2. Structural Models of Credit Risk

- Intensity-based models do not provide any fundamental explanation for the arrival of default, providing only a consistent description of the distribution of arrival times.

- Structural models provide this theoretical background, explaining the arrival of default by using company's balance sheet and stock market data.

- **In structural models, the default time is determined endogenously by the behavior of the company's asset** ⇔ default occurs when the market value of assets falls below the debt face value - **1st passage of assets by a default boundary**.

- These models became quite popular in the last decades, also due to the drawbacks of traditional credit risk models and rating updates by agencies.

- The rationale of structural models is that **market prices are the best assessment available on the companies' capital or debt value**, notwithstanding the higher volatility of market prices, namely for stocks, leading to false alarms of defaults.

Altman Z-score (1968)

- The first attempt to incorporate market prices in a credit risk model was done in the Z-Score model, by Altman (1968), which is an ad-hoc specification for credit risk as a function of several financial ratios, being one of them dependent on stock market capitalization.

- 22 financial ratios from 66 companies between 1946 and 1965 were used, evenly split between defaulting and non-defaulting companies.

Altman Z-score (1968)

- For defaulting companies, financial statements one year before the default were used, having been obtained the following model (PD decrease along with the Z-score):

$$Z = 1.2\ X_1 + 1.4\ X_2 + 3.3\ X_3 + 0.6\ X_4 + 1.0\ X_5$$

where:

X_1 = working capital (net) / total assets;

X_2 = retained earnings/ total assets

X_3 = EBIT / total assets;

X_4 = market capitalization/book value of long-term liabilities

X_5 = sales/total assets

- $Z<1,81$ – defaulting companies
- $Z>2,99$ – non-defaulting companies

Merton Model

- Later, Black and Scholes (1973) and Merton (1974) developed a corporate valuation approach based on financial options.

Black, F. and Scholes, M. (1973) The pricing of options and corporate liabilities. *Journal of Political Economy*, 81: 637–654.

Merton, R.C. (1974) On the pricing of corporate debt: the risk structure of interest rates. *Journal of Finance*, 29: 449–470.

- **This approach became known in the literature as the Merton Model.**
- The model is based on the assumption that, when the company issues debt, shareholders transfer the control of the company to creditors.
- However, they retain an option of recovering that control if the company reimburses the debt.

Merton Model

- Therefore, the equity value may be seen as the price of a call-option on the company assets, with a strike equal to the debt value.

- At the debt redemption date (time *T*): $V_E = Max(V_A - X, 0)$

- Before redemption date (at time *t*): $V_E = E\left[e^{-r(T-t)} Max(V_A - X, 0)\right]$

where

V_E = market value of the company's own funds

V_A = market value of company's assets

X = nominal value of the company's total debt payable at time T

r = risk-free interest rate for the maturity *T-t.*

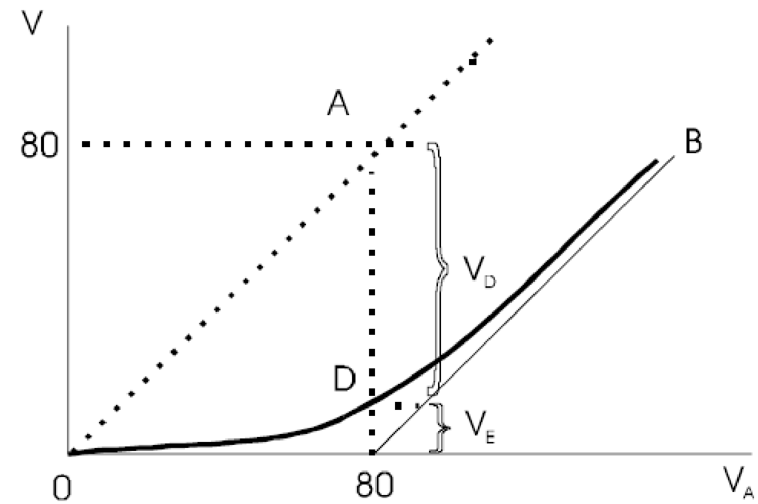
Merton Model

- Consequently, at the maturity date, the equity value is positive only if the market value of assets exceeds the value of liabilities.
- However, this is not true at a date before the maturity, as even when the $V_A < X$, there is a chance of $V_A > X$ at T.
- The PD at any time corresponds to the $P[V_A < X]$ at T, which depends on the current V_A , but also on the density function of V_A at T.
- Therefore, the expected value and the volatility of V_A have to be estimated.

Merton Model

- Assuming the following balance sheet, the equity value at *T* will be zero if $V_A < X$.
- However, at a date before the maturity, the equity value can be positive even if $V_A < X$, as the probability of $V_A > X$ at *T* may be >0.
- The equity value before T follows the bold line in the chart on the RHS.

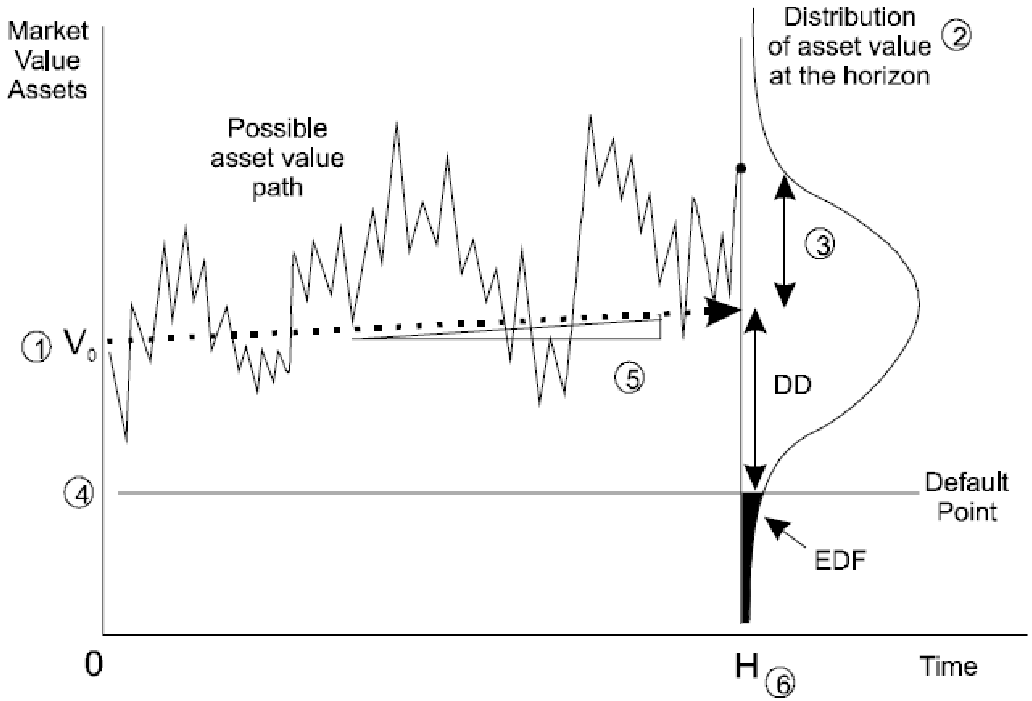
Assets	Liabilities
100	80
	20



Merton Model

- If on the redemption date, the market value of assets is lower than the debt value, the shareholders don't exercise the call option, i.e. the debt is not repaid =>

PD= P[Market Value of Assets < Debt value].



1. The current asset value.
2. The distribution of the asset value at time H.
3. The volatility of the future assets value at time H.
4. The level of the default point, the book value of the liabilities.
5. The expected rate of growth in the asset value over the horizon.
6. The length of the horizon, H.

Source: Crosbie and Bohn (2003), "Modeling Default Risk", KMV.

Merton Model

- The value of the debt can also be seen as a derivative, as its payoff corresponds to:

(i) the face value – if there is no default

(ii) the market value of assets – if there is a default (in this case, the recovery will provide bondholders a payment that will stem from the asset liquidation)

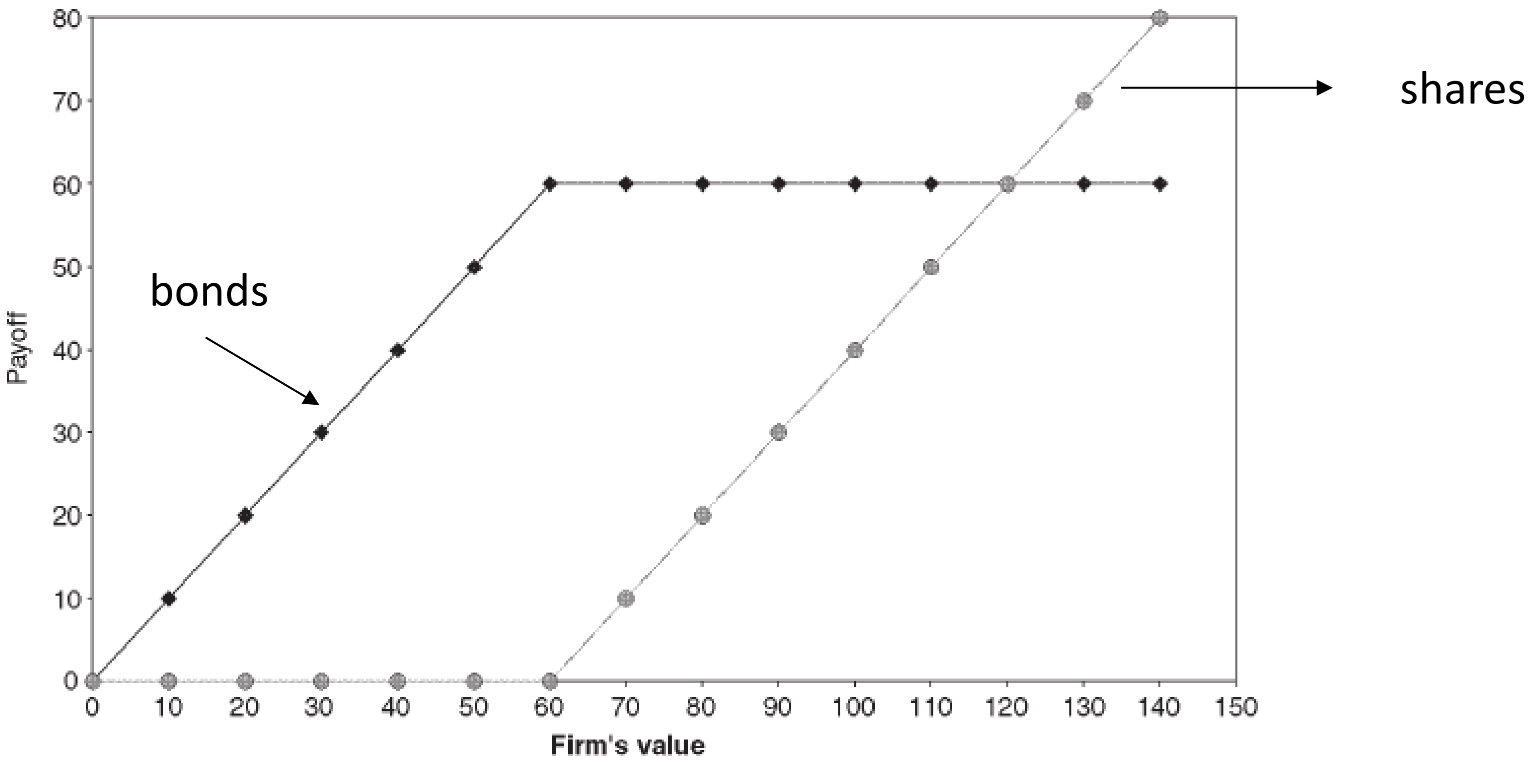
$$V_B = Min(X, V_A)$$

where

V_B = market value of the company's debt

Merton Model

- Payoffs of shares and bonds for $X = 60$:



Source: Schonbucher, Philipp J. (2003), "Credit Derivatives Pricing Models – Models, Pricing and Implementation", Wiley.

Merton Model

- Consequently, if the call-option can be valued, **the PD will be obtained from the distribution function resulting from the stochastic process of the company's asset market value**.
- Assuming that the option is European and the market value of assets may be taken as the price of a non-paying dividend asset, **one can use the Black-Scholes formula and calculate the PD from the implied volatility of the company's asset value** and an estimate for the respective growth rate.
- The Merton model is based on the assumption of the growth rates of the company's market value of assets (V_A) being normally distributed:

$$dV_A = \mu V_A dt + \sigma_A V_A dz \Leftrightarrow \quad dV_A / V_A = \mu dt + \sigma_A dz$$

where V_A is the company's market value of assets, μ *and* σ_A the respective trend and instantaneous volatility and *dz* is a Wiener process (random shocks normally distributed).

Merton Model

- Given that this is exactly the stochastic process of the underlying asset of an European option under the assumptions taken in the Black-Scholes pricing formula, the pricing formula for the European *call-option* on the company's market value of assets that corresponds to the stock price is:

$$V_E = V_A N(d1) - e^{-rT} X N(d2)$$

where

V_E is the market value of the company's own funds

N is the cumulative normal distribution function

r is the risk-free interest rate for the maturity T

X is the nominal value of the company's total debt payable in maturity T.

$$d1 = \frac{\ln\left(\frac{V_A}{X}\right) + \left(r + \frac{\sigma_A^2}{2}\right)T}{\sigma_A \sqrt{T}} \qquad d2 = d1 - \sigma_A \sqrt{T}.$$

Merton Model

- Therefore, the option valuation involves the calculation of the market value of assets and equity.
- Accounting identity => $V_A = V_B + V_E$
- However, most debt in V_B is not observable for most firms (most firms don't even have traded debt).

⬇

- V_A is not observable neither.

⬇

- In the option pricing formula, there are 2 unknowns: V_A and σ_A.

Merton Model

- Consequently, an additional equation is required, in order to determine the values for those 2 variables.
- This equation will result from the relationship between the volatility of assets and the volatility of capital:

$\delta = N(d_1)$

(1) $\sigma_E = \frac{V_A}{V_E} N(d_1) \sigma_A$ (from $\sigma_E = \frac{\partial V_E}{\partial V_A} \frac{V_A}{V_E} \sigma_A$)

- In Jarrow and Rudd (1983), it is shown that the stock volatility is a multiple of the volatility of the market value of assets:

(2) $\sigma_E = \eta \sigma_A$

Merton Model

Given (1) and that

(3) $$\delta = \frac{\partial C(X)}{\partial V_A} = N(d_1)$$

one gets (from (1) and (2):

(4)

$$\frac{V_A}{V_E} N(d1)\sigma_A = \eta\sigma_A \Leftrightarrow$$

$$\Leftrightarrow \frac{V_A}{V_E}\delta = \eta \Leftrightarrow$$

$$\Leftrightarrow \delta = \frac{\eta}{V_A/V_E} \Leftrightarrow \delta = \frac{\sigma_E/\sigma_A}{V_A/V_E} \Leftrightarrow$$

$$\Leftrightarrow \delta = \frac{\sigma_E}{\sigma_A}\frac{V_E}{V_A} \Leftrightarrow V_A = \frac{\sigma_E}{\sigma_A}\frac{V_E}{\delta}$$

From inputs V_E, σ_E , X, r and T, the equation system including the option pricing formula and (4) allows to estimate V_A and σ_A .

Merton Model

- Therefore, the PD is the probability of the market prices of assets falling below the nominal value of debt at the expiry date:

$$p_t = \Pr\left[V_A^t \le X_t \mid V_A^0 = V_A\right] = \Pr\left[\ln V_A^t \le \ln X_t \mid V_A^0 = V_A\right]$$

- Given that the market value of assets follows a log-normal distribution, one gets (with μ = expected asset returns):

$$\ln V_A^t = \ln V_A + \left(\mu - \frac{\sigma_A^2}{2}\right)t + \sigma_A \sqrt{t}\varepsilon$$

- Therefore, the PD is:

$$p_t = \Pr\left[\ln V_A + \left(\mu - \frac{\sigma_A^2}{2}\right)t + \sigma_A\sqrt{t}\varepsilon \le \ln X_t\right] = \Pr\left[-\frac{\ln\frac{V_A}{X_t} + \left(\mu - \frac{\sigma_A^2}{2}\right)t}{\sigma_A\sqrt{t}} \ge \varepsilon\right] \Leftrightarrow p_t = N\left[-\frac{\ln\frac{V_A}{X_t} + \left(\mu - \frac{\sigma_A^2}{2}\right)t}{\sigma_A\sqrt{t}}\right]$$

- Risk-neutral PD ($\mu = r$): $p_t = N\left[-d_2\right]$

Merton Model

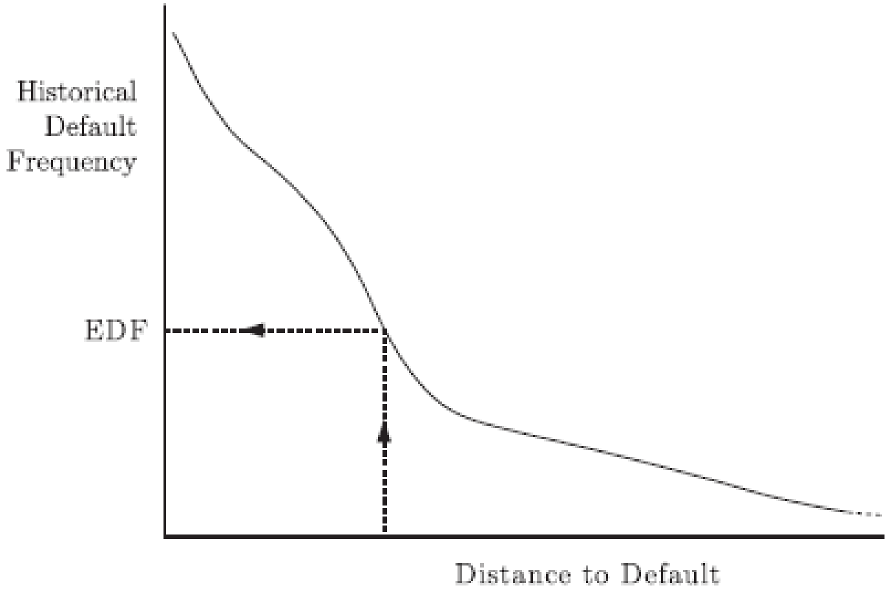
- Open issues:
  - How to obtain values for μ and σ_E?
  - How to deal with complex debt structures, with different maturities, seniority degrees and installments?
  - How to deal with the sensitivity of PDs to the leverage ratio?
  - How to solve the kurtosis problem in the market value of assets?
  - How to use the PD estimates as a leading indicator of rating changes?

- Estimation – non-linear least squares, minimizing the sum of the squared differences between the market value and the estimated value of the stocks (through the option pricing formula) and the assets.

Moody's KMV Model

- Moody's KMV overcomes the distribution problems motivated by the normality assumption through a database of loans providing empirical PDs as a function of the distance-to-default measure:

$$DD = \left[\frac{\ln \frac{V_A}{X_t} + \left(\mu - \frac{\sigma_A^2}{2} \right) t}{\sigma_A \sqrt{t}} \right]$$



Source: Duffie, Darrell and Kenneth J. Singleton (2003), "Credit Risk", Princeton University Press.

Moody's KMV Model

- In this model, σ_A **is a linear combination of a modeled and an empirical volatility,** the latter weighting 70% (80% for Financial Institutions).

- **Empirical vols -** calculated as the annualized standard deviation of the growth rates of the nominal value of assets, using 3 years of weekly observations for US companies (5 years of monthly data for European companies), excluding extreme values and adjusting for effects of M&A.

- **Modeled vols -** obtained from a regression between the observed vol and size, revenues, profitability, sector and region variables.

Moody's KMV Model

- **For banks, the PD is harder to estimate, given the diversity and uncertainty of the liabilities' maturities.**

- Moreover, by definition, banks are highly leveraged companies.

↓

- **Moody's KMV proposes the *default point* (the value of the payable liabilities in the maturity considered) to be calculated as a % of the total liabilities,** being that % differentiated according to the type of institution.

- According to Kerry (2019),* equity-market-based capital ratios signaled better the run-up to the global financial crisis than regulatory capital ratios.

* Kerry, Will (2019), "Finding the Bad Apples in the Barrel: Using the Market Value of Equity to Signal Banking Sector Vulnerabilities", IMF WP/19/180.

- According to Oderda *et al*. (2002), Moody's KMV model anticipates defaults with a lead of around 15 months, but also produces false alarms in 88% of the cases.

3. Reduced Form Models

- A reduced-form model (aka intensity-based models) assumes that hazard rates for the different debtors are stochastic processes correlated with macroeconomic variables.

- **A default may happen at any time, regardless the fundamental reason** ⇔ unlike structural model approaches, reduced-form models don't attempt to predict default by looking at its underlying causes, as they are essentially statistical and are calibrated using credit spreads that are observable in financial markets.

- The reduced-form approach is less intuitive than a structural model and employs mostly credit instrument prices derived from markets, e.g. corporate bonds and credit derivatives (vs. equity prices from stock markets employed by the Merton/KMV approach).
- Therefore, reduced-form models depend mainly on credit market spreads, but they may also use other input factors, including equity prices (as in Jarrow (2001)) and balance sheet data, to better disentangle the estimation of PDs from LGDs.

Term Structure of PDs

- However, world's credit markets are very imperfect sources of data, as credit risk is not the only determinant of price for credit risky securities, as bonds may differ due to:

(i) liquidity – many corporate bonds are quite illiquid and their prices are much less transparent than share prices in an equity market (also because many bond transactions are done at over-the-counter market, rather than on a formal exchange).

(ii) embedded options (e.g., convertible bonds or callable bonds, and so on);

(iii) different regulations and taxes in local markets.

- Given that credit spreads can be decomposed in default risk (PD, or l) and recovery risk (LGD, or ϕ), the PD can be modeled from the credit spreads and LGDs.
- Taking several maturities, one can obtain a term structure of PDs.

Pricing a zero-coupon bond

- 2 equivalent ways to calculate the price of a risky zero coupon bond (assuming one-period maturity and redemption value of one monetary unit):

(i) Expected value of the future cash-flows, discounted at the risk-free rate:

$$P = \frac{E_0(X_1)}{1+r} = \frac{\lambda\phi + (1-\lambda)\cdot 1}{1+r}$$

(ii) Future cash-flows, discounted at the risk-free rate plus the credit spread:

$$P = \frac{1}{1+r+s}$$

Credit spreads

- Equalizing both expressions =>

$$s = \frac{\lambda(1-\phi)(1+r)}{1-\lambda(1-\phi)} \cong \lambda(1-\phi)$$

=> Credit spread:

- Increases with the probability of default λ;
- Decreases with the recovery rate ϕ;
- Increases with the risk-free rate r;
- In reality, these spreads may also be impacted by risk premium due to uncertainty about risk-free interest rates, PDs and LGDs.

Credit spreads

- This relationship can be generalized for any maturity:

(i) Expected value of the future cash-flows, discounted at the risk-free rate:

$$P_0 = \sum_{i=1}^{T} \frac{E_0(X_i)}{(1+r)^i} = \sum_{i=1}^{T} \frac{\left[\lambda\phi + (1-\lambda)\right]C_i}{(1+r)^i} + \frac{\left[\lambda\phi + (1-\lambda)\right]NV}{(1+r)^T}$$

(ii) Future cash-flows, discounted at the risk-free rate plus the credit spread:

$$P_0 = \sum_{i=1}^{T} \frac{E_0(X_i)}{(1+r+s)^i} = \sum_{i=1}^{T} \frac{C_i}{(1+r+s)^i} + \frac{NV}{(1+r+s)^T}$$

Modeling PDs

- Consequently, the (risk-neutral) PD can be obtained by modeling the risk-free and the hazard rate, instead of the market value of the company's assets.
- From the spreads of similar bonds for different maturities, one can obtain the PD term structure, that can be compared to the statistics of rating agencies (the "true" PDs).
- The initial and most popular reduced form models were presented in Jarrow and Turnbull (1995) and Duffie and Singleton (1995).
- A commercial version of reduced form models is KRIS (Kamakura Risk Information Services) Credit Portfolio Model, developed by Kamakura.
- The approach proposed is quite general and can be applied to the construction of default models for all types of borrowers.
- The fitting process involves hazard rate modeling as a logistic regression between the probability of default $P(t)$ for a given time period t, provided the firm has survived until that time and a set of explanatory variables, X_i (i = 1, ..., n): $P(t) = 1 / [1 + \exp(-\alpha - \Sigma\beta_i X_i)]$
- X_i include financial ratios (e.g. return on assets and leverage ratio), macroeconomic factors (e.g. unemployment rate), stock market data (e.g. monthly excess return over a stock market index, equity volatility or the company's size relative to the total market capitalization of the relevant country and industry variables).

Modeling PDs

- Zero recovery defaultable bond price:

$$d_0(t, T) = E_t^* \left[e^{-\int_t^T (r(u)+\lambda^*(u))\, du} \right]$$

being λ^* the risk-neutral hazard rate

- Risk-free interest and hazard rates depend on a set of macroeconomic variables ($X(t)$):

$$r(t) = a_r(t) + b_r(t) \cdot X(t)$$

$$\lambda^*(t) = a_{\lambda^*}(t) + b_{\lambda^*}(t) \cdot X(t)$$

As both depend on X(*t*), the hazard rate becomes correlated with the interest rates.

Modeling PDs

- From the equations in the previous slide, we get **prices for the defaultable and the risk-free bond**, respectively:

$$d_0(t, T) = e^{\alpha_d(t,T)+\beta_d(t,T)\cdot X(t)}$$

$$\delta(t, T) = e^{\alpha_\delta(t,T)+\beta_\delta(t,T)\cdot X(t)}$$

- **Credit risk spread**:

$$s(t, T) = -\frac{\log d_0(t, T) - \log \delta(t, T)}{T - t}$$

$$s(t, T) = -\frac{\alpha_s(t, T) + \beta_s(t, T) \cdot X(t)}{T - t}$$

for α_s and β_s obtained by subtraction of the respective α's and β's

Pros and Cons

Advantages:

- ✔ Credit spreads are directly modelled: the intensity of the default process *is* the credit spread.
- ✔ Can be fitted to observed credit spreads.
- ✔ Realistic credit spreads through discontinuous dynamics.
- ✔ Suitable for the pricing of credit derivatives.

Disadvantages:

X There is no explicit link to the company's fundamentals

4. CREDIT RATING MODELS

- Credit risk may be assessed for different levels of credit ratings, by modelling rating changes.

- Stylized facts:

(i) Frequencies for low-probability events are usually based on a very small number of observations

(ii) Ratings momentum – rating changes tend to be more frequent for entities whose ratings have been revised recently

(iii) Ratings delay – rating changes tend to lag market prices for several months

(iv) Credit spreads are often misaligned with PDs, as the latter are just historical information.

(v) Credit spreads change along time.

4. Credit Rating Models

- An important source about rating frequencies corresponds to the regular reports published by rating agencies.
- These reports include information about cumulative PDs, 1y PDs along time and transition matrices, namely for 1y.

	A	B	D
A	$p_{AA} = 0.80$	$p_{AB} = 0.15$	$p_{AD} = 0.05$
B	$p_{BA} = 0.10$	$p_{BB} = 0.80$	$p_{BD} = 0.10$
D	$p_{DA} = 0.00$	$p_{DB} = 0.00$	$p_{DD} = 1.00$

Source: Schonbucher, Philipp J. (2003), "Credit Derivatives Pricing Models – Models, Pricing and Implementation", Wiley.

- From these 1y rating transition matrices, it is possible to calculate frequencies of default for larger periods.

4. Credit Rating Models

- 2y PD for rating A?
- As this default may occur either in the 1st or in the 2nd year, the most straightforward answer would be to calculate the Cumulative Probability of Default as 1 - Cumulative Probability of Survival, being the latter the joint probability of surviving in both years:
- $(1 - p_{AD})(1 - p_{AD}) = 0{,}95^2 = 0{,}9025$ => $PD_{2y} = 1 - 0{,}9025 = 0{,}0975$
- However, this answer would be valid only with no ratings (or no rating transitions besides defaults, or a single non-default rating).
- We need to take into account all rating transitions during the whole period before default, not only the transitions to default, but also the rating changes before default (i.e. during the 1st year).

4. Credit Rating Models

- Therefore, if a default occurs in the 2nd year, the rating transition to default may be either from rating A or from B, as defaults are often preceded by rating downgrades.
- Actually, during the 1st year , a rating A may be kept or may be downgraded to B, or even move straight to default:

A -> A -> D => $p_{AA}p_{AD} = 0{,}80 \cdot 0{,}05 = 0{,}04$

A -> B -> D => $p_{AB}p_{BD} = 0{,}15 \cdot 0{,}10 = 0{,}015$

A -> D (-> D) => $p_{AD}p_{DD} = 0{,}05 \cdot 1 = 0{,}05$

- $PD_{2y} = p_{AA}p_{AD} + p_{AB}p_{BD} + p_{AD}p_{DD} = 0{,}04 + 0{,}015 + 0{,}05 = 0{,}10$ (which compares to 0,0975 when rating transitions before default were discarded).
- All 2y transition frequencies will result from the squared rating transition matrix.

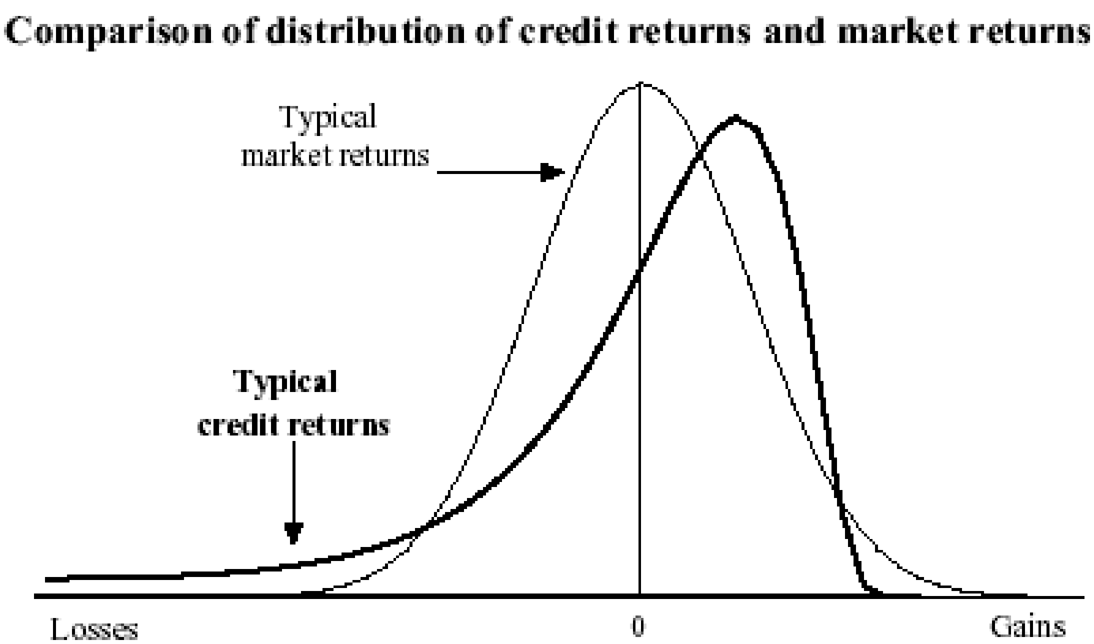
Types of Models

- Default mode (DM) – take into consideration only the changes in the value of bonds due to defaults.
- Marked-to-market (MTM) – allows to assess the impact on the credit value of any change in its risk.

- Individual models – focus on the changes of a credit value, regardless the correlations with other credits in the portfolio.
- Portfolio models – incorporate the correlations between the several assets of a credit portfolio.

Challenges in Estimating Portfolio Credit Risk

- Non-normal returns - credit returns are highly skewed and fat-tailed.
- Difficulty in modeling correlations - lack of data, contrary to equities.



Source: Riskmetrics Group (2007), “CreditMetrics – Technical Document”.

Credit-VaR

- CreditMetrics is a portfolio model that estimates the Credit-Var, taking into account the current ratings of bonds, the transition frequencies and the correlation between bond prices.

- We will start by considering a very simple portfolio, comprising just one BBB 5y-bond at par (redemption value of 100), with an annual coupon rate of 6%, paid annually.

- To calculate the potential losses 1 year ahead, due to rating downgrades, we need to obtain the value of the bond after rating migrations in 1 year.

↓

- This can be done by using forward interest rates for each rating level to discount the remaining cash-flows 1 year ahead (the 4 remaining coupons and the redemption value).

Credit-VaR

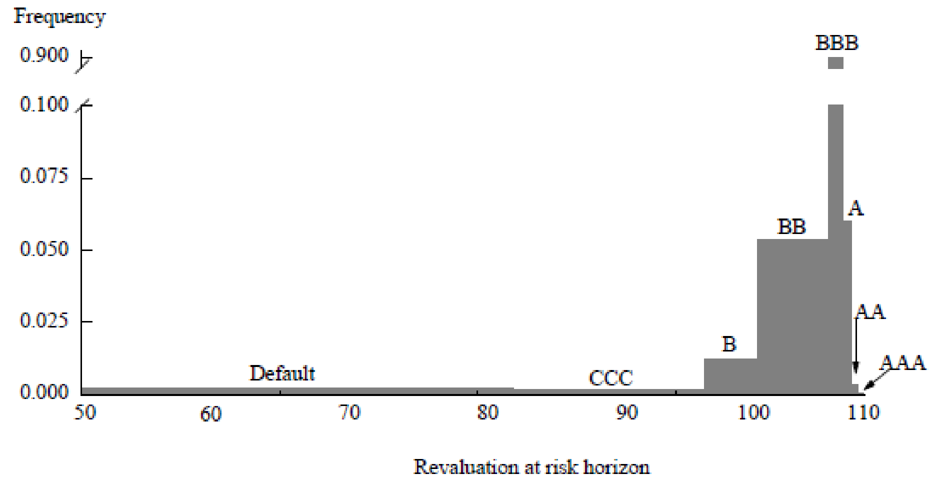
- Credit-Var = Unexpected loss due to credit risk increase = Difference between the price after a very unlikely unfavorable event and the expected value of the future price at the risk horizon.

- Credit-Var = Difference between the mean of the distribution and a value at the left tail

Distribution of value of a BBB par bond in one year

Year-end rating	Value ($)	Probability (%)
AAA	109.37	0.02
AA	109.19	0.33
A	108.66	5.95
BBB	107.55	86.93
BB	102.02	5.30
B	98.10	1.17
CCC	83.64	0.12
Default	51.13	0.18

Distribution of value for a 5-year BBB bond in one year



Note: The default price is the expected recovery rate.

Source: JPMorgan (1997), "CreditMetrics - Technical document"

Credit-VaR

- Calculation of prices for the risk horizon requires:

(i) Obtain the forward interest rate curves for each rating (m=risk horizon; n=maturity at the risk horizon of the remaining cash-flows)

(ii) Calculate the NPV at the risk horizon of the remaining cash-flows until maturity.

Category	Year 1	Year 2	Year 3	Year 4
AAA	3.60	4.17	4.73	5.12
AA	3.65	4.22	4.78	5.17
A	3.72	4.32	4.93	5.32
BBB	4.10	4.67	5.25	5.63
BB	5.55	6.02	6.78	7.27
B	6.05	7.02	8.03	8.52
CCC	15.05	15.02	14.03	13.52

- Forward Price if the upgrade from BBB to rating A occurs:

$$V = 6 + \frac{6}{(1+3.72\%)} + \frac{6}{(1+4.32\%)^2} + \frac{6}{(1+4.93\%)^3} + \frac{106}{(1+5.32\%)^4} = 108.66$$

Source: JPMorgan (1997), “CreditMetrics - Technical document”

Credit-VaR

- **1-year 99% Credit-Var** = Mean-$P_{1,B}$ (as the probability of having 1 year ahead a rating not above B = P(B) + P(CCC) + P(D)) = 1,17 + 0,12 + 0,18 ≈ 1% = 107,09-98,1 = **9**.

$$Mean = p_1 \cdot V_1 + p_2 \cdot V_2 + \ldots + p_{64} \cdot V_{64}$$

Ratings	Probability of Transition (%) (1)	Loan Value at year-end (2)	Difference to the mean (3)=(2)-μ	Contribution to the variance (4)=(1)x(3)^2
AAA	0.02	109.37	2.27	0.00
AA	0.33	109.19	2.09	0.0
A	5.96	108.66	1.56	0.15
BBB	86.93	107.55	0.45	0.18
BB	5.3	102.02	-5.08	1.37
B	1.17	98.10	-9.00	0.95
CCC	0.12	83.64	-23.46	0.66
Default	0.18	51.13	-55.97	5.64
Mean (μ)		107.09		
Variance (Σ(4))		8.95		
Standard-dev.		2.99		

Credit-VaR

- Now we add a single-A 3y bond, with annual coupon rate of 5%.
- Year-end price of the single-A 3y bond, after the several potential rating migrations 1 year ahead:

	Year-end Bond Price	Probability of Transition (%)
AAA	106.59	0.09
AA	106.49	2.27
A	106.30	91.05
BBB	105.64	5.52
BB	103.15	0.74
B	101.39	0.60
CCC	88.71	0.01
Default	51.13	0.06

The rating where the cumulative probability of 1% is crossed is BB (0,74+0,60+0,01+0,06=1,41), where the price is 103,15.

1-year 99% Credit-Var of the standalone A-bond = Mean - $P_{1,BB}$ = 106,54 – 103,15 = 3,39 (lower than the B-bond, as the credit risk is lower).

Source: Riskmetrics Group (2007), "CreditMetrics – Technical Document".

Credit-VaR

- All potential values of the portfolio will result from the combination of the 8 potential values for each bond (8x8):

All possible 64 year-end values for a two-bond portfolio ($)

Obligor #1 (BBB)		Obligor #2 (single-A)							
		AAA	AA	A	BBB	BB	B	CCC	Default
		106.59	106.49	106.30	105.64	103.15	101.39	88.71	51.13
AAA	109.37	215.96	215.86	215.67	215.01	212.52	210.76	198.08	160.50
AA	109.19	215.78	215.68	215.49	214.83	212.34	210.58	197.90	160.32
A	108.66	215.25	215.15	214.96	214.30	211.81	210.05	197.37	159.79
BBB	107.55	214.14	214.04	213.85	213.19	210.70	208.94	196.26	158.68
BB	102.02	208.61	208.51	208.33	207.66	205.17	203.41	190.73	153.15
B	98.10	204.69	204.59	204.40	203.74	201.25	199.49	186.81	149.23
CCC	83.64	190.23	190.13	189.94	189.28	186.79	185.03	172.35	134.77
Default	51.13	157.72	157.62	157.43	156.77	154.28	152.52	139.84	102.26

Source: Riskmetrics Group (2007), “CreditMetrics – Technical Document”.

Credit-VaR

- The joint probabilities would just be product of the rating migration probability for each bond, if these ratings were independent.

Joint migration probabilities with zero correlation (%)

Obligor #1 (BBB)		Obligor #2 (single-A) AAA	AA	A	BBB	BB	B	CCC	Default
		0.09	2.27	91.05	5.52	0.74	0.26	0.01	0.06
AAA	0.02	0.00	0.00	0.02	0.00	0.00	0.00	0.00	0.00
AA	0.33	0.00	0.01	0.30	0.02	0.00	0.00	0.00	0.00
A	5.95	0.01	0.14	5.42	0.33	0.04	0.02	0.00	0.00
BBB	86.93	0.08	1.98	79.15	4.80	0.64	0.23	0.01	0.05
BB	5.30	0.00	0.12	4.83	0.29	0.04	0.01	0.00	0.00
B	1.17	0.00	0.03	1.06	0.06	0.01	0.00	0.00	0.00
CCC	0.12	0.00	0.00	0.11	0.01	0.00	0.00	0.00	0.00
Default	0.18	0.00	0.00	0.16	0.01	0.00	0.00	0.00	0.00

Source: Riskmetrics Group (2007), “CreditMetrics – Technical Document”.

Credit-VaR

- However, ratings do not tend to be independent, as they may be moved by the same macroeconomic factors.

- **Joint rating migration probabilities with correlated bonds:**

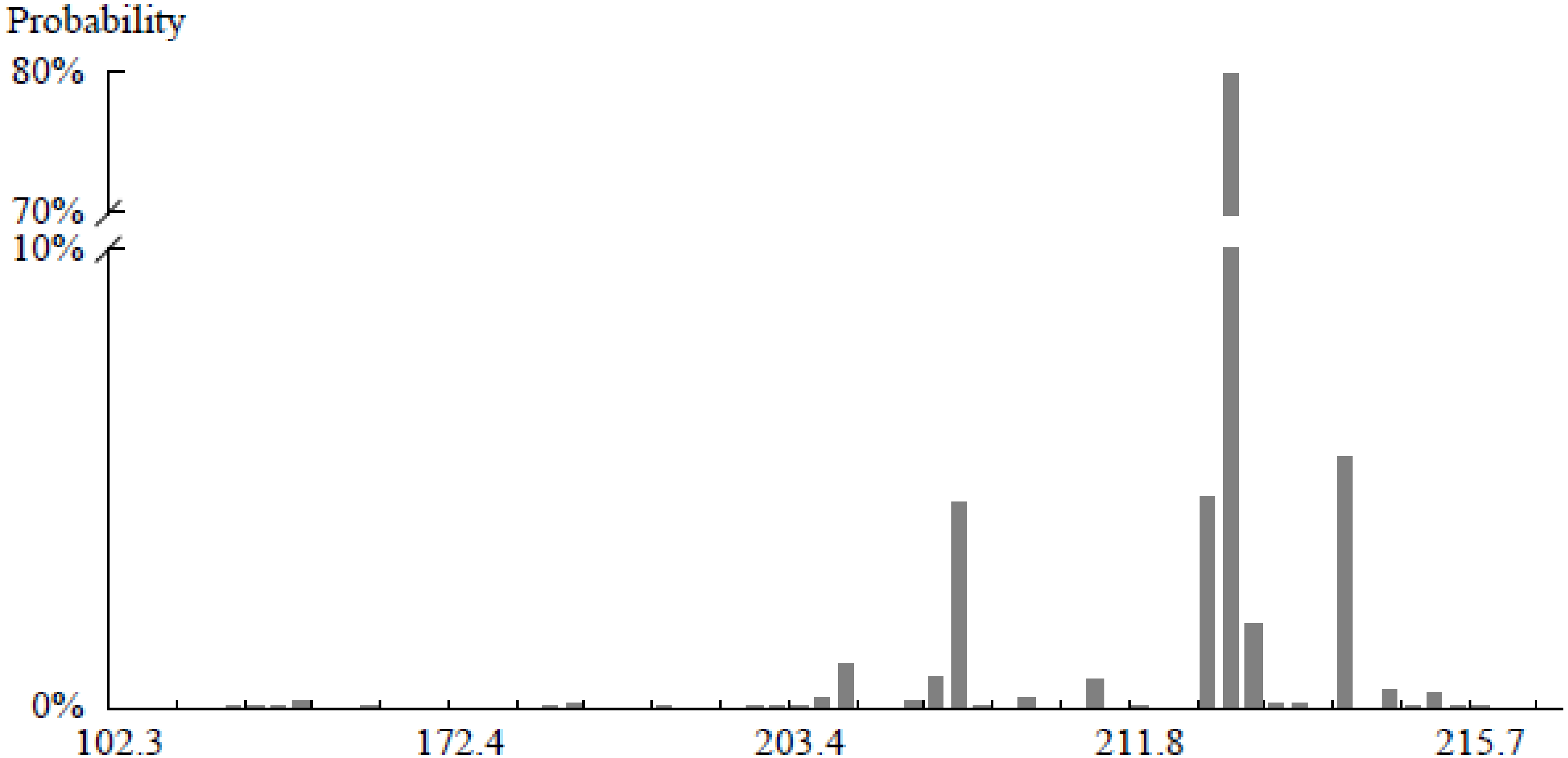
Joint migration probabilities with 0.30 asset correlation (%)

Obligor #1 (BBB)		Obligor #2 (single-A)							
		AAA	AA	A	BBB	BB	B	CCC	Default
		0.09	2.27	91.05	5.52	0.74	0.26	0.01	0.06
AAA	0.02	0.00	0.00	0.02	0.00	0.00	0.00	0.00	0.00
AA	0.33	0.00	0.04	0.29	0.00	0.00	0.00	0.00	0.00
A	5.95	0.02	0.39	5.44	0.08	0.01	0.00	0.00	0.00
BBB	86.93	0.07	1.81	79.69	4.55	0.57	0.19	0.01	0.04
BB	5.30	0.00	0.02	4.47	0.64	0.11	0.04	0.00	0.01
B	1.17	0.00	0.00	0.92	0.18	0.04	0.02	0.00	0.00
CCC	0.12	0.00	0.00	0.09	0.02	0.00	0.00	0.00	0.00
Default	0.18	0.00	0.00	0.13	0.04	0.01	0.00	0.00	0.00

Source: Riskmetrics Group (2007), “CreditMetrics – Technical Document”.

Credit-VaR

Distribution of value for a portfolio of two bonds



Source: Riskmetrics Group (2007), “CreditMetrics – Technical Document”.

Credit-VaR

There are at least four interesting features in the joint likelihood table above:

1. The probabilities across the table necessarily sum to 100%.
2. The most likely outcome is that both obligors simply remain at their current credit ratings. In fact, the likelihoods of joint migration become rapidly smaller as the migration distance grows.
3. The effect of correlation is generally to increase the joint probabilities along the diagonal drawn through their current joint standing (in this case, through BBB-A).
4. The sum of each column or each row must equal the chance of migration for that obligor standing alone. For instance, the sum of the last row must be 0.18%, which is the default likelihood for Obligor #1 (BBB) in isolation.

Source: Riskmetrics Group (2007), "CreditMetrics – Technical Document".

Credit-VaR

$$\text{Mean:}\quad \mu_{Total} = \sum_{i=1}^{S=64} p_i \mu_i = 213.63$$

$$\text{Variance:}\quad \sigma^2_{Total} = 11.22$$

	BBB Bond	A Bond	Portfolio
Mean	107,10	106,53	213,63
St.-Dev.	2,99	1,49	3,35

sq. root of 11.22

Conclusion: The means of the BBB and the A bonds sum directly, but the risk (standard deviations) is much less than the summed individual risks, due to diversification.

1-year 99% Credit-VaR = Mean-$P^P_{1,(B,A)}$ (as the probability of having 1 year after a rating not above B in the 1st bond and A in the 2nd bond = P(B,A)+P(B,BBB)+…+P(D,D) = 0,92+0,18+…+0 =1,45 ≈ 1%) = 213,63-204,4 = 9,23.

Source: Riskmetrics Group (2007), "CreditMetrics – Technical Document".

Credit-VaR

With zero correlation, the ratings considered to calculate the 99% Credit-VaR would be the same, as the cumulative probability = 1% is only achieved at the rating combination (B,A), but the actual degree of freedom would be slightly smaller:

1-year 99% Credit-VaR = Mean-$P^{P}_{1,(B,A)}$ (as the probability of having 1 year after a rating not above B in the 1st bond and A in the 2nd bond = P(B,A)+P(B,BBB)+…+P(D,D) = 0,92+0,18+…+0 =1,45 ≈ 1%) = 213,63-204,4 = **9,23**.

- Assuming a normal distribution, **the VaR would be**:

$$VaR(\alpha\%) = N(1-\alpha\%)*\sigma =>$$

$$VaR(5\%) = 1.65*\sigma = 1.65*3.35=5.53$$

VaR(1%) = 2.33*σ = 2.33*3.35=7.81 (lower than the observed value - 9,23 - due to fat tails)

Credit-VaR

- The decision to hold a bond or not is likely to be made within the context of some existing portfolio.
- Thus, the more relevant calculation is the **marginal increase to the portfolio risk** that would be created by adding a new bond to it = **0,36** in standard-deviation and **0,24** in Credit-VaR.
- This increase in Credit-VaR is much smaller than the A-Bond 99% Credit-Var (3,39) due to the diversification effect.

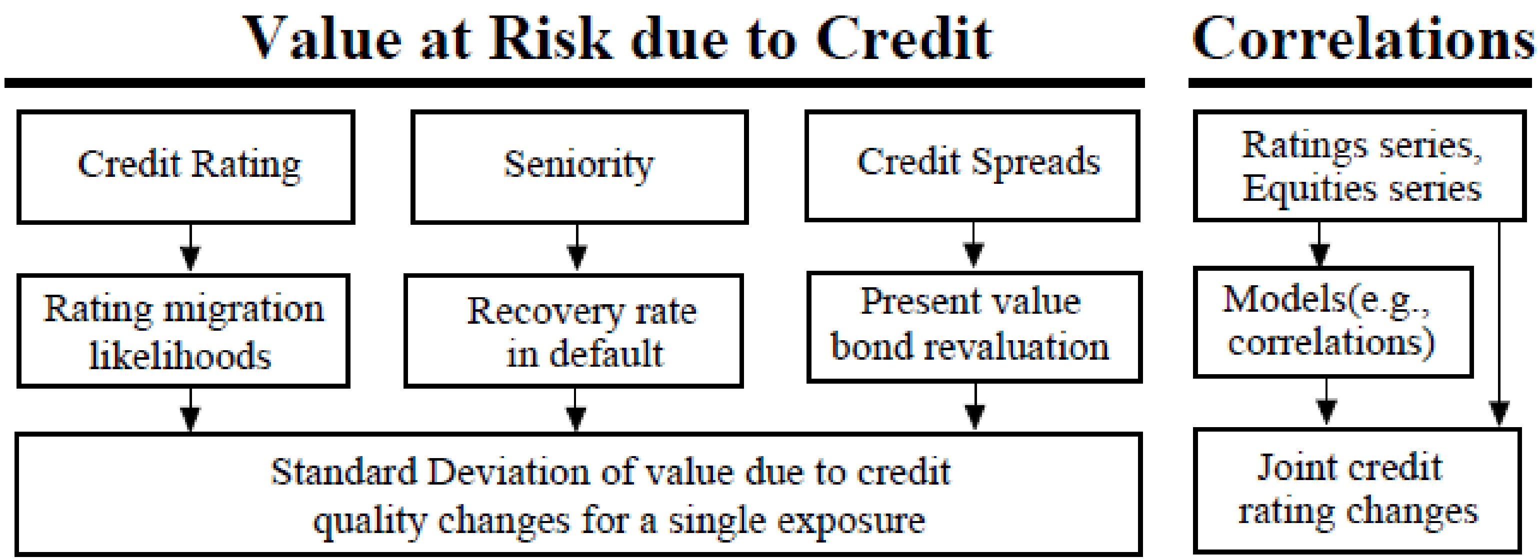
	BBB-Bond (1)	A-Bond (2)	Portfolio (3)	A-Bond Marginal Risk (4) = (3)-(1)
Standard-deviation	2,99	1,49	3,35	**0,36**
99% Credit-Var	8,99	3,39	9,23	**0,24**

Source: Riskmetrics Group (2007), "CreditMetrics – Technical Document".

Creditmetrics

In our examples of one and two bond portfolios, we have been able to specify the entire distribution of values for the portfolio. We remark that this becomes inconvenient, and finally impossible, to do this in practice as the size of the portfolio grows. Noting that for a three asset portfolio, there are 512 (that is, 8 times 8 times 8) possible joint rating states. For a five asset portfolio, this number jumps to 32,768, and in general, for a portfolio with N assets, there are 8^N possible joint rating states.

Information required:



Source: Riskmetrics Group (2007), “CreditMetrics – Technical Document”.