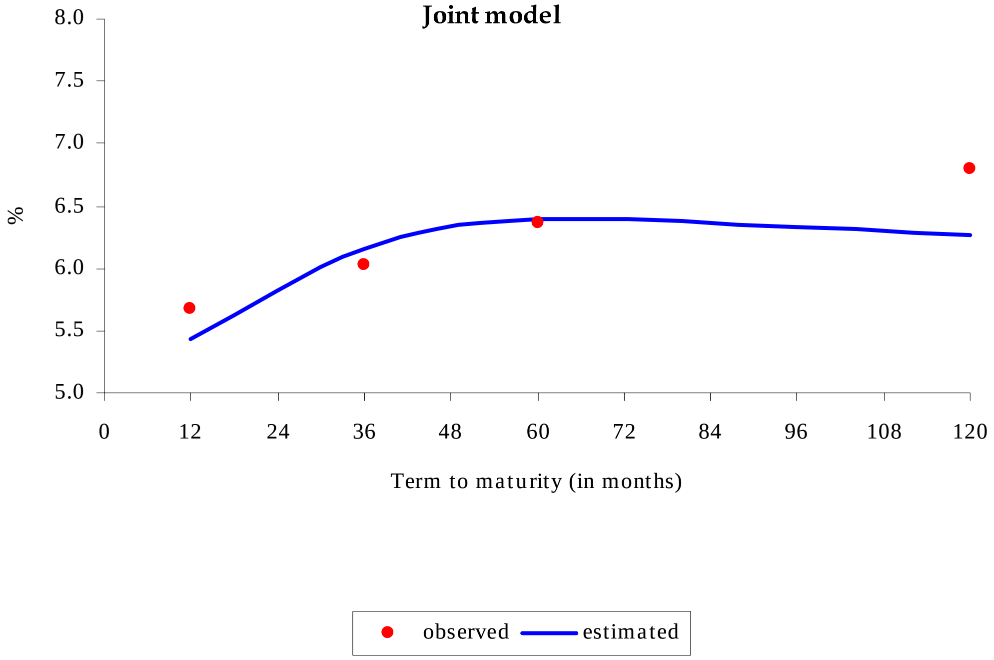


Chart 3.13b
France Average Yield Curves 1986-1998:
Joint model

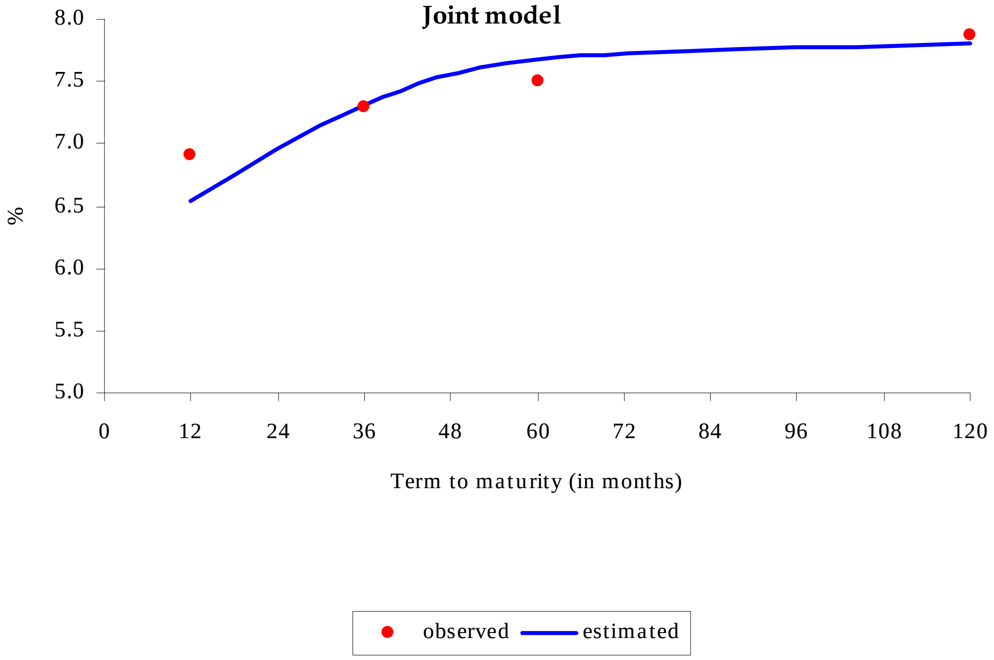


Chart 3.14a
Germany Volatility Curve 1986-1998:
Joint model

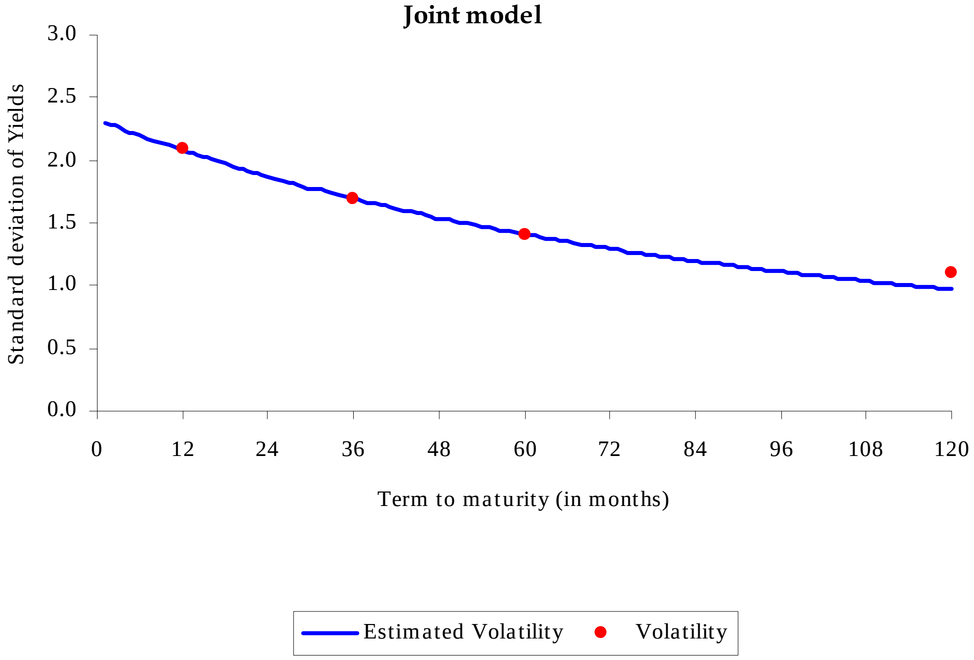


Chart 3.14b
France Volatility Curve 1986-1998:
Joint model

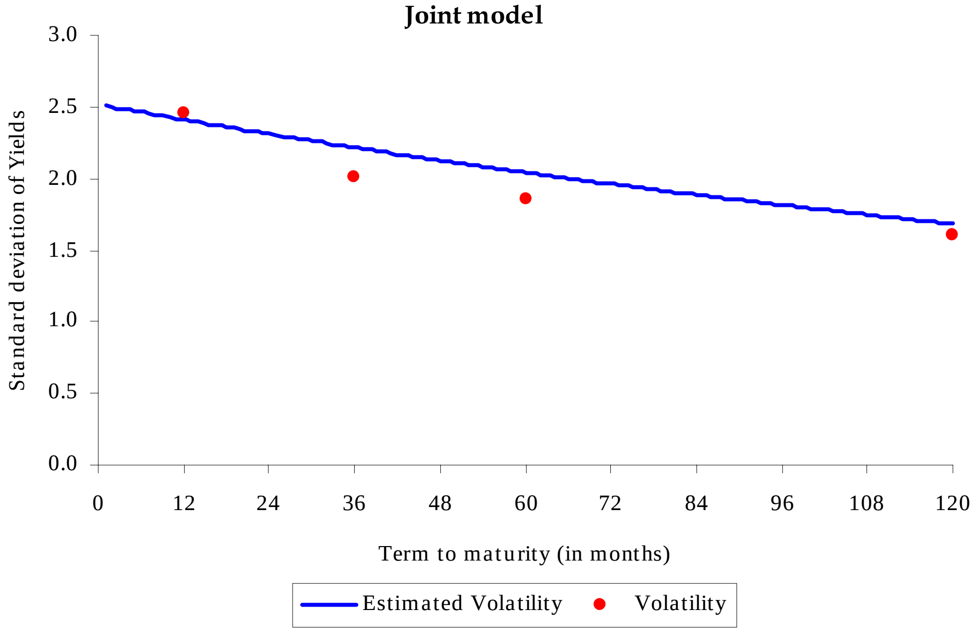
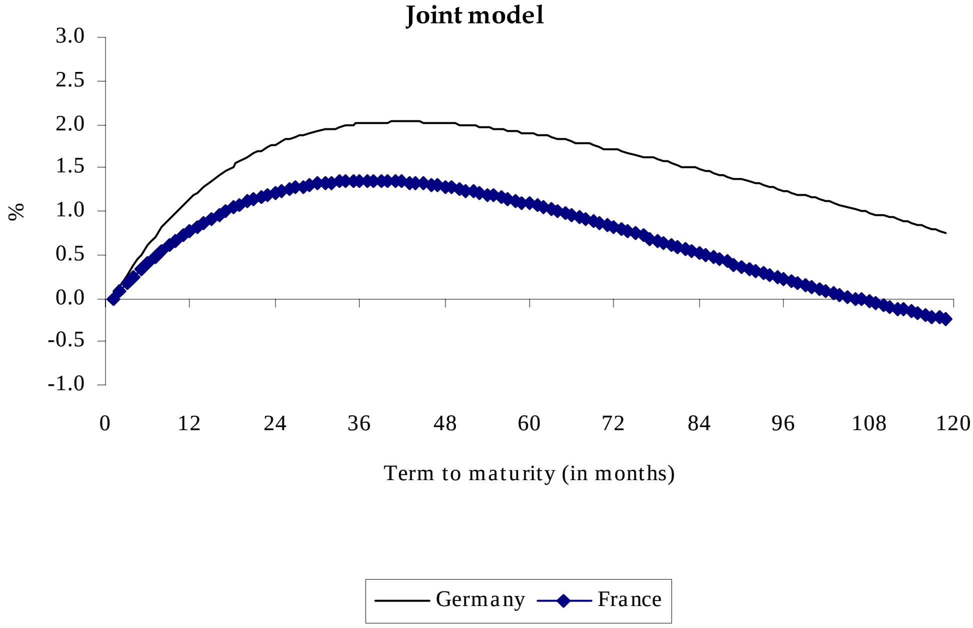


Chart 3.15
Term Premium Curve 1986-1998:
Joint model



The estimated forward rate and expected short-term interest rate curves are shown in 3.16 and 3.17. Comparing to chart 3.7a, the joint model provides lower forward and expected short-term interest rates for Germany.

Chart 3.16
Average Forward Curve 1986-1998:
Joint model

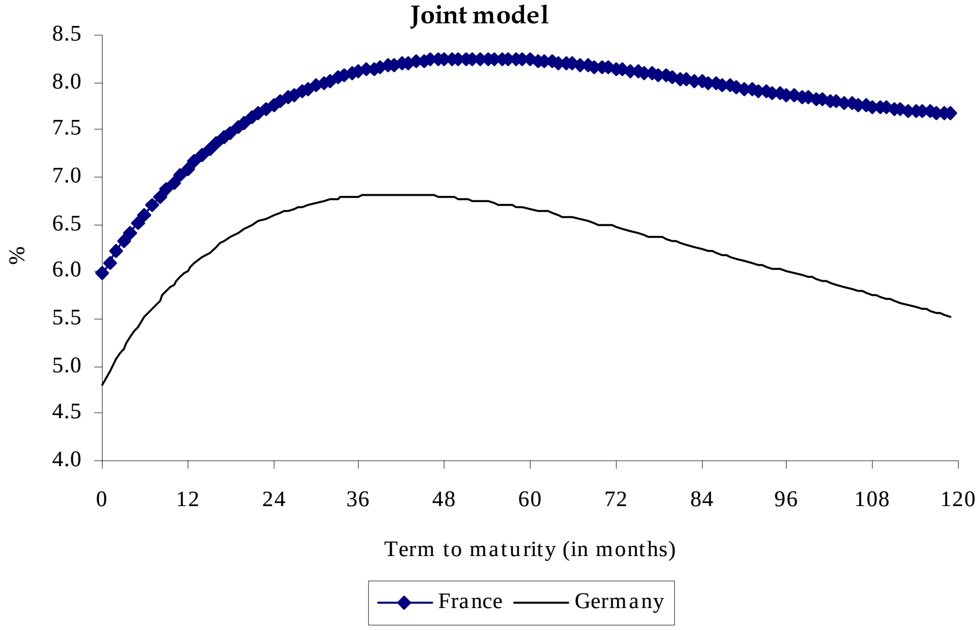
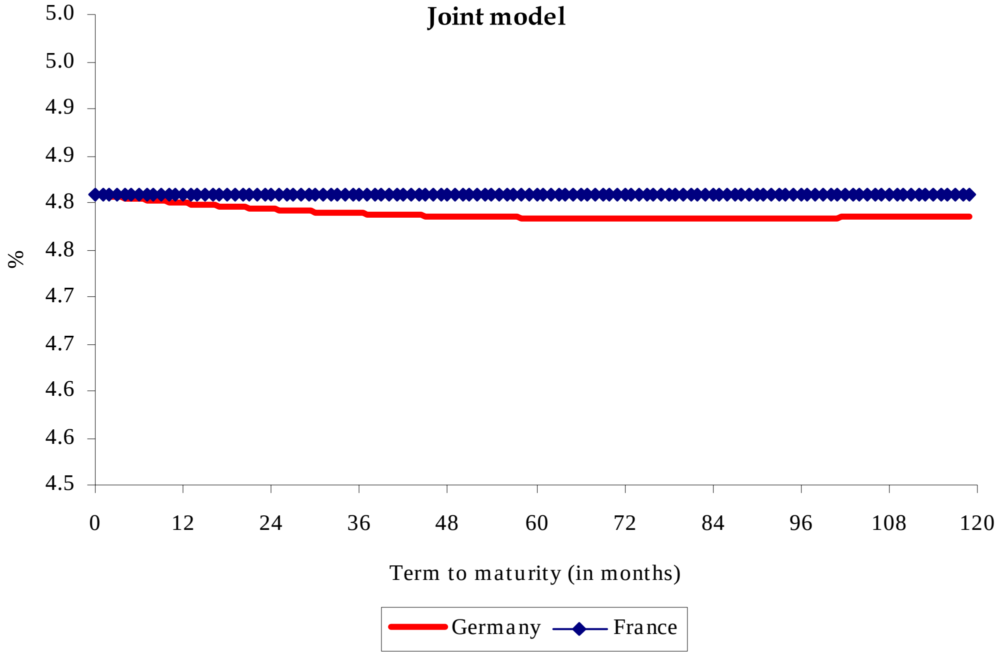


Chart 3.17
Average Expected Short-term Interest Rate 1986-1998:
Joint model



In charts 3.18a, 3.18b and 3.19, the time-series results for the yields, the yield spreads and the inflation confirm the low quality of the econometric adjustment provided by the model used. However, the estimated latent factors and the economic variables with which they are supposed to be related still exhibit high correlation coefficients, as illustrated in charts 3.20a and 3.20b.

Actually, the correlation coefficients between *ex-post* real interest rates, on one hand, and the first factor for German and French yield curves are close to 0.9 and 0.75 respectively. Conversely, the correlation between the German inflation rate and the second factor is around 0.8.

The factor loadings estimated are presented in charts 3.21a and 3.21b. They are in line with charts 3.10a and 3.10b, in what concerns to the higher relative weight of the first and the second factors respectively in the shorter and in the longer terms.

Chart 3.18a - Time-series yield estimation results 1986-1998: Joint model

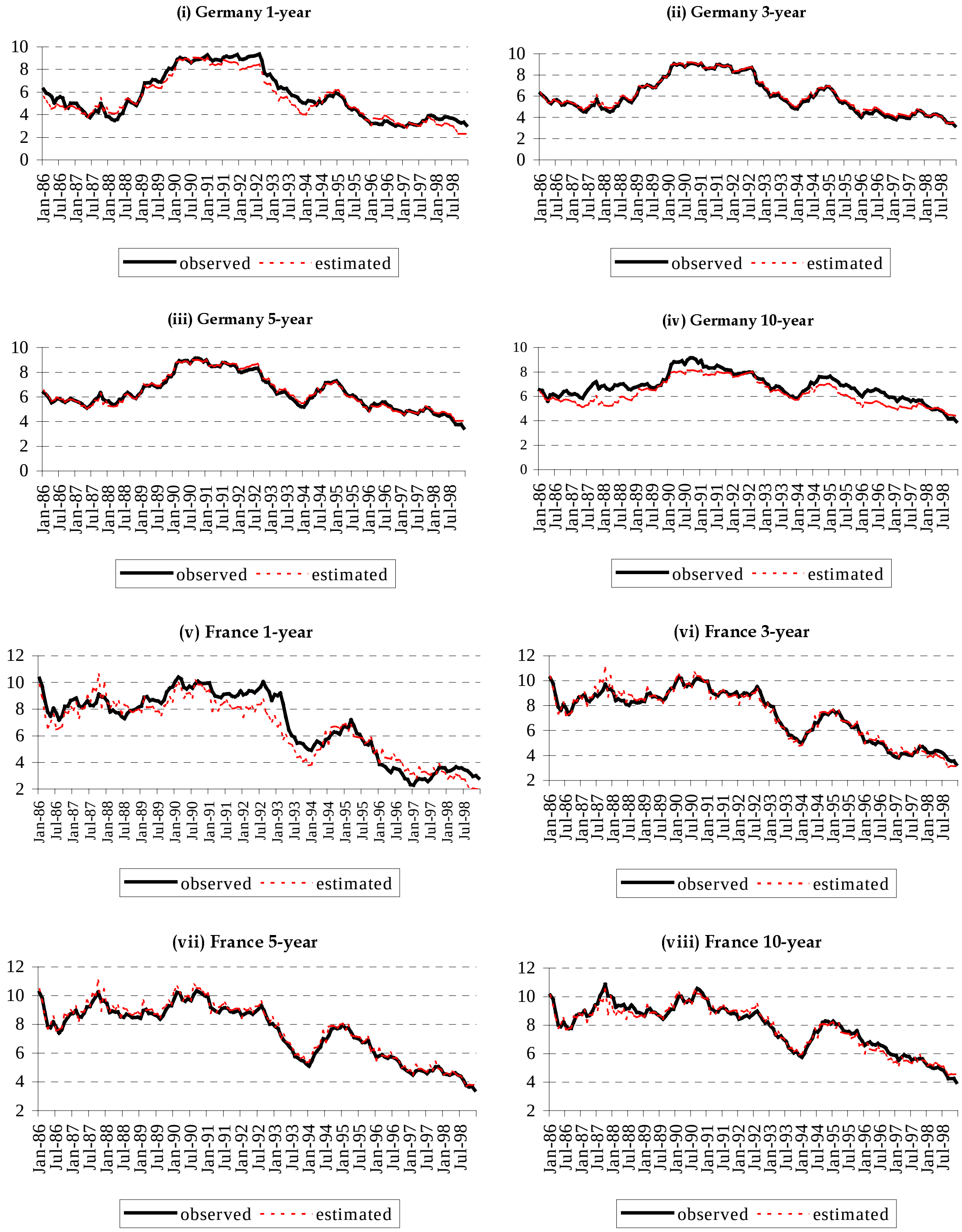


Chart 3.18b - Time-series estimation results for the yield spread France-Germany 1986-1998: Joint model

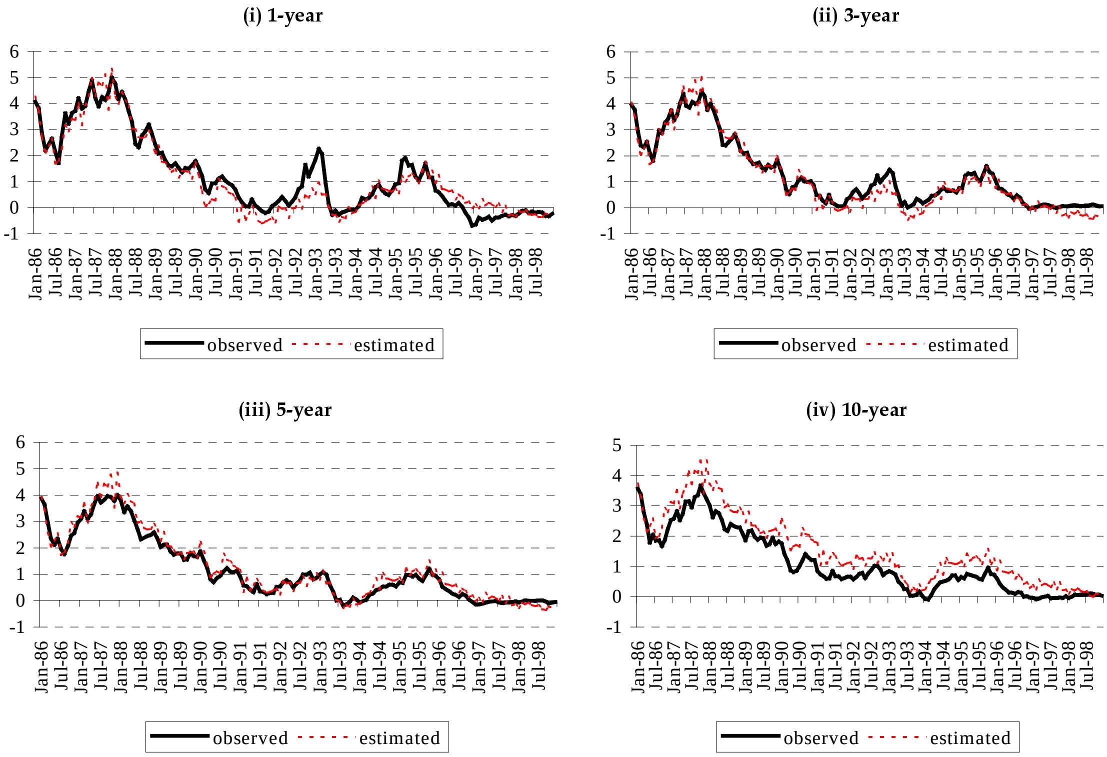


Chart 3.19
Observed and Estimated Germany Yearly Inflation 1986-1998: Joint model

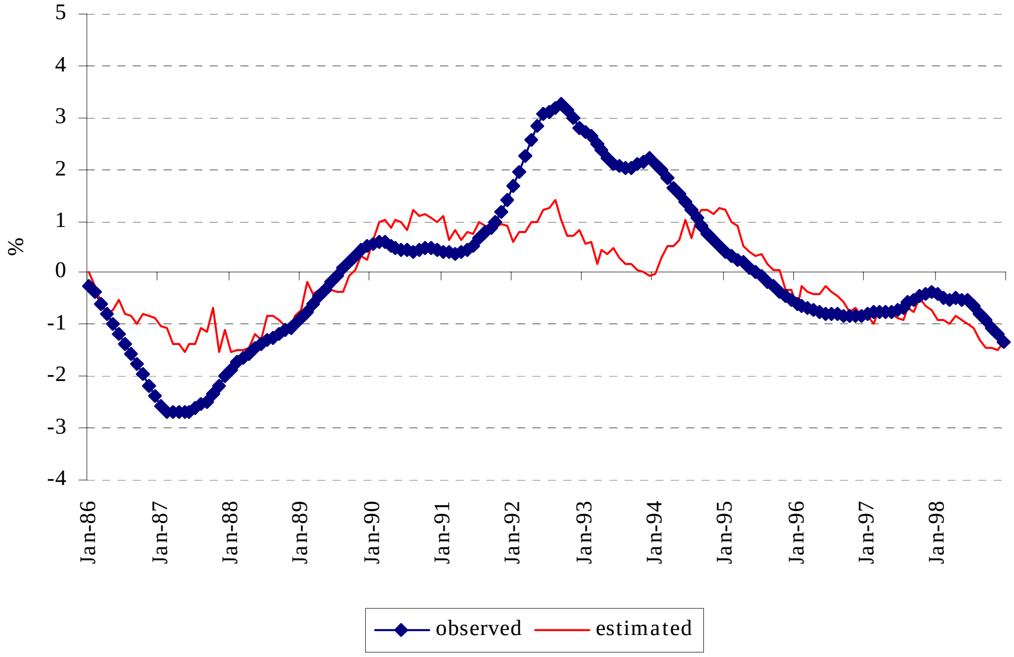


Chart 3.20a
Time-series evolution of the 1st. factor 1986-1998

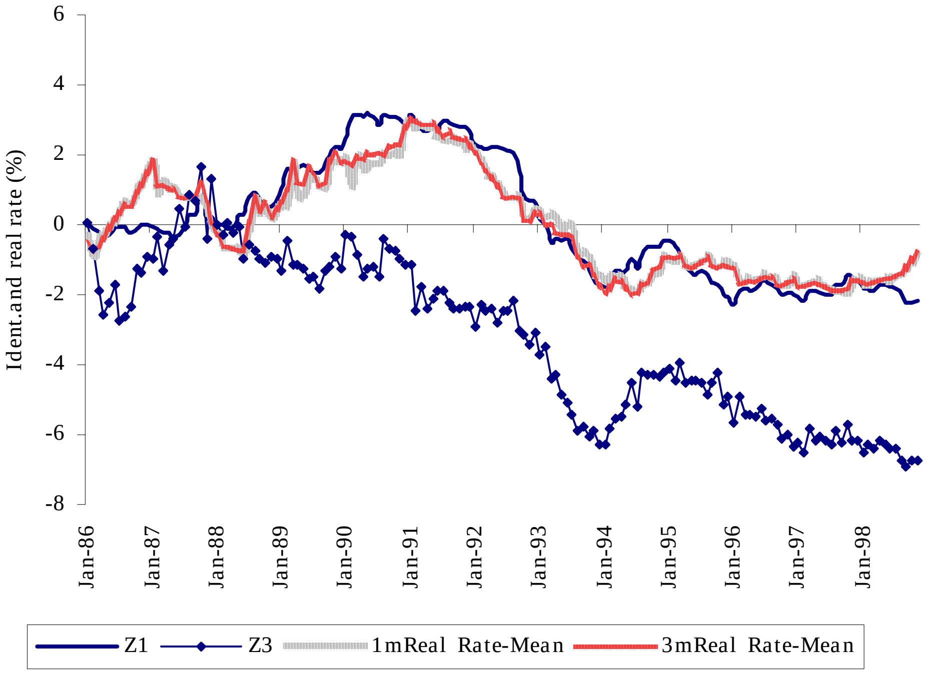


Chart 3.20b
Time-series evolution of the 2nd. factor 1986-1998

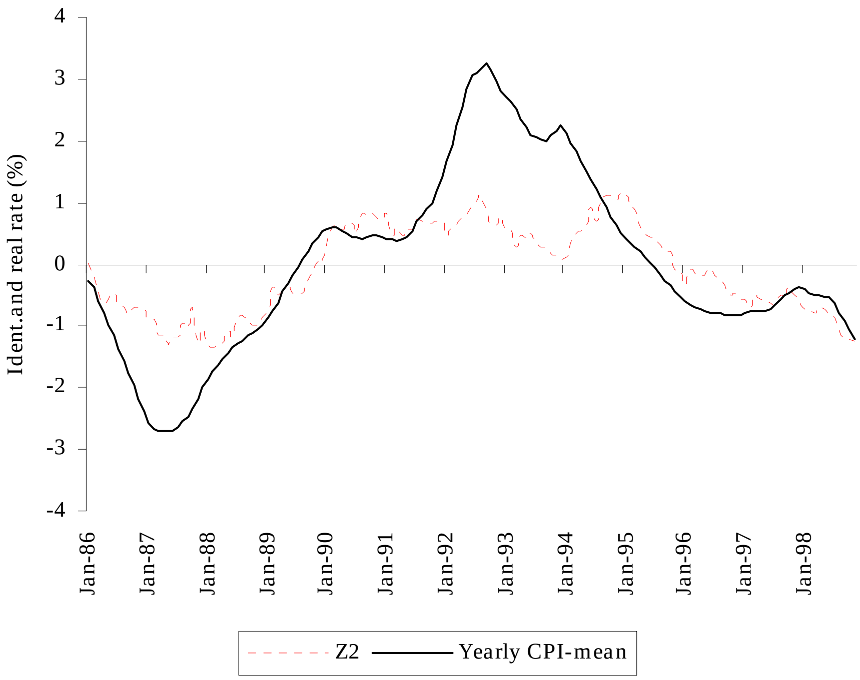


Chart 3.21a
Factor loadings in the joint model for Germany 1986-1998:
Joint model

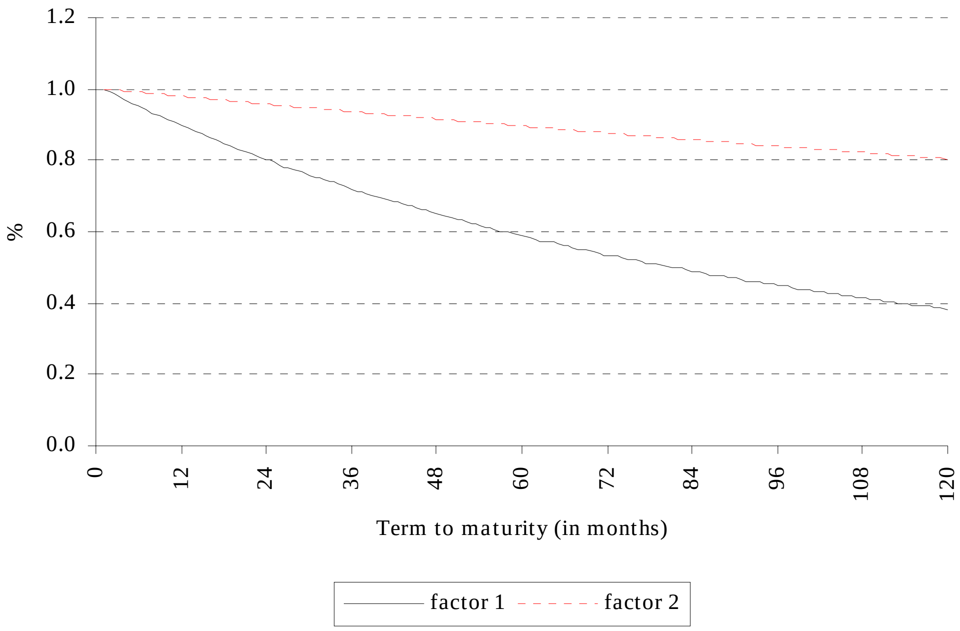
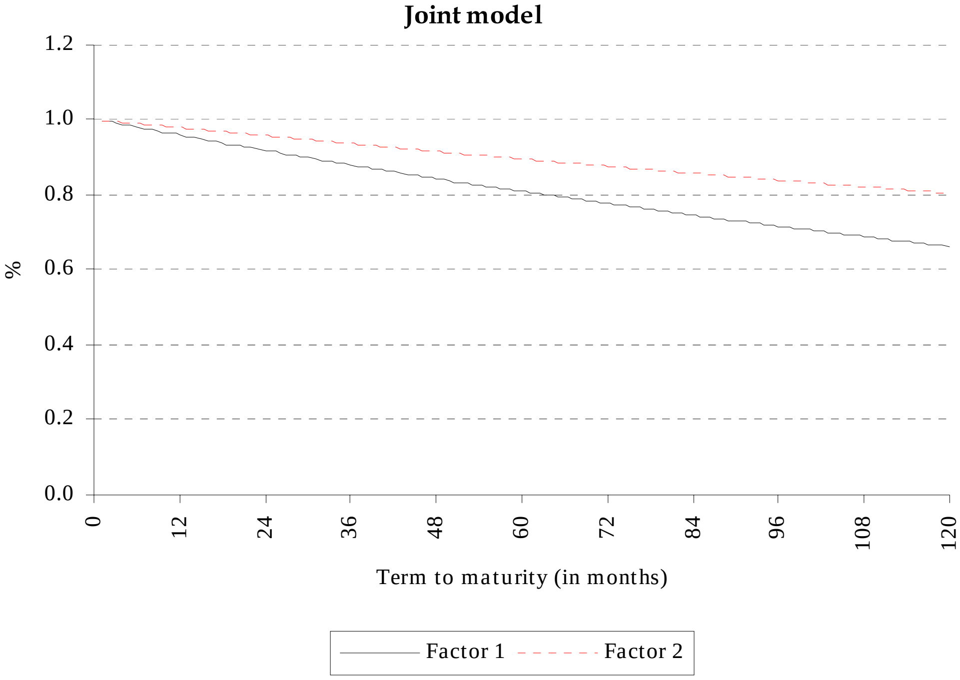


Chart 3.21b
Factor loadings in the joint model for France 1986-1998:
Joint model



3.7. Conclusions

The identification of the factors that determine the time series and cross section behaviour of the term structure of interest rates is one of the most challenging research topics in finance. In this chapter, it was shown that a two-factor constant volatility model describes quite well the dynamics and the shape of the German yield curve between 1986 and 1998.

The data supports the expectations theory with constant term premiums and thus the term premium structure can be calculated and short-term interest rate expectations derived from the adjusted forward rate curve. The estimates obtained for the term premium curve are not inconsistent with the figures usually conjectured. Nevertheless, poorer results are obtained if the second factor is directly linked to the inflation rate, given that restrictions on the behaviour of that factor are imposed, generating less plausible shapes and figures for the term premium curve.

We identified within the sample two periods of poorer model performance, both related to world wide gyrations in bond markets (Spring 1994 and 1998), which were characterised by sharp changes in long-term interest rates while short-term rates remained stable.

As to the evolution of bond yields in Germany during 1998, it seems that it is more the (low) level of inflation expectations as compared to the level of real interest rates that underlies the dynamics of the yield curve during that year. However, there still remains substantial volatility in long bond yields to be explained. This could be related to the spillover effects of international bond market developments on the German bond market in the aftermath of the Russian and Asian crisis.

It was also shown that one of those factors seems to be related to the *ex-ante* real interest rate, while a second factor is linked to inflation expectations. This conclusion is much in accordance with the empirical literature on the subject and it suggests that a Central Bank has a decisive role concerning the bond market moves, given that it influences both the short and the long ends of the yield curve, respectively by influencing the real interest rate and the inflation expectations. Accordingly, the second factor may be used as an indicator of monetary policy credibility.

Lastly, a joint model for the German and the French term structure was presented. Though different model specifications and samples were tried, the estimation results obtained are poorer than those provided by the single model for the German term structure, though they are in line with the previous results on the correlation between the factors and the relevant economic variables, as well as the relative weight of the factors in the whole maturity spectrum of the yield curve.