2.2. Interest Rate Risk

Interest Rate Risk

- Definition: sensitivity of the P&L to interest rate shifts in different maturities (i.e. changes in the term structure of interest rates).

- Types of interest rate risk:

(i) Marked-to market financial assets (assets valued according to market prices) - market risk of interest rate-sensitive financial assets (e.g. bonds).

(ii) Non-marked-to market financial assets (e.g. loans and deposits):

- Risk of fluctuation of the Net Interest Income of a bank stemming from the impact of interest rate changes on the cash-flows generated by assets and liabilities.
- Risk of optionality embedded in assets and liabilities, impacting on the volume of assets and liabilities that generate cash-flows, e.g. prepayment of loans and early redemption of deposits.

Interest Rate or Repricing Gaps

- Most assets and liabilities in the banking book are not marked-to-market.

- Their value does not change due to interest rate moves.
- Nonetheless, interest rate moves still impact on the Net Income (NI) of banks, because many of these assets and liabilities generate cash-flows that are sensitive to interest rates.
- These changes in the cash-flows impact on the Net Interest Income (NII, the difference between interest charged and interest paid by banks) and therefore on NI.

2.2.1. The Term Structure of Interest Rates

2.2.1.1. Main Concepts

Bond Prices and Yields-to-Maturity

- Price of a coupon-paying bond with discrete compounding:

(1) $$P = \sum_{i=1}^{N} \frac{C_i}{(1+y)^i} + \frac{M}{(1+y)^N}$$

where:
P = price;
C_i = coupons paid on the bond;
M = repayment value of the bond;
y = yield to maturity;
i = periods in which the cash-flows are generated.

- Price of a coupon-paying bond with continuous compounding:

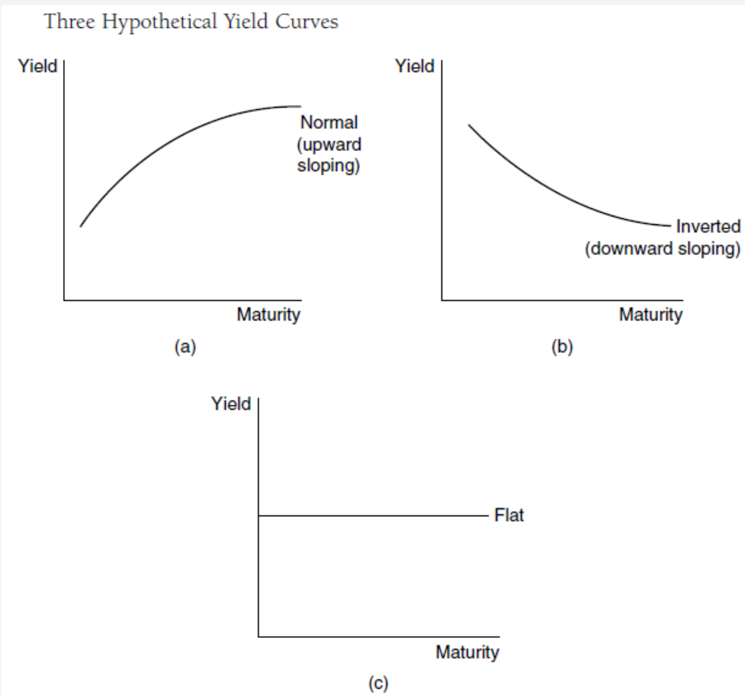
(2) $$P = \sum_{i=1}^{N} (C_i + M) \cdot e^{-yi}$$

- Price of a coupon-paying bond with continuous compounding and payments:

(3) $$P = \int_{i=0}^{N} (C_i + M) \cdot e^{-yi} \cdot di$$

Term Structure of Interest Rates

- TSIR – relationship between "risk-free" interest rates and maturities.
- Yield Curve – graphical representation of the TSIR, which may assume many different shapes (monotonic or non-monotonic), namely regarding the slope and the convexity.



Source: Fabozzi, Frank J., (2012), "The Handbook of Fixed Income Securities", 8th Edition, McGraw-Hill Education

Term Structure of Interest Rates

- Problems with using YTM to characterise the TSIR:
  - interest earned on different bonds but paid in the same periods are discounted at different rates
  - interest earned on a bond paid at different times is discounted at the same rate (flat yield curve).
  - there aren't bonds for all maturities

  - yield curves are usually designed from linear interpolations of YTM, exhibiting a very irregular shape, being therefore hardly plausible and hampering the extraction of information about expectations of market participants on future interest rates, namely for maturities that do not coincide with the maturities of the existing securities.

- The yield curve changes in response to:
  - Economic shocks
  - Market-specific events
  - Policy decisions

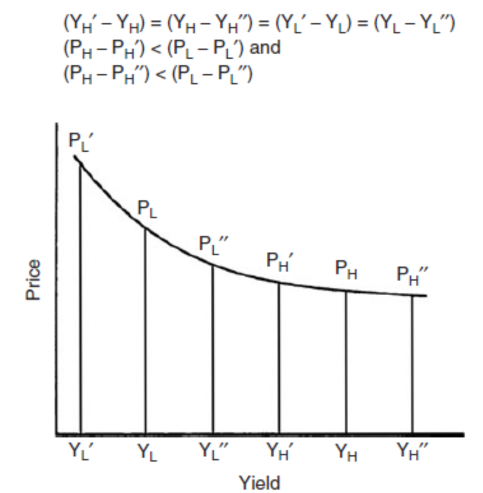
Stylized Facts

- Volatility
- Correlation
- Standard Movements
  - Shift Movements
  - Twist Movements
  - Butterfly Movements

Volatility

- Yields and bond prices are typically much less volatile than prices in other asset classes.
- Even though volatility of short-term rates is usually higher, volatility of bond prices in longer maturities is higher, due to the higher impact of interest rate shifts on the net present value of the cash-flows of bonds with higher maturities (as it will be seen).
- With higher yields, the volatility of bond prices (P_H) due to yield changes is lower.
- As it is illustrated below, $P_H' - P_H'' < P_L' - P_L''$.
- Changes in bond prices are close to symmetric for small yield changes, but for larger symmetric yield changes, price increases are higher than price decreases.
- Prices of bonds with higher coupon rates are less volatile, given the higher weight of intermediate cash-flows => zero-coupon bonds are the most volatile.

The Effect of Yield Level on Price Volatility



Source: Fabozzi, Frank J. (2012), "The Handbook of Fixed Income Securities", 8th Edition, McGraw-Hill Education

Correlation

- Rates with different maturities are
  - positively but not perfectly correlated, meaning that there is more than one factor behind the yield curve dynamics
  - correlation decreases with differences in maturity

- Example:

	1M	**3M**	**6M**	**1Y**	**2Y**	**3Y**	**4Y**	**5Y**	**7Y**	**10Y**
1M	1									
3M	0.999	1								
6M	0.908	0.914	1							
1Y	0.546	0.539	0.672	1						
2Y	0.235	0.224	0.31	0.88	1					
3Y	0.246	0.239	0.384	0.808	0.929	1				
4Y	0.209	0.202	0.337	0.742	0.881	0.981	1			
5Y	0.163	0.154	0.255	0.7	0.859	0.936	0.981	1		
7Y	0.107	0.097	0.182	0.617	0.792	0.867	0.927	0.97	1	
10Y	0.073	0.063	0.134	0.549	0.735	0.811	0.871	0.917	0.966	1

Standard Movements

- The evolution of the interest rate curve can be split into 3 standard movements, regardless the time period or the market:

  - **Shift movements** (changes in level), which account for 70 to 80% of observed movements on average.
  - **Twist movements** (changes in slope), which accounts for 15 to 30% of observed movements on average.
  - **Butterfly movements** (changes in curvature), which accounts for 1 to 5% of observed movements on average.

=> 1 or 2-factor models tend to be enough to explain the behavior of the yield curve in most occasions.

The Term Structure of Interest Rates

- As it will be seen, there are 3 alternative ways to represent TSIR:
  - Spot rates - rates set today for deals occurring also today and with a given maturity, with no intermediate cash-flows (e.g. coupons).
  - Forward rates – rates set today for deals to occur at a given future time and with a given maturity.
  - Discount factors – factors to discount the future cash-flows, to be comparable to current cash-flows, computed usually from the spot rates.
- In order to overcome the shortcomings of YTM, the TSIR must be characterised by the spot curve, which must be constructed from current zero-coupon "risk-free" bond yields.
- However, in most countries, zero-coupon government bonds are limited or restricted to the shortest maturities (up to 1y, Treasury Bills) => the TSIR will have to be estimated from the available information, usually money market rates and Government bond yields for maturities > 1y.

Spot and Discount Rates

- Replacing the YTM by the spot rates in (1), one gets:

$$(4)\quad P=\sum_{i=1}^{N}\frac{C_i}{(1+s_i)^i}+\frac{M}{(1+s_N)^N}$$

$$(1)\quad P=\sum_{i=1}^{N}\frac{C_i}{(1+y)^i}+\frac{M}{(1+y)^N}$$

- Discount factors (discrete interest compounding):

$$(5)\quad d_i=\frac{1}{(1+s_i)^i}$$

- Rewriting (4) by using (5), one obtains:

$$(6)\quad P=\sum_{i=1}^{N}C_i\cdot d_i+M\cdot d_N$$

Spot and Discount Rates

- Continuous interest compounding:

(7) $d_i = e^{-s_i \cdot i}$

- Spot rate can be deducted from the discount function as follows:

- Discrete interest compounding:

(8) $s_i = \left(\frac{1}{d_i}\right)^{1/i} - 1$

- Continuous interest compounding:

(9) $s_i = -\left(\frac{ln(d_i)}{i}\right)$

Forward Rates

- Given that the value of an amount invested at rate *s*, continuously compounded, during the term *i*, is equivalent to:

(10) $$\lim_{N\to\infty}\left(1+\frac{s_i}{N}\right)^{N\cdot i} = e^{s_i\cdot i}$$

- With continuous compounding, the amount invested at rate *s* for a maturity *m*+*n* is equivalent to:

(11) $$e^{s_{m+n}(m+n)} = e^{s_m\cdot m}\cdot e^{{}_m f_n\cdot n}$$

(12) $$s_{m+n}\cdot(m+n) = s_m\cdot m + {}_m f_n\cdot n$$

(13) $${}_m f_n = \frac{s_{m+n}\cdot(m+n) - s_m\cdot m}{n}$$

Forward Rates

- Using (9) and (13) becomes:

(14) $${}_m f_n = \frac{-\ln(d_{m+n}) + \ln(d_m)}{n}$$

(9) $$s_i = -\left(\frac{ln(d_i)}{i}\right)$$

- n->0 =>

(13) $${}_m f_n = \frac{s_{m+n} \cdot (m+n) - s_m \cdot m}{n}$$

(15) $${}_m f_0 = \lim_{n \to 0}\left(\frac{-\ln(d_{m+n}) + \ln(d_m)}{n}\right) = -\frac{\partial[\ln(d_m)]}{\partial m} = \frac{\partial(m \cdot s_m)}{\partial m}$$

or

(16) $${}_m f_0 = s_m + m \cdot \frac{\partial(s_m)}{\partial m}$$

(17) $$s_m = \frac{1}{m} \cdot \int_0^m {}_\mu f_0 d\mu$$ => Spot rate = simple average of the instantaneous forward rates (the forward rates with a very short time to settlement)

Forward Rates

- With discrete interest rate compounding, we have:

(18) $$(1+s_{m+n})^{m+n} = (1+s_m)^m \cdot \left[1+E_m(s_n)\right]^n$$

- Assuming that the expected value corresponds to the forward rate:

(19) $${}_m f_n = \left[\frac{(1+s_{m+n})^{m+n}}{(1+s_m)^m}\right]^{\frac{1}{n}} - 1$$

2.2.1.2. Explanatory Theories

Introduction

- Explanatory theories of the TSIR attempt to explain the relationship between “risk-free” interest rates and the corresponding maturities.

- These theories depend mostly on:
  - the preferences of market participants for maturities, namely their credit, liquidity and interest rate risk aversion.
  - the expectations on the future behavior of short-term interest rates, i.e. monetary policy.

Explanatory Theories

- Explanatory theories:
  - Expectations – Fisher, Irving (1896)
  - Preferred Habitat – Modigliani and Sutch (1966)
  - Liquidity Premium (or Preference) – Hicks (1939)
  - Market Segmentation – Culbertson (1957), Fama (1984) and Mankiw and Summers (1984)

Fisher, I., 1896. Appreciation and interest. Publications of the American Economic Association, 23-29.

Modigliani, F., Sutch, R. 1966. Innovations in interest rate policy. American Economic Review 56, 178-197.

Hicks, J., 1939. Value and Capital. Oxford University Press, London.

Culbertson, J., 1957. The term structure of interest rates. Quaterly Journal of Economics 71, 485-517.

Fama, E., 1984. The information in the term structure. The Journal of Financial Economics, 509-528.

Mankiw, N., Summers, L., 1984. Do long-term interest rates overreact to short term interest rates. Brooking Paper in Economic Activity, 223-247.

Expectations theory

- **This theory postulates that long term rates depend on the current short-term rates and the expectations on their future path.**

- Let us assume that an investor has 2 investment alternatives for a horizon = *T*:
  - A long term (zero-coupon) bond, with maturity = *T*;
  - A set of bonds with short term maturities (=1), with the last investment done at T-1. The investment in these several bonds can be done by rolling over the initial investment.

- The expected returns for these 2 alternatives must be equal (being r(t,T) the yield in time = t of a bond maturing at a later period T):
  - $[1+r(t,T)]^T = (1+r(t,1)) \times (1+E(t)\ (r(t+1),1)) \times (1+E(t)\ (r(t+2),1)) \times \ldots \times (1+E(t)\ (r(T-1),1))$
  - $r(t,T)=[(1+r(t,1)) \times (1+E(t)\ (r(t+1),1)) \times (1+E(t)\ (r(t+2),1)) \times \ldots \times (1+E(t)\ (r(T-1),1))]^{1/T}-1$

Expectations theory

- If one assumes that there is no risk premium (i.e. investors are risk-neutral regarding investing in short or in long term interest rates), expected interest rates = forward rates.
- According to this theory, the yield curve may have different shapes and positively (negatively) sloped curves correspond to expectations of short-term interest rate increases (decreases).
- Therefore, changes in yield curves are interpreted as changes in market expectations.
- **2 versions of the expectations theory:**

(i) pure – there is no risk premium => forward rates correspond to the expected future interest rates => $f_t^j = E_t(s_{t+j})$

(ii) non-pure – there is risk premium, but it's constant along time => forward rates do not correspond to expected future interest rates, but changes in forward rates correspond to changes in expectations about future interest rates.

Preferred habitat theory

- This theory sustains that investors have preferred maturities, but they accept to invest in different maturities if they are compensated for that.

- The risk premium paid to attract investors to maturities different from those preferred do not necessarily increase with the maturity.

- Moves in the yield curve do not correspond necessarily to changes in investors' expectations about the future path of short-term interest rates and the yield curve may have different shapes.

- Forward rates (or their changes) cannot be used to gauge expectations about future interest rates.

Liquidity premium theory

- It is a particular case of the preferred habitat theory
- Investors always prefer short to long maturities
- Investors always demand a premium to invest in longer maturities => Long term interest rates > short term interest rates
- The yield curve will always be positively sloped (unless we assume that long-term rates still reflect interest rate expectations and these point to sharp decreases)
- A positively sloped curve is usually considered as a regular curve, given that investors tend to be risk-averse => premium to invest in longer maturities due to the uncertainty on the future path of interest rates.

Market segmentation theory

- This theory postulates that interest rates in each maturity stem only from the supply and demand in that maturity.
- As a consequence, there is no relationship between interest rates in different maturities and the yield curve may have very irregular shapes.

- Main conclusions:

(i) the yield curve shape is explained by a mix of all these theories, even though market participants usually consider that a normal yield curve is a positively sloped one.

(ii) the risk premium is usually considered as increasing with maturities.

(iii) even though the risk premium is not nil, changes in long-term interest rates may be considered as changes in expectations on future short-term interest rates' behavior if one assumes that risk premium is constant along time, which tends to happen, at least, in short periods of time.

2.2.1.3. Static fitting methods

Introduction

FUNDAMENTAL ASSET PRICING FORMULA

$$P = \sum_{t=1}^{N} \frac{C_t}{(1+s_t)^t} + \frac{M}{(1+s_N)^N}$$

Main question: Where do we get s_t from?

- Any relevant information concerning how to price a financial asset must be primarily obtained from market sources.
- Spot rate - annualized rate of a pure risk-free discount (or zero-coupon) bond.
- As there aren't enough zero-coupon bonds for most countries and currencies, this information will have to be extracted from coupon-paying bonds.

Main Methods

- Bootstrapping method
- Carleton and Cooper (1976)
- Interpolation methods:
  - Linear
  - Polynomial
    - Simple
    - Splines
- Deterministic methods:
  - Nelson-Siegel (1987)
  - Svensson (1994)
  - Bjork and Christensen (1999)

Bootstrapping

- Consider 2 securities (nominal value = 100€):
  - 1-year pure discount bond, with P = 95€.
  - 2-year coupon-paying bond, with coupon rate = 8% and P = 99€.

- 1-year spot rate:

$$95 = \frac{100}{(1 + R_{0,1})}; \; R_{0,1} = 5.26\%$$

- 2-year spot rate:

$$99 = \frac{8}{(1 + R_{0,1})} + \frac{108}{(1 + R_{0,2})^2} = \frac{8}{1.0526} + \frac{108}{(1 + R_{0,2})^2};$$

$$R_{0,2} = 8.7\%$$

Bootstrapping

- The same type of reasoning can be developed for any number of bonds, e.g. 4 bonds (*d*(*k*) is the discount factor for cash-flows to be paid *k* years from now):

- Solve the following system recursively to obtain d(k):

$$101 = 105d(1)$$

$$101.5 = 5.5d(1) + 105.5d(2)$$

$$99 = 5d(1) + 5d(2) + 105d(3)$$

$$100 = 6d(1) + 6d(2) + 6d(3) + 106d(4)$$

- R(k) is obtained from d(k): d(k)=1/{[1+R(k)]^k} => R(k)=[1/d(k)]^(1/k)-1:

Maturity (k)	Price	Coupon rate	d(k)	R(k)
1	101	5,0%	0,9619	3,960%
2	101,5	5,5%	0,9119	4,717%
3	99	5,0%	0,8536	5,417%
4	100	6,0%	0,7890	6,103%

Bootstrapping

- Limitation: usually, one cannot find enough bonds with coincident coupon payment dates and longer maturities.
- Moreover, bond maturities are not round figures (when measured in years) often.
- A usual practical way to estimate the yield curve by bootstrapping involves the employment of interbank money market rates for different maturities:

Maturity	Price	Coupon rate	R(k)
O/N			4,40%
1m			4,50%
2m			4,60%
3m			4,70%
6m			4,90%
9m			5,00%
1y			5,10%
1y2m	103,7	5,00%	5,41%
1y9m	102,0	6,00%	5,69%
2y	99,5	5,50%	5,79%

→ 1y2m rate:

$$103.7 = \frac{5}{(1+4.6\%)^{1/6}} + \frac{105}{(1+x)^{1+1/6}}$$

Bootstrapping

- Conclusions:

(i) If one can find different bonds with coincident cash-flow dates and one of them only has one remaining cash-flow date, then one can get the spot rates directly.

(ii) These are spot rates instead of yields (for the shortest bond, the yield is equal to the spot rate, as this is a zero-coupon) and consequently they do not face the consistency problems of yields.

⇓

(iii) Therefore, we have a single spot rate for each maturity.

(iv) One can also calculate spot rates by using money market rates.

Carlton and Cooper

- Estimation of the discount factors by OLS method if the number of bonds is larger than the number of discount factors to be estimated: $\underset{(ix1)}{P} = \underset{(ixt)}{CF} \cdot \underset{(tx1)}{d}$

where

$i = 1, \ldots, k$ - riskless government bonds considered

$t = 1, \ldots, n$ - the cash-flows for which the discount factors are calculated.

P = vector of the prices of the *i* bonds (a column vector with *i* rows);

CF = matrix of the cash-flows of the *i* bonds for the *t* cash-flows (*i* rows and *t* columns);

d = vector of the discount factors for the *t* cash-flows (a column vector with *t* cash-flows).

- This method has several drawbacks:

(i) it only allows the estimation of some points of the discount function (for the maturities of the cash-flows considered);

(ii) it does not impose any smoothness on the discount function, allowing meaningless shapes; and

(iii) It faces multicolinearity problems resulting from the linear dependence between the cash-flows of, at least, some of the securities considered.

Interpolation - Linear

- Interpolations may be useful if we don't have all market information required to get spot rates for the same maturities.
- Simplest approach - linear interpolations:
  - Assuming that we know discount rates for maturities t_1 and t_2:
    - the rate for maturity t, being $t_1 < t < t_2$ = weighted average of the adjacent rates, being the weights higher for the maturity closer to t (e.g. if $t = t_2$, t_1 will not have any relevance to calculate t):

$$R(0,t) = \frac{(t_2 - t)R(0,t_1) + (t - t_1)R(0,t_2)}{(t_2 - t_1)}$$

  - Linear interpolations provide good proxies for near maturities.
  - However, for distant maturities, the shape of the resulting yield curve tends to be kinked.
  - By definition, linear interpolation doesn't allow to get estimates for maturities longer than those observed.

Interpolation - Polynomial

- Polynomial interpolations of the interest rates allow to obtain smoother yield curves, with interest rates as polynomial functions of maturities.
- A very common polynomial interpolation is the cubic => one can estimate the full term structure just by knowing the spot rates for 4 maturities.
- Therefore, if $R(0, t_1)$, $R(0, t_2)$, $R(0, t_3)$ and $R(0, t_4)$ are known, one can solve the following system in order to the 4 coefficients of the 3rd order polynomial.

$$\begin{cases} R(0,t_1) = at_1^3 + bt_1^2 + ct_1 + d \\ R(0,t_2) = at_2^3 + bt_2^2 + ct_2 + d \\ R(0,t_3) = at_3^3 + bt_3^2 + ct_3 + d \\ R(0,t_4) = at_4^3 + bt_4^2 + ct_4 + d \end{cases} \Rightarrow R = T \cdot A, \text{ being } R = \begin{bmatrix} R(0,1) \\ R(0,2) \\ R(0,3) \\ R(0,4) \end{bmatrix}, T = \begin{bmatrix} t_1^3 & t_1^2 & t_1 & 1 \\ t_2^3 & t_2^2 & t_2 & 1 \\ t_3^3 & t_3^2 & t_3 & 1 \\ t_4^3 & t_4^2 & t_4 & 1 \end{bmatrix}, A = \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$$

- If one uses more than 4 spot rates, these coefficients are estimated by econometric techniques (as we will have degrees of freedom), e.g. OLS (as the functions are linear in the coefficients).
- Otherwise: $R = T \cdot A \Leftrightarrow A = T^{-1} \cdot R$

Interpolation - Polynomial

- The calculation of *a*, *b*, *c* and *d* allows to obtain the spot rate for any maturity *t*:

$$R(0,t) = at^3 + bt^2 + ct + d$$

- Assuming the following rates are known:
  - R(0,1) = 3%
  - R(0,2) = 5%
  - R(0,3) = 5.5%
  - R(0,4) = 6%

$$\begin{cases} R(0,1) = a \cdot 1^3 + b \cdot 1^2 + c \cdot 1 + d \\ R(0,2) = a \cdot 2^3 + b \cdot 2^2 + c \cdot 2 + d \\ R(0,3) = a \cdot 3^3 + b \cdot 3^2 + c \cdot 3 + d \\ R(0,4) = a \cdot 4^3 + b \cdot 4^2 + c \cdot 4 + d \end{cases}$$

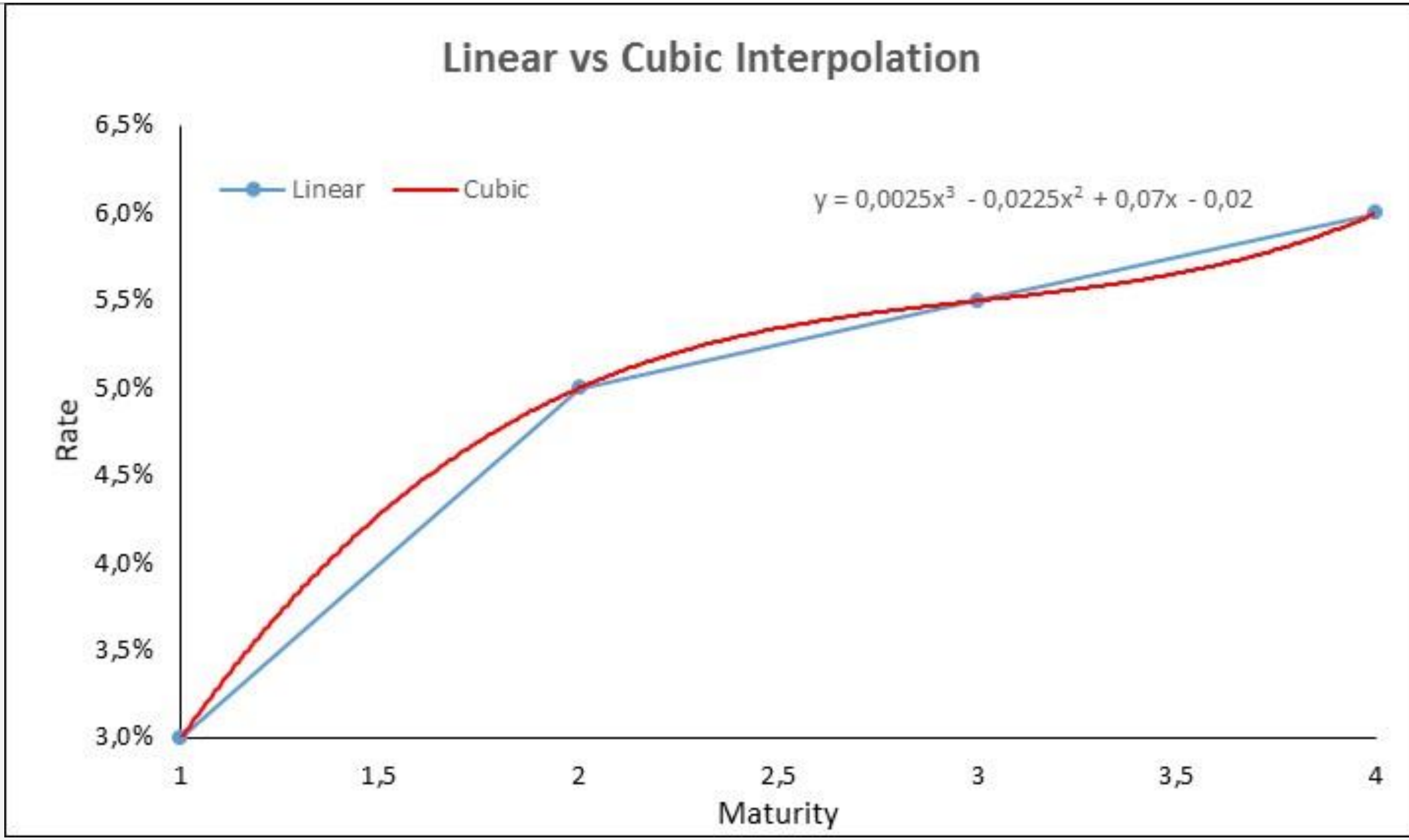
$$\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 8 & 4 & 2 & 1 \\ 27 & 9 & 3 & 1 \\ 64 & 16 & 4 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 3\% \\ 5\% \\ 5.5\% \\ 6\% \end{pmatrix} = \begin{pmatrix} 0.0025 \\ -0.0225 \\ 0.07 \\ -0.02 \end{pmatrix}$$

- **Goal - Compute the 2.5 year rate:**

$$R = T \cdot A \Leftrightarrow A = T^{-1} \cdot R$$

R(0,2.5) = a x 2.5^3 + b x 2.5^2 + c x 2.5^1 + d = 5.34375%

Interpolation - Polynomial



Interpolation - Polynomial

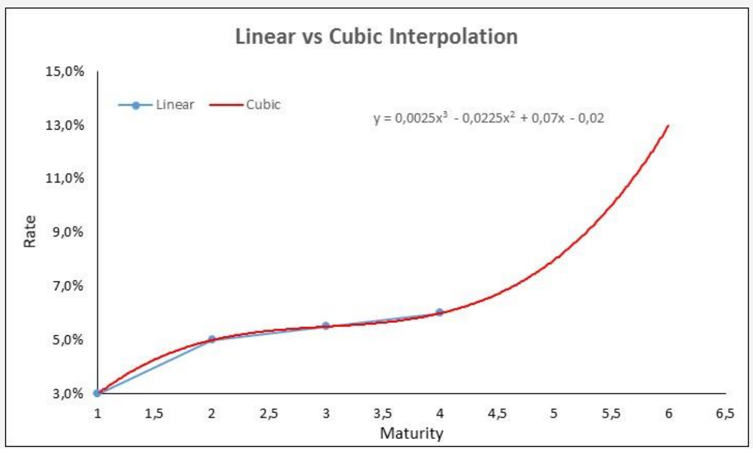
- **Conclusions:**

(i) Polynomials provide very good in-sample fittings

(ii) However, the estimates out-of-sample tend to be too irregular (e.g. in the previous example the 10-year would be 93%!)

(iii) **Polynomial splines** improve the adjustment, by allowing different specifications for the polynomials in different maturity buckets.

- Nonetheless, the explosive behavior of the resulting curves is kept.



Spline Methods

- **Definition:** Discount factors (p) as polynomial functions of the maturity (s), with **all coefficients differing in the different maturity buckets**:

$$p(s) = \begin{cases} p_0(s) = d_0 + c_0 s + b_0 s^2 + a_0 s^3, s \in [0,5] \\ p_5(s) = d_1 + c_1 s + b_1 s^2 + a_1 s^3, s \in [5,10] \\ p_{10}(s) = d_2 + c_2 s + b_2 s^2 + a_2 s^3, s \in [10,20] \end{cases}$$

- Imposing continuity constraints and given the fact that the discount factor for zero maturity is 1, the number of parameters is reduced:

$$p_0(5) = p_5(5)$$
$$p_5(10) = p_{10}(10)$$
$$p_0(0) = 1$$

- The No. of parameters may be even further reduced if one assumes that only 1 of the parameters is different in the several maturity buckets => McCulloch (1971, 1975) splines.

McCulloch Splines

- Dividing the maturity spectrum in *k-2* intervals, with *k-3* knots, the discount function can be defined as a cubic function, adding a factor (spline) to the 3rd order component, being *k* = No. of parameters:

$$d(t) = 1 + a_{2,1}t + a_{3,1}t^2 + a_{4,1}t^3 + \sum_{h=1}^{k-3} a_{4,h+1}(t - t_h)^3 \cdot D_h(t)$$

where *D*h(t) for *h*=1,2,…, *k*-3 are functions defined on the basis of the knots of the intervals, as follows:

$D_h(t)$ =0, if $t<t_h$, $D_h(t)$ =1, if $t \geq t_h$, for h=1,….,k-3.

- The discount function is continuous ⇔ for all knots, the values for the discount function are given as:

$$d(t) = a_0 + \sum_{h=1}^{k} a_h g_h(t)$$

McCulloch Splines

- **How to choose the number of parameters/intervals and the vertices:**

– If the number of intervals is very low, the spline adjustment becomes close to the simple polynomial.

– K-3 = 1 =>

- No. of intervals (k-2) =2
- No. of knots (k-3) =1
- No. of parameters =4

McCulloch Splines

- McCulloch proposes the No. intervals (k-2) = square root of the number of observations (bonds/maturities), rounded to the nearest integer, with the knots chosen to ensure all intervals have the same No. observations (or the difference between the No. observations in each interval is not higher than 1).

↓

- With 10 interest rates observed, we should have 3 intervals => 2 knots.

- Alternative methodology (used more often) - fixing the knots of the intervals in maturity dates corresponding to the maturities in which the market is traditionally "divided": 1, 3, 5 and 10 years.

McCulloch Splines

- If the vertices of the intervals correspond to the maturities in which the market is traditionally "divided" - 1, 3, 5 and 10 years – we have:
  - No. Intervals: k-2 = 5 (0-1, 1-3, 3-5, 5-10 and > 10y)
  - No. Vertices: k-3 = 4 (1, 3, 5 and 10)
  - No. Parameters: k = 7

$$d(t) = 1 + a_{2,1}t + a_{3,1}t^2 + a_{4,1}t^3 + \sum_{h=1}^{k-3} a_{4,h+1}(t - t_h)^3 \cdot D_h(t)$$

$$d(t) = 1 + a_{2,1}t + a_{3,1}t^2 + a_{4,1}t^3 + a_{4,2}(t-1)^3 \cdot D_1(t) + a_{4,3}(t-3)^3 \cdot D_2(t)$$
$$+ a_{4,4}(t-5)^3 \cdot D_3(t) + a_{4,5}(t-10)^3 \cdot D_4(t)$$

$D_1(t) = 0$, if $t<1$, $D_1(t) = 1$, if $t \geq 1$, $D_2(t) = 0$, if $t<3$, $D_2(t) = 1$, if $t \geq 3$,

$D_3(t) = 0$, *if* $t<5$, $D_3(t) = 1$, if $t \geq 5$, $D_4(t) = 0$, if $t<10$, $D_4(t) = 1$, if $t \geq 10$

McCulloch Splines

- The polynomial splines method provides better estimates in sample, i.e. up to the longest observed maturity, comparing to polynomial functions.
- However, the estimation problems outside the sample remain, as the discount function tends to assume irregular shapes from the longest maturity onwards.
- Whenever the yield curve assumes complex shapes, the use of a high number of parameters leads the estimated curve to adjust excessively to outliers => yield curve becomes even more irregular.
- This is particularly inconvenient if the goal is, as it often happens, the estimation of the term structure of interest rates for a fixed or standardised range of maturities, or to calculate forward rates.
- Therefore, more complex specifications will be required.

Deterministic Methods

- 3 steps:
  - **Step 1**: select a set of K bonds with prices P^j paying cash-flows $F^j(t_i)$ at dates $t_i>t$
  - **Step 2**: select a deterministic interest rate model for the functional form of the discount factors $p(t,t_i;ß)$, or the discount rates $R(t,t_i;ß)$ (or alternatively spot or forward rates), where $ß$ is a vector of unknown parameters, and generate prices.

$$\hat{P}^j(t) = \sum_{i=1}^{N} CF^j(t_i) p(t,t_i;\beta) = \sum_{i=1}^{N} CF^j(t_i) e^{-\left(t_j - t\right) R(t,t_i;\beta)}$$

  - **Step 3**: estimate the parameters $ß$ as the ones making the theoretical prices as close as possible to market prices: $\beta = \arg\min \sum_{j=1}^{K} \left(\hat{P}^j(t) - P^j(t) \right)^2$

- In reality, these methods are usually employed to fit the interest rates (e.g. YTM or spot rates), instead of bond prices.

Deterministic Methods

- **Key advantages:**
  - Parsimonious models, i.e. do not involve many parameters
  - Ensure stable functions
  - Adjust to many possible shapes of the TS
  - Some parameters have economic interpretation

Nelson and Siegel

- Nelson and Siegel (1987) proposed to fit the term structure using a flexible and smooth parametric function (Nelson, Charles and Andrew F Siegel, “Parsimonious Modeling of Yield Curves”, The Journal of Business, 1987, vol. 60, issue 4, 473-89).
- They demonstrated that the proposed model is capable of capturing many of the typically observed shapes that the yield curve assumes over time:

$$s_m = \beta_0 + (\beta_1 + \beta_2) \cdot \left[1 - e^{(-m/\tau)}\right] / (m/\tau) - \beta_2 \cdot \left[e^{(-m/\tau)}\right]$$

$${}_m f_0 = \beta_0 + \beta_1 \cdot e^{(-m/\tau)} + \beta_2 \cdot \left[(m/\tau) \cdot e^{(-m/\tau)}\right]$$

$$d_m = e^{\left[-\beta_0 m - (\beta_1 + \beta_2)\tau\left(1 - e^{-\frac{m}{\tau}}\right) + \beta_2 m \cdot e^{\left(-\frac{m}{\tau}\right)}\right]}$$

β_0 : level parameter - long-term spot or instantaneous forward rate ($lim_{m \to \infty} s$ or $lim_{m \to \infty} f$)

$\beta_0 + \beta_1$: short-term rate ($lim_{m \to 0} s$ or $lim_{m \to 0} f$); β_1 : (-) slope parameter´; β_2: curvature parameter;

τ : influences the speed of convergence of the curve towards the asymptotic value.

$\left(1 - \frac{\beta_1}{\beta_2}\right)\tau$: point of inflection of the slope of the forward curve

$\left(2 - \frac{\beta_1}{\beta_2}\right)\tau$: point of inflection of the concavity of the forward curve

Svensson

- Nelson-Siegel model faces estimation difficulties whenever the yield curve has more than one point of inflection of the slope or concavity.
- This is usually observed after disturbances in money markets.

↓

- Several more flexible NS specifications have been proposed in the literature to improve the fit to more complex shapes, namely with multiple inflection points, introducing additional factors and parameters.
- A popular term-structure estimation method among central banks (see BIS, 2005) to address is the 4-factor Svensson (1994) model, that accommodates 2 changes in the slope or in the concavity.

↓

- Svensson (1994) proposes to increase the flexibility and fit of the NS model by adding a 2nd hump-shape factor with a separate decay parameter (Svensson, Lars (1994), "Estimating and interpreting forward interest rates: Sweden 1992-94". IMF WP No.114).

Svensson

- The resulting 4-factor spot and forward curves are given by:

$$s_m = b_0 + b_1 \cdot \left[1 - e^{(-m/\tau_1)}\right] / (m/\tau_1) +$$

$$+ b_2 \cdot \left\{\left[1 - e^{(-m/\tau_1)}\right] / (m/\tau_1) - e^{(-m/\tau_1)}\right\} +$$

$$+ b_3 \cdot \left\{\left[1 - e^{(-m/\tau_2)}\right] / (m/\tau_2) - e^{(-m/\tau_2)}\right\}$$

$$_m f_0 = \beta_0 + \beta_1 \cdot e^{(-m/\tau_1)} + \beta_2 \cdot \left[(m/\tau_1) \cdot e^{(-m/\tau_1)}\right] + \beta_3 (m/\tau_2) e^{(-m/\tau_2)}$$

with

β_3: additional curvature parameter

τ_2 : additional parameter to govern the speed of convergence of the curve towards the asymptotic value

Svensson

- Even though the Svensson method is more adequate to estimate the TSIR for monetary policy purposes, given its higher adjustment capacity in the segment of the shorter maturities, when the yield curve assumes simple shapes in the short segment, the estimation by the NS method seems preferable, since it is more parsimonious.
- In fact, the NS model is a restricted version of the Svensson model with the restriction $\beta_3 = 0$ and/or $\tau_2 \to 0$.
- We can test the null hypothesis corresponding to those restrictions:
  - $H_0: \beta_0 = \beta_1 = \ldots = \beta_q = 0$

where: v = likelihood function of the adjustment with restrictions; v^* = likelihood function of the adjustment without restrictions; q = number of restrictions.

- The test is based on the following log-likelihood ratio test: $\lambda = -2 \cdot (\ln v - \ln v^*) \approx \chi^2(q)$

Svensson

- In this case, *v* corresponds to the likelihood function of the NS model (the restricted model), while *v** is the likelihood function of the Svensson model.

- If the logarithm of the likelihood function of the Svensson model is large enough (i.e., statistically above that of NS model), the Svensson model will be selected.

- H_0 is rejected if $\lambda > \chi^2 \Leftrightarrow$ Svensson model must be chosen.
- A potential problem with the Svensson model is that it is highly non-linear, which can make the estimation of the model difficult.
- Nonetheless, one can implement it even in a spreadsheet!

Bjork and Christensen

- One alternative model to Nelson-Siegel and Svensson models was developed by Bjork, T. and Christensen B.J. (1999): ”Interest rate dynamics and consistent forward rate curves”, Mathematical Finance.

- Very similar to Svensson, by adding a 4th factor to the instantaneous forward curve, but with a different specification for this 4th factor, that depends on a parameter (τ) that is the same in the 3rd factor:

$$f_t(\tau) = \beta_{1,t} + \beta_{2,t}\exp\left(-\frac{\tau}{\lambda_t}\right) + \beta_{3,t}\left(\frac{\tau}{\lambda_t}\right)\exp\left(-\frac{\tau}{\lambda_t}\right) + \beta_{4,t}\exp\left(-\frac{2\tau}{\lambda_t}\right)$$

$${}_m f_0 = \beta_0 + \beta_1 \cdot e^{(-m/\tau_1)} + \beta_2 \cdot \left[(m/\tau_1)\cdot e^{(-m/\tau_1)}\right] + \beta_3 (m/\tau_2) e^{(-m/\tau_2)}$$

← Svensson

$$y_t(\tau) = \beta_{1,t} + \beta_{2,t}\left[\frac{1-\exp\left(-\frac{\tau}{\lambda_t}\right)}{\left(\frac{\tau}{\lambda_t}\right)}\right] + \beta_{3,t}\left[\frac{1-\exp\left(-\frac{\tau}{\lambda_t}\right)}{\left(\frac{\tau}{\lambda_t}\right)} - \exp\left(\frac{\tau}{\lambda_t}\right)\right] + \beta_{4,t}\left[\frac{1-\exp\left(-\frac{2\tau}{\lambda_t}\right)}{\left(\frac{2\tau}{\lambda_t}\right)}\right]$$

- The 4th component resembles the 2nd, as it also mainly affects short-term maturities.
- The difference is that it decays to zero at a faster rate.

Bjork and Christensen

- Properties:

- The factor in $\beta_{4,t}$ can be interpreted as a second slope factor.
- As a result, Björk and Christensen model captures the slope of the term structure by the (weighted) sum of $\beta_{2,t}$ and $\beta_{4,t}$.
- The instantaneous short rate in this case is given by : $y_t(0) = \beta_{1,t} + \beta_{2,t} + \beta_{4,t}$

Bliss (1997)

- A second option to make the Nelson-Siegel model more flexible is by relaxing the restriction that the slope and curvature component must be governed by the same decay parameter τ.

- Bliss (1997) estimates the term structure of interest rates with a 3-factor model that allows for 2 different decay parameters τ_1 and τ_2 (Bliss, Robert R. 1997. “Testing Term Structure Estimation Methods.” Advances in Futures and Options Research 9:197–231).

- The forward and spot curves are then given by:

$$_m f_0 = \beta_0 + \beta_1 \cdot e^{(-m/\tau_1)} + \beta_2 \cdot \left[(m/\tau_2) \cdot e^{(-m/\tau_2)}\right]$$

$$s_m = \beta_0 + \beta_1 \cdot \left[1 - e^{(-m/\tau_1)}\right] / (m/\tau_1) + \beta_2 \cdot \left[\left[1 - e^{(-m/\tau_2)}\right] / (m/\tau_2) - \left[e^{(-m/\tau_2)}\right]\right]$$

Diebold, Piazzesi, and Rudebusch (2005)

- Conversively, some authors argue that even the NS model has to many parameters to be estimated, as the variation in interest rates can be explained mostly by 2 common factors:

- Diebold, Piazzesi, and Rudebusch (2005)* examined a 2-factor NS model, even though they recognize that more than 2 factors may “be needed in order to obtain a close fit to the entire yield curve at any point in time”.

- Compared to the 3-factor NS model, the 2-factor model only contains the level and slope factor => only 3 parameters have to be estimated:

$$s_m = \beta_0 + \beta_1 \cdot \left[1 - e^{(-m/\tau)}\right] / (m/\tau)$$

* Diebold, Francis X., Monika Piazzesi and Glenn D. Rudebusch (2005), "Modelling Bond Yields in Finance and Macroeconomics“, American Economic Review, 95, pp. 415-420.

Diebold, Piazzesi, and Rudebusch (2005)

- Despite the lack of theoretical background of deterministic interest rate models, the BIS concluded that 9 out of 13 central banks which reported their curve estimation methods to the BIS use these models (BIS (2005), "Zero-coupon yield curves: technical documentation", BIS Papers, No 25, Monetary and Economic Department, Oct.2005).
- According to this study, **most central banks have adopted either the NS (1987) model or the extended version by Svensson (1994),** with the exception of Canada, Japan, Sweden, UK and the US, which all apply variants of the "smoothing splines" method.
- Deterministic interest rate models are also widely used among market practitioners.
- Given that these models are usually non-linear in the parameters, attention has to be paid to their starting values.

2.2.2. Marked-to-market financial assets

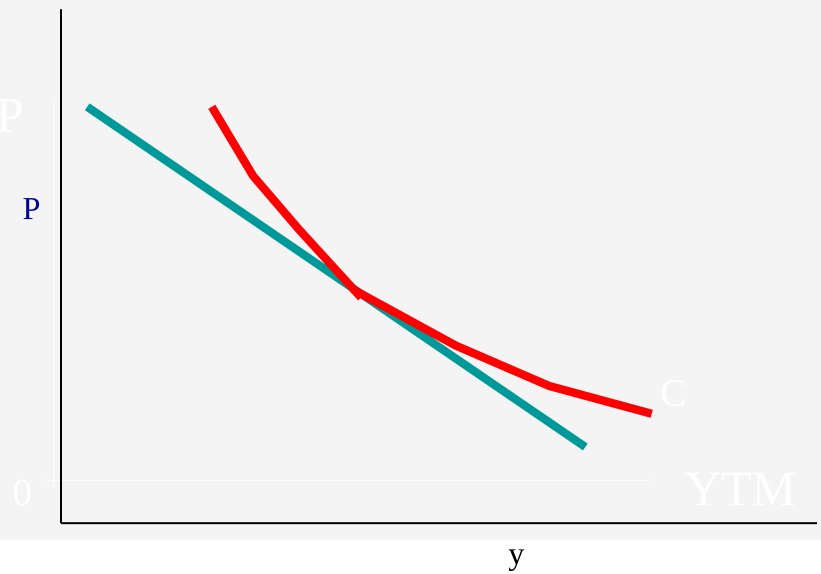
Introduction

- These assets face market risk, as adverse prices changes impact negatively on their value.
- Market risk stems either from credit and interest rate risk.
- In this section, we will focus on how to measure and hedge market risk in bonds.
- Basic principle: attempt to reduce as much as possible the dimensionality of the problem, i.e. to hedge risk with as few factors as possible.
- First step: duration hedging
  - Consider only one risk factor
  - Assume only small changes in the risk factor
- Second step: convexity hedging
  - Relax the assumption of small interest rate changes

2.2.2.1. Risk Measures

Introduction

- Focus: sensitivity of the bond price to changes in yield - Interest rate risk:
  - Rates change from *y* to *y*+*dy*
  - *dy* - small variation in yields, e.g. 1 bp (e.g., from 5% to 5.01%)
  - *dP* - variation in bond price due to *dy*
- The relationship between bond prices and the yields is not linear.
- However, for small changes in yields, a good proxy for *dP* is the first derivative of the bond price in order to *y*.



Macaulay Duration

- With continuously compounded interest rates and assuming a flat yield curve (same yields for all maturities), we have:

$$P^c = FVe^{-yT} + \sum_{n=1}^{T} ce^{-yn}$$

$$\frac{\partial P^c}{\partial y} = \frac{\partial[FVe^{-yT} + \sum_{n=1}^{T} ce^{-yn}]}{\partial y} = -T \cdot FVe^{-yT} - \sum_{n=1}^{T} n \cdot ce^{-yn}$$

- **Macaulay Duration (Frederick Macaulay, 1938) –** aka effective maturity: **Average maturity (measured in years) of all cash-flows weighted by the relevance of their NPV on the bond price** (while residual maturity is just the maturity of the final cash-flow), as follows.

Macaulay Duration

- Calculated as (the absolute value of) the partial derivative of the bond price with respect to yield, divided by the bond price:

$$D = -\frac{\frac{\partial P^c}{\partial y}}{P^c} = \frac{\sum_{n=1}^{T} n \cdot ce^{-yn} + T \cdot FVe^{-yT}}{P^c}$$

$$= 1 \cdot \frac{ce^{-y}}{P^c} + 2 \cdot \frac{ce^{-2y}}{P^c} + 3 \cdot \frac{ce^{-3y}}{P^c} + \cdots + T \cdot \frac{ce^{-yT}}{P^c} + T \cdot \frac{FVe^{-yT}}{P^c}$$

$$\frac{\partial P^c}{\partial y} = -DP^c \Rightarrow \frac{dP^c}{P^c} = -Ddy$$

Duration: (absolute value of a) percentage impact (%) on bond price of a given small change (percentage points) in the yield with continuously compounded interest rates.

Modified Duration

- With **discrete compounding interest rates**:

$$P^c = \frac{FV}{(1+y)^T} + \sum_{n=1}^{T} \frac{c}{(1+y)^n} = FV \cdot (1+y)^{-T} + \sum_{n=1}^{T} c(1+y)^{-n}$$

$$\frac{\partial P^c}{\partial y} = \frac{\partial\left[FV \cdot (1+y)^{-T} + \sum_{n=1}^{T} c(1+y)^{-n}\right]}{\partial y} = -T \cdot FV(1+y)^{-T-1} - \sum_{n=1}^{T} c \cdot n(1+y)^{-n-1} =$$

$$= -\frac{T \cdot FV(1+y)^{T-1}}{(1+y)^{2T}} - \sum_{n=1}^{T} \frac{c \cdot n(1+y)^{n-1}}{(1+y)^{2n}} = -\frac{T \cdot FV}{(1+y)^{T+1}} - \sum_{n=1}^{T} \frac{c \cdot n}{(1+y)^{n+1}}$$

$$= -\frac{1}{(1+y)}\left[\frac{T \cdot FV}{(1+y)^T} + \sum_{n=1}^{T} \frac{c \cdot n}{(1+y)^n}\right]$$

Weighted-average maturity of all cash-flows (weighted by the relative weight of their NPV on the bond price)

$$D = \frac{\frac{T \cdot FV}{(1+y)^T} + \sum_{n=1}^{T} \frac{c \cdot n}{(1+y)^n}}{P^c}$$

$$\frac{\partial P^c}{\partial y} = -\frac{1}{(1+y)} D P^c \Rightarrow \frac{dP^c}{P^c} = -\frac{1}{(1+y)} D dy$$

$$\frac{1}{(1+y)} D = MD$$

→ Modified duration: (absolute value of a) percentage impact (%) on bond price of a given change (percentage points) in the yield with discretely compounded interest rates

Macaulay Duration

T = 10, c = 5%, y = 5% (bond at par)

Time of Cash Flow	Cash Flow F_n	$w_n = \frac{1}{P^C} \cdot \frac{F_n}{(1+y)^n}$	$n \cdot w_n$	$n^2 \cdot ce^{-yn}$
1	50	0,047619048	0,047619	47,56147123
2	50	0,045351474	0,090703	180,9674836
3	50	0,04319188	0,129576	387,3185894
4	50	0,041135124	0,16454	654,9846025
5	50	0,039176308	0,195882	973,5009788
6	50	0,03731077	0,223865	1333,472797
7	50	0,035534067	0,248738	1726,48582
8	50	0,033841968	0,270736	2145,024147
9	50	0,032230446	0,290074	2582,394014
10	1050	0,644608916	6,446089	3032,653299
Duration (Σ n x w_n)			8,107822	13064,3632
Modified Duration (D/(1+y))			7,721735	

$$D = \frac{\frac{T \cdot FV}{(1+y)^T} + \sum_{n=1}^{T} \frac{c \cdot n}{(1+y)^n}}{P^c} \cong 8$$

- The lower the coupon rate, the higher (and closer to residual maturity) the duration will be, as the relative weight of the final cash-flow will be higher => zero-coupon bonds have duration equal to residual maturity.
- For a given coupon rate and yield, duration increases with the maturity: $\frac{\partial D}{\partial n} \geq 0$
- For a given maturity and coupon rate, duration increases as the yield decreases, given that the net present value of the cash-flows increase more in longer than in shorter maturities:

$$\frac{\partial D}{\partial y} \leq 0$$

Duration Hedging

- Principle: immunize the value of a bond portfolio to changes in yield:
  - P = value of the portfolio
  - H = value of the hedging instrument

- Duration hedging is very simple to do, but it is only valid for small changes and parallel shifts of the yield curve.
- The impact of these small changes is often provided by a measure usually employed in financial markets – basis point value (BPV) or price value of a basis point (PVBP):

$$PVBP = |\text{ initial price} - \text{price if yield is changed by 1 basis point }|$$

Duration Hedging

- Changes in value:
  - Portfolio: $dP \approx P'(y)dy$
  - Hedging instrument: $dH \approx H'(y)dy$
- Strategy: hold *q* units of the hedging instrument so that
  $dP + qdH = \left(qH'(y) + P'(y)\right)dy = 0$
- Solution: $q = -\frac{P'(y)}{H'(y)} = -\frac{P \times D_P}{H \times D_H}$

  given that *P'(y) = dP/dy* and *dP/P = -D x dy* (with continuously compounded interest rates) ⇔ $dP/dy = -P \times D_P$

  ⬇

  we can calculate the number of hedging instruments to implement a duration hedging strategy just by knowing the current prices of the bond and the hedging instrument, as well as both durations.

Duration Hedging

Example:

- At date t, a portfolio P has a price of €328635, a 5.143% yield and a 7.108 duration.
- Hedging instrument – bond with price = €118.786, yield = 4.779% and duration = 5.748.

$$q = -\frac{P'(y)}{H'(y)} = -\frac{P \times D_P}{H \times D_H}$$

- Hedging strategy involves taking a short position (i.e. selling futures contracts) as follows:

  q = -(328635x7.108)/(118.786x5.748) = - 3421

- Therefore, 3421 units of the hedging bond should be sold.

Duration Hedging

- Problems:

(i) High cost of duration immunization for very large portfolios

(ii) Duration is always changing => dynamic rebalancing of the portfolio

(iii) Relationship between prices and yields is not linear => the 2^{nd} derivative has to be considered.

Convexity

Convexity

- Considering a second order Taylor approximation:

$$\frac{dP^c}{P^c} = \underbrace{\frac{dP^c}{dy}\frac{1}{P^c}}_{-D}(dy) + \frac{1}{2}\underbrace{\frac{d^2P^c}{dy^2}\frac{1}{P^c}}_{C}(dy)^2$$

- With continuous compounding:

$$\frac{\partial^2 P^c}{\partial y^2} = T^2 \cdot FVe^{-yT} + \sum_{n=1}^{T} n^2 \cdot ce^{-yn} \quad \text{as} \quad \frac{\partial P^c}{\partial y} = -T \cdot FVe^{-yT} - \sum_{n=1}^{T} n \cdot ce^{-yn}$$

$$\Downarrow$$

$$C = \frac{\partial^2 P^c}{\partial y^2} \cdot \frac{1}{P^c} = \frac{T^2 \cdot FVe^{-yT} + \sum_{n=1}^{T} n^2 \cdot ce^{-yn}}{P^c} \geq 0$$

Convexity

- With discrete compounding, convexity may be written as a function of MD and its first derivative in order to the yield:

$$C = \frac{\partial^2 P^c}{\partial y^2} \cdot \frac{1}{P^c} \leftrightarrow C \cdot P^c = \frac{\partial^2 P^c}{\partial y^2}$$

$$\frac{\partial P^c}{\partial y} = -MD \cdot P^c \Rightarrow C \cdot P^c = \frac{\partial(-MD \cdot P^c)}{\partial y} = \frac{\partial(-MD)}{\partial y} \cdot P^c + \frac{\partial P^c}{\partial y} \cdot (-MD)$$

$$= \frac{\partial(-MD)}{\partial y} \cdot P^c + \left(-\frac{1}{(1+y)} DP^c\right) \cdot (-MD)$$

$$= -\frac{\partial(MD)}{\partial y} \cdot P^c + \underbrace{(-MD \cdot P^c)}\cdot(-MD) \leftrightarrow$$

$$\leftrightarrow C = -\frac{\partial(MD)}{\partial y} + MD^2$$

- As MD decreases with the yield => C≥0

Convexity

- For a given maturity and yield, convexity increases when the bond provides regular payments along time => convexity increases with the coupon rate and with maturity => the yield curve assumes a convex shape in longer maturities.
- But if the coupon rate increases, the yield also increases => convexity and duration down.
- For a given maturity and coupon rate, convexity increases when the yield decreases.
- A bond with higher convexity is always preferred, as its price benefits more from yield decreases and its less impacted by yield increases => bonds with low coupon rates.

$$C = \frac{\partial^2 P^c}{\partial y^2} \cdot \frac{1}{P^c} = \frac{T^2 \cdot FVe^{-yT} + \sum_{n=1}^{T} n^2 \cdot ce^{-yn}}{P^c} \geq 0$$

2.2.3. Non-marked-to-market financial assets

Interest Rate or Repricing Gaps

- Most assets and liabilities in the banking book are not marked-to-market.

↓

- Their value does not change due to interest rate moves.
- Nonetheless, interest rate moves still impact on the Net Income (NI) of banks, because many of these assets and liabilities generate cash-flows that are sensitive to interest rates.
- These changes in the cash-flows impact on the Net Interest Income (NII, the difference between interest charged and interest paid by banks) and therefore on NI.

Interest Rate Risk in the Balance Sheet

- Measurement:

(i) Interest Rate or Repricing Gap analysis:

- **Definition:** differences between assets and liabilities to be repriced in different time buckets (usually up to 1 year, with usual time bands being 1 week, 2 weeks, 1m, 2, 3, 6 and 12m).
- **Data:** All principal balances must be included in the gap report, along with interest flows. However, there is a trade-off between technical accuracy and practicality, as banks should include interest payments on tranches of principal that have not yet been repaid or repriced and the spread component of floating-rate instruments, but capturing and reporting this data is difficult.

↓

For amortizing loans, installments should by allocated to the time period in which they are scheduled to occur, but in most cases, gaps are calculated using principal rather than interest flows.

- **Variable-rate products:** they are normally linked to a benchmark rate (e.g. 1m, 3m or 6m Libor or Euribor, being the 6m Euribor the most usual reference rate for residential mortgage loans in Portugal) and the repricing frequency often corresponds to the maturity of the reference rate chosen.

Interest Rate Risk in the Balance Sheet

- **Bullet repayment loans:** the entire principal balance should be allocated to a time bucket corresponding to their maturity.
- **Fixed rate retail bank products:** as these tend to be homogenous and face early repayments, banks usually build models to estimate the behavioral run-off profile of loans and deposits.
- **Non-interest rate bearing balance sheet items** (e.g. non-interest bearing deposits, fixed assets and capital, even though capital may be considered as a fixed rate liability) - banks often decide to represent a proportion (e.g. 20%) of non-interest bearing deposits as notionally repricing in the short term and spread the remainder between 1 month and 3 or 5 years.
- **Types of gaps:** static or dynamic and marginal or cumulative.
- **Management:**

(i) banks usually impose internal limits on these gaps

(ii) banks may also decide to hedge against the interest-rate risk, by entering into interest-rate derivative transactions or by changing the pricing structure of their balance sheet, in order to mitigate their exposure.

(iii) banks may also decide to keep their gaps (at least up to a given magnitude) if they expect to benefit from interest rate changes.

Interest Rate Risk in the Balance Sheet

Example 1

- The Bank is negatively impacted by interest rate increases.
- Impact of a change in the yield curve on the NI in the following year:

$$\Delta NI_{1y} = \sum_{j=1}^{k} \Delta i_j \cdot gap_j \cdot (12 - m_j)$$

being i the interest rate for the mid-point maturity of each gap (m), j the order number of the gap and k the total number of gaps up to 1y.

Repricing Bucket	Assets	Liabilities	Interest Rate Gap	Cumulative Gap
Currency (£m)				
0 – 1 month	500	4,600	–4,100	–4,100
1 – 2 months	443	324	119	–3,981
2 – 3 months	156	1,781	–1,625	–5,606
3 – 4 months	342	430	-88	–5,694
4 – 5 months	213	24	189	–5,505
5 – 6 months	224	69	155	–5,350
6 – 9 months	356	17	339	–5,011
9 – 12 months	324	46	278	–4,733
12 – 15 months	614	32	582	–4,151
15 – 18 months	459	123	336	–3,815
18 – 24 months	875	275	600	–3,215
2 years – 3 years	1,365	135	1,230	–1,985
3 years – 4 years	845	86	759	–1,226
4 years – 5 years	725	58	667	–559
5 years – 6 years	413	0	413	–146
6 years – 7 years	45	0	45	–101
7 years – 10 years	89	0	89	–12
10 years +	12	0	12	0
Total	8,000	8,000		

Source: Choudhry, Moorad (2018) "The Moorad Choudhry Anthology: Past, Present and Future Principles of Banking and Finance", Wiley.

Interest Rate Risk in the Balance Sheet

- The sensitivity of NII to interest rate shocks are usually based on parallel yield curve shifts.
- **1y impact of a 1 pp upward parallel shift in the yield curve = -48.1m£.**

- Problems with this calculation:

(i) Bank balance sheets are not constant over time

(ii) Parallel shifts in the yield curve are rare

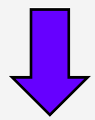
(iii) Some assets and liabilities won't reprice by the exact amount of the shock in rates and on the exact dates assumed.

Repricing Bucket Currency (£m)	Interest Rate Gap	IR Gap x Rate Shock x Remaining Months/12		(£m)
0 – 1 month	–4,100	–4,100 × 1% × 11.5/12	=	–39.29
1 – 2 months	119	119 × 1% × 10.5/12	=	1.04
2 – 3 months	–1,625	–1,625 × 1% × 9.5/12	=	–12.86
3 – 4 months	–88	–88 × 1% × 8.5/12	=	–0.62
4 – 5 months	189	189 × 1% × 7.5/12	=	1.18
5 – 6 months	155	155 × 1% × 6.5/12	=	0.84
6 – 9 months	339	339 × 1% × 4.5/12	=	1.27
9 – 12 months	278	278 × 1% × 1.5/12	=	0.35
12 – 15 months	582			
15 – 18 months	336			
18 – 24 months	600			
2 years – 3 years	1,230			
3 years – 4 years	759			
4 years – 5 years	667			
5 years – 6 years	413			
6 years – 7 years	45			
7 years – 10 years	89			
10 years +	12			
				–48.10

Source: Choudhry, Moorad (2018) "The Moorad Choudhry Anthology: Past, Present and Future Principles of Banking and Finance", Wiley.

Interest Rate Risk in the Balance Sheet

- This calculation can be simplified if only the cumulative interest rate or repricing gap (CGAP) is considered, getting a rougher but faster estimate.
- In this example, the cumulative 1y gap = -4.733 m£



- 1y impact of a 1 pp upward parallel shift in the yield curve =-4.733x0,01=47,33.
- This figure is very close to the one obtained by using the several marginal gaps (48,10).

Interest Rate Risk in the Balance Sheet

Example 2:

- For the 1st gap in the table below, the impact of a 1 pp increase in interest rates is:

$$\Delta NII_i = (-\$10 \text{ million}) \times .01 = -\$100{,}000$$

	(1) Assets	(2) Liabilities	(3) Gaps	(4) Cumulative Gap
1. One day	\$ 20	\$ 30	\$−10	\$−10
2. More than one day–three months	30	40	−10	−20
3. More than three months–six months	70	85	−15	−35
4. More than six months–twelve months	90	70	+20	−15
5. More than one year–five years	40	30	+10	−5
6. Over five years	10	5	+5	0
	\$260	\$ 260		0

Source: Saunders, Anthony and Marcia Millon Cornett (2018), Financial Institutions Management – A Risk Management Approach, 9th Edition, McGraw-Hill International.

- 1y CGAP:

$$CGAP = (-\$10) + (-\$10) + (-\$15) + \$20 = -\$15 \text{ million}$$

- Assuming a parallel upward shift in the yield curve up to 1y :

$$\Delta NII_i = (CGAP)\,\Delta R_i$$
$$= (-\$15 \text{ million})(.01) = -\$150{,}000$$

Interest Rate Risk in the Balance Sheet

(ii) Earnings-at-risk (EaR) – impact on earnings - NI or Economic Value of Equity (EVE) - from several very unfavorable scenarios for interest rates.

- EVE sensitivity calculation is also based on interest rate gaps, by computing the sum of the NPV of each bucket gap, assuming the current interest rates and then assessing the impact of different shifts in the yield curve.
- Typically, banks assess their EVE sensitivities to different shock scenarios.
- The interest rate shocks assumed should reflect a stressful rate environment that is both plausible and severe.
- The bank's ALCO committee usually set limits also on the change in EVE, based on the bank's risk appetite.

Interest Rate Risk in the Balance Sheet

- Key steps:

(i) Develop a bottom-up forecast of NII for the next 1–5 years;

(ii) Capture assumptions for all conceivable interest rate environments on:
   - (i) How all products would be repriced;
   - (ii) New business volumes;
   - (iii) Forecast prepayments / early redemptions;
   - (iv) The level of loan defaults.

(iii) Run a simulation to evaluate the impact of multiple different interest rate paths on NII and EVE;

(iv) Review the distribution of NII and EVE outputs;

(v) Focus on outlying values, particularly on the downside.

(vi) If these are of concern to management, prepare strategies to implement, in order to reduce the exposure.

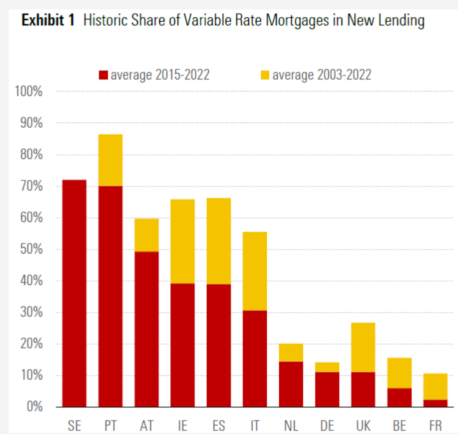
Interest Rate Risk in the Balance Sheet

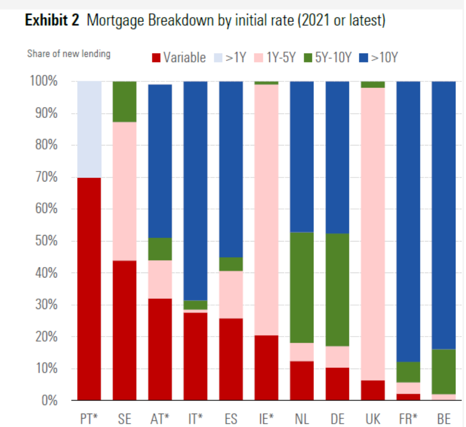
- EBA guidelines:

- FIs must measure their exposure to IRR in the banking book, in terms of both potential changes to EV, and changes to expected NII or earnings, considering:
  - different scenarios for potential changes in the level and shape of the yield curve, and to changes in the relationship between different market rates (i.e. basis risk);
  - assumptions made on non-interest bearing assets and liabilities of the banking book (including capital and reserves);
  - assumptions made on customer behaviour for ‘non-maturity deposits’ (i.e. the maturity assumed for liabilities with short contractual maturity but long behavioural maturity);
  - behavioural and automatic optionality embedded in assets or liabilities, considering:
    (a) impacts on current and future loan prepayment speeds from the underlying economic environment, interest rates and competitor activity;
    (b) the speed/elasticity of adjustment of product rates to changes in market interest rates; and
    (c) the migration of balances between product types, due to changes in their features.

Interest Rate Risk

- Portuguese banks usually have positive interest rate gaps, as credit rates are mostly indexed to money market rates (e.g. Euribor), while among liabilities only bonds issued are usually indexed, as term deposits are mostly short term liabilities (though may be renewed) with interest rates fixed by the bank => short term interest rate decreases are, *ceteris paribus*, unfavorable to banks.
- However, we must also bear in mind that higher rates may increase credit risk, being Portugal one of the countries with the highest % of variable rate loans, namely in residential mortgage loans.





Source: DBRS (2022), " European Banks Face an Increase in Residential Mortgage Risks as Interest Rates Rise".