2nd part → 2nd February 2023

1 a) $A\begin{pmatrix}0\\1\\-1\end{pmatrix} = \begin{bmatrix}2\alpha & \alpha & \alpha\\ \alpha & 2\alpha & 0\\ \alpha & 0 & 2\alpha\end{bmatrix}\begin{pmatrix}0\\1\\-1\end{pmatrix} = \begin{bmatrix}0\\2\alpha\\-2\alpha\end{bmatrix} = 2\alpha\begin{bmatrix}0\\1\\-1\end{bmatrix}$

Eigenvalue: 2α

b) $\alpha = 1$

i) $A = \begin{bmatrix}2 & 1 & 1\\ 1 & 2 & 0\\ 1 & 0 & 2\end{bmatrix}$

$\Delta_1 = 2 > 0$

$\Delta_2 = 3 > 0$

$\Delta_3 = 8 - 2 - 2 = 4 > 0$

By the method of leading minors, we conclude that Q is positively defined.

ii) $X^T A X = 0 \Leftarrow X = (0,0,0)$
$\uparrow$
A is positively defined.

2a) $(x,y) \neq (0,0) \Rightarrow f$ is continuous because it is the product of continuous functions whose denominator is different from 0.

$\therefore f$ is continuous in $\mathbb{R}^2 \setminus \{(0,0)\}$.

$(x,y)=(0,0)$

$$0\le\left|\frac{2x^2y^2}{\sqrt{x^2+y^2}}\right|\le\frac{2.|x^2+y^2|.y^2}{\sqrt{x^2+y^2}}=2\sqrt{x^2+y^2}.y^2$$

Since $\lim\limits_{(x,y)} 2\sqrt{x^2+y^2}.y^2=0$, then by the squeezing theorem, we have $\lim\limits_{(x,y)\to(0,0)} f(x,y)=0$

Since $\lim\limits_{(x,y)\to(0,0)} f(x,y)=f(0,0)$, then f is continuous on $(0,0)$.

b)

$$D_{(0,1)}f(1,1)=\frac{\partial f}{\partial y}(1,1)$$

$$\frac{\partial f}{\partial y}(x,y)=\frac{4x^2y.\sqrt{x^2+y^2}-2x^2y^2.\frac{1}{\cancel{2}}.\cancel{2}y.\frac{1}{\sqrt{x^2+y^2}}}{x^2+y^2}$$

$$\frac{\partial f}{\partial y}(1,1)=\frac{4.\sqrt{2}-\frac{2}{\sqrt{2}}}{2}=2\sqrt{2}-\frac{1}{\sqrt{2}}=\frac{4\sqrt{2}}{2}-\frac{\sqrt{2}}{2}=\frac{3\sqrt{2}}{2}$$

3 a) $f(x,y) = xy e^{x+y}$

$$\nabla f(x,y) = \left(y e^{x+y} + xy e^{x+y} ; x e^{x+y} + xy e^{x+y} \right)$$

$$\nabla f(x,y) = \bar{0} \Leftrightarrow \begin{cases} e^{x+y} y(1+x) = 0 \\ e^{x+y} . x(1+y) = 0 \end{cases} \Leftrightarrow \begin{cases} y = 0 \\ x = 0 \end{cases} \vee \begin{cases} x = -1 \\ y = -1 \end{cases}$$

$(0,0)$ and $(-1,-1)$

$$Hf(x,y) = \begin{pmatrix} y e^{x+y} + y e^{x+y} + xy e^{x+y} & e^{x+y} + y e^{x+y} + x e^{x+y} + xy e^{x+y} \\ e^{x+y} + y e^{x+y} + x e^{x+y} + xy e^{x+y} & x e^{x+y} + x e^{x+y} + xy e^{x+y} \end{pmatrix}$$

$$Hf(0,0) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$\Delta_1 = 0$

$\Delta_2 = -1 \neq 0$

$\therefore (0,0)$ is a saddle-point

$$Hf(-1,-1) = \begin{pmatrix} -e^{-2} & 0 \\ 0 & -e^{-2} \end{pmatrix}$$

$\Delta_1 = -e^{-2}$

$\Delta_2 = (e^{-2})^2 > 0$

$(-1,-1)$ is a maximizer.

3 b).

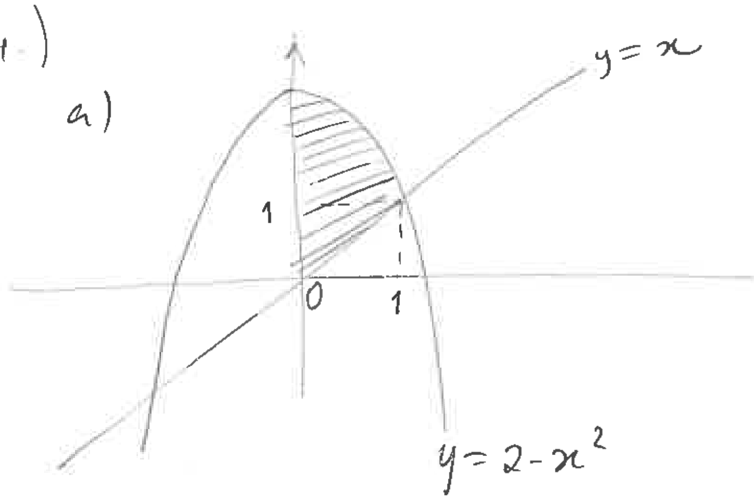
$$\begin{cases} y = 1 \\ \lim\limits_{x \to +\infty} f(x,y) = \lim\limits_{x \to +\infty} x e^{x+1} = +\infty \end{cases}$$

$$\begin{cases} \lim\limits_{x \to +\infty} f(x,y) = \lim\limits_{x \to +\infty} -x e^{x-1} = -\infty \\ y = -1 \end{cases}$$

$\therefore$ f does not have global extrema.

4.)

a)



b) $\iint_\Omega xy \, dx\, dy = \int_0^1 \int_x^{2-x^2} xy \, dy \, dx =$

$= \int_0^1 \left[x \frac{y^2}{2} \right]_x^{2-x^2} dx =$

$= \int_0^1 x . \frac{(2-x^2)^2}{2} - \frac{x^3}{2} \, dx =$

$$= \int_0^1 \frac{x.(4-4x^2+x^4)}{2} - \frac{x^3}{2}\,dx$$

$$= \frac{1}{2}\int_0^1 4x - 5x^3 + x^5 \quad dx$$

$$= \frac{1}{2}\left[2x^2 - \frac{5}{4}x^4 + \frac{x^6}{6}\right]_0^1 = \frac{1}{2}\left(\frac{2}{1} - \frac{5}{4} + \frac{1}{6}\right) = \left(\frac{24-15+2}{6.2}\right)$$

$$= \frac{11}{24}$$

5.

$$\begin{cases} y'' + 2y' + y = x^2 \\ y'(0) = 1 \\ y(0) = 0 \end{cases}$$

$P(\lambda) = (\lambda+1)^2 \qquad P(\lambda) = 0 \Leftrightarrow \lambda = -1_{(2)}$

$y_{hom}(x) = C_1 e^{-x} + C_2\, x e^{-x} \quad , C_1, C_2 \in \mathbb{R}.$

$y_{part}(x) = Ax^2 + Bx + C$

$\Downarrow$ solving system

$$\begin{cases} A = 1 \\ B = -4 \\ C = 6. \end{cases} \qquad y_{part}(x) = x^2 - 4x + 6.$$

$$y(x) = C_1 e^{-x} + C_2 x e^{-x} + (x^2 - 4x + 6)$$

$$\begin{cases} y'(0) = 1 \\ y(0) = 0 \end{cases} \Rightarrow \begin{cases} -C_1 e^{-0} + C_2 e^{-0} - C_2 \cdot 0 \cdot e^{-0} + 2x - 4 = 1 \\ C_1 + 0 + 6 = 0 \end{cases}$$

$$\Leftrightarrow \begin{cases} C_2 = -1 \\ C_1 = -6 \end{cases}$$

$$\therefore y(x) = -6 e^{-x} - x e^{-x} + (x^2 - 4x + 6)$$

$x \in \mathbb{R}$.