Decision Making and Optimization

Master in Data Analytics for Business

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Decision Analysis

Decision analysis

Decision analysis is a systematic approach to making informed decisions, especially in complex and uncertain situations. It involves identifying, evaluating and comparing different options to determine the best possible choice.

Decision analysis is widely used in areas such as business, engineering, health and public policy to ensure that decisions are based on a thorough and rational assessment of the available options

Decision analysis: main components

- Problem Definition: Clearly identify the problem or decision to be made.
- Alternatives: List all possible options/actions.
- Nature States: List all possible outcomes.
- Decision Criteria: Define the criteria that will be used to help with decisions.
- Evaluation: Define the payoffs or costs that will be used to evaluate the alternatives.
- Visualization: Use tools such as decision trees and influence diagrams to visualize the options and their possible outcomes.
- Probabilities: Assign probabilities to the different outcomes and calculate the expected value of each alternative.
- Sensitivity Analysis: Evaluate how changes in assumptions can affect the final decision.

Decision analysis: problem definition

1. Alternative actions: $A = \{a_1, a_2, \ldots, a_m\}$
   - Identify and enumerate **ALL** actions so that
   - No action is ignored, **EXHAUSTIVE**
   - Avoid duplication or multiple choice, **EXCLUSIVE**
   - Scope: select one, and only **ONE** action in A
2. Nature States: $\Theta = \{\theta_1, \theta_2, \ldots, \theta_n\}$ all the possible outcomes of the actions
   - Identify and enumerate **ALL** nature states so that
   - No nature state is ignored, **EXHAUSTIVE**
   - Avoid duplication or ambiguity, **EXCLUSIVE**
   - One and only **ONE** state occurs!
   - The decision maker only knows the state after the action has been chosen

Decision analysis: problem definition

3. Payoffs or Costs matrix
   - Evaluate actions according to their consequences and the decision maker's preferences for those consequences
   - $p(a_i, \theta_j)$ cost or payoff from making decision $a_i \in A$ when the nature state is $\theta_j \in \Theta$, for $i = 1, \ldots, m;\ j = 1, \ldots, n$

Eliminate any dominated action:

- for the expected return, action $a_u \in A$ is dominated by action $a_v \in A$ if
  $p(a_u, \theta_j) \leq p(a_v, \theta_j)$ for all nature state $\theta_j \in \Theta$
  and for one θ_j we have $p(a_u, \theta_j) < p(a_v, \theta_j)$
- for the expected costs, action $a_u \in A$ is dominated by action $a_v \in A$ if
  $p(a_u, \theta_j) \geq p(a_v, \theta_j)$ for all nature state $\theta_j \in \Theta$
  and for one θ_j we have $p(a_u, \theta_j) > p(a_v, \theta_j)$

this is, a_u is never better than a_v

Example

The National Outdoors School (NOS) is preparing a summer camp in the heart of Alaska to train people in wilderness survival. NOS estimates that attendance will fall into one of four categories: 200, 250, 300 and 350 people. The cost of the campsite will be lowest if it is exactly the right size. Deviations above or below the ideal level of demand will result in additional costs due to building more capacity than needed or lost income opportunities when demand is not met.

		demand			
		200	250	300	350
Size built	200	5	10	18	25
	250	8	7	12	23
	300	21	18	12	21
	350	30	22	19	15

alternatives, nature states and payoff matrix

Actions:

a_1 built a camp with size 200
a_2 built a camp with size 250
a_3 built a camp with size 300
a_4 built a camp with size 350

Nature States:

θ_1 camp satisfy a demand of 200
θ_2 camp satisfy a demand of 250
θ_3 camp satisfy a demand of 300
θ_4 camp satisfy a demand of 350

the costs $p(a_i, \theta_j)$ are

		demand			
		θ_1	θ_2	θ_3	θ_4
Size built	a_1	5	10	18	25
	a_2	8	7	12	23
	a_3	21	18	12	21
	a_4	30	22	19	15

Decision making criteria

When probability distribution in Θ is either unknown or cannot be determined, use Decision Criteria without probabilities such as:

- Maxmin/Minmax
- Savage regret
- Laplace
- Hurwicz

When probability distribution in Θ is known or can be determined, use Decision Criteria with probabilities such as:

- Maximum Likelihood criteria
- Bayes probabilistic criteria

Decision making without probabilities

Maxmin/Minmax

Conservative $\longleftarrow$ The best from the worst

decision: take a_i that

$$\max_{a_i \in A}\{\min_{\theta_j \in \Theta} p(a_i, \theta_j)\}$$

when P is a payoffs matrix

$$\min_{a_i \in A}\{\max_{\theta_j \in \Theta} p(a_i, \theta_j)\}$$

when P is a costs matrix

Savage regret

Moderate the degree of conservatism of the Maxmin/Minmax criteria

calculate

$$r(a_i, \theta_j) = \max_{a_k \in A}\{p(a_k, \theta_j)\} - p(a_i, \theta_j)$$

when P is a payoffs matrix

$$r(a_i, \theta_j) = p(a_i, \theta_j) - \min_{a_k \in A}\{p(a_k, \theta_j)\}$$

when P is a costs matrix

decision: take a_i that

$$\min_{a_i \in A}\{\max_{\theta_j \in \Theta}\{r(a_i, \theta_j)\}\}$$

Laplace

Laplace assumes that all state occur with the same probability

decision: take a_i that

$$\max_{a_i \in A}\{\frac{1}{n}\sum_{j=1}^{n} p(a_i, \theta_j)\}$$

when P is a payoffs matrix

$$\min_{a_i \in A}\{\frac{1}{n}\sum_{j=1}^{n} p(a_i, \theta_j)\}$$

when P is a costs matrix

Example

		demand				Minmax	SavReg	Laplace
		θ_1	θ_2	θ_3	θ_4	$\max_{\theta_j \in \Theta}$	$r(a_i, \theta_j)$	$\frac{1}{n}\sum_{j=1}^{n} p(a_i, \theta_j)$
Size built	a_1	5	10	18	25	25	10	14.5
	a_2	8	7	12	23	23	8	12.5
	a_3	21	18	12	21	21	16	18
	a_4	30	22	19	15	30	25	21.5
$\min_{a_k \in A}\{p(a_k, \theta_j)\}$		5	7	12	15			

Minmax: $\min_{a_i \in A}\{\max_{\theta_j \in \Theta} p(a_i, \theta_j)\} = 21$ take a_3

Savage Regret: $\min_{a_i \in A}\{\max_{\theta_j \in \Theta}\{r(a_i, \theta_j)\}\} = 8$ take a_2

Laplace: $\min_{a_i \in A}\{\frac{1}{n}\sum_{j=1}^{n} p(a_i, \theta_j)\} = 12.5$ take a_2

Hurwick Index of optimism

Index of optimism $0 \leq \alpha \leq 1$ ($\alpha = 0$ minimax criteria; $\alpha = 1$ liberal)

decision: take a_i that

$$\max_{a_i \in A}\{\alpha \max_{\theta_j \in \Theta} p(a_i, \theta_j) + (1-\alpha) \min_{\theta_j \in \Theta} p(a_i, \theta_j)\}$$

when P is a payoffs matrix

$$\min_{a_i \in A}\{\alpha \min_{\theta_j \in \Theta} p(a_i, \theta_j) + (1-\alpha) \max_{\theta_j \in \Theta} p(a_i, \theta_j)\}$$

when P is a costs matrix

Example

	demand				Hurwick			
	θ_1	θ_2	θ_3	θ_4	$\min_{\theta_j}$	$\max_{\theta_j}$	$\alpha \min_{\theta_j} + (1-\alpha)\max_{\theta_j}$	$\alpha = 0.5$
a_1	5	10	18	25	5	25	25-20α	15
a_2	8	7	12	23	7	23	23-16α	15
a_3	21	18	12	21	12	21	21- 9α	16.5
a_4	30	22	19	15	15	30	30-15α	22.5

Hurwick: $\min_{a_i \in A}\{\alpha \min_{\theta_j \in \Theta} p(a_i, \theta_j) + (1-\alpha)\max_{\theta_j \in \Theta} p(a_i, \theta_j)\} = 15$

take a_1 or a_2

$\alpha = 0.25$ the best action is a_3 the minmax action!

Decision making with probabilities

Maximum Likelihood

An *a priori* probability distribution for Θ is known.
The decision maker knows probability *a priori* of θ_k: $h_\theta(\theta_k) = P(\theta = \theta_k)$,

$$\sum_{k=1}^{n} h_\theta(\theta_k) = 1$$

The action that maximizes the expected return is

$$\max_{a_i \in A}\{p(a_i, \theta_k)\} \text{ with } \theta_k \text{ having } \max_{\theta_j \in \Theta} h(\theta_j)$$

(when P is a payoff matrix)

The action that minimizes the expected cost is

$$\min_{a_i \in A}\{p(a_i, \theta_k)\} \text{ with } \theta_k \text{ having } \max_{\theta_j \in \Theta} h(\theta_j)$$

(when P is a costs matrix)

Example

a priori distribution of θ_k: $h_\theta(\theta_k) = P(\theta = \theta_k)$ is known

		demand			
		θ_1	θ_2	θ_3	θ_4
Size built	a_1	5	10	18	25
	a_2	8	7	12	23
	a_3	21	18	12	21
	a_4	30	22	19	15
	$h_\Theta(\theta_j)$	0.1	0.2	0.3	0.4

Maximum Likelihood: $\min_{a_i \in A}\{p(a_i, \theta_k)\} = 15$ take a_4

Bayes criteria

An *a priori* probability distribution for Θ is known.
The decision maker knows probability *a priori* of θ_k: $h_\theta(\theta_k) = P(\theta = \theta_k)$,

$$\sum_{k=1}^{n} h_\theta(\theta_k) = 1$$

The Bayes action that maximizes the expected return is

$$\max_{a_i \in A}\{\sum_{j=1}^{n} h(\theta_j) p(a_i, \theta_j)\} = \max_{a_i \in A}\{E(p(a_i, \theta))\}$$

(when P is a payoff matrix)

The Bayes action that minimizes the expected cost is

$$\min_{a_i \in A}\{\sum_{j=1}^{n} h(\theta_j) p(a_i, \theta_j)\} = \min_{a_i \in A}\{E(p(a_i, \theta))\}$$

(when P is a costs matrix)

Example

a priori distribution of θ_k: $h_\theta(\theta_k) = P(\theta = \theta_k)$ is known

		demand				Bayes
		θ_1	θ_2	θ_3	θ_4	$\sum_{j=1}^{n} h(\theta_j)p(a_i, \theta_j)$
Size built	a_1	5	10	18	25	17.9
	a_2	8	7	12	23	15
	a_3	21	18	12	21	17.7
	a_4	30	22	19	15	19.1
	$h_\theta(\theta_j)$	0.1	0.2	0.3	0.4	

Bayes: $\min_{a_i \in A}\{\sum_{j=1}^{n} h(\theta_j)p(a_i, \theta_j)\} = 15$ take a_2

Exercise

A company owns a tract of land that may contain oil. A consulting geologist has reported that she believes there is a 1 chance in 4 of oil. Because of this prospect, another oil company offered to purchase the land for $90,000. However, the company is considering holding the land in order to drill for oil itself. The cost of drilling is $100,000. If oil is found, the resulting expected revenue will be $800,000, so the company's expected profit (after deducting the cost of drilling) will be $700,000. A loss of $100,000 (the drilling cost) will be incurred if the land is dry (no oil).

Determine the action that should be selected using the following criterias: Minimax/Maximin; Savage Regret; Laplace; Hurwicz Index (take $\alpha = 0.5$); Maximum Likelihood; Bayes Criteria.

Expected Value of decision alternative a_i

An *a priori* probability distribution for Θ is known.

The decision maker knows *a priori* the probability of θ_k:

$$h_\theta(\theta_k) = P(\theta = \theta_k),$$

$$\sum_{k=1}^{n} h_\theta(\theta_k) = 1$$

The Expected Value of decision alternative a_i, $i = 1, \ldots, m$, is defined as

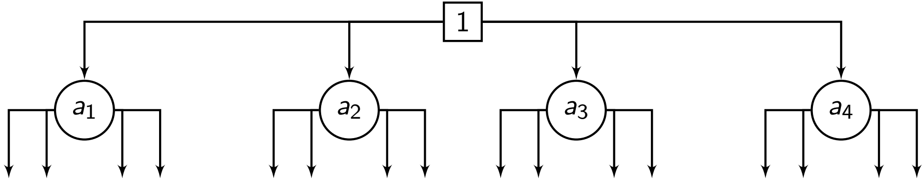
$$EV(a_i) = \sum_{j=1}^{n} h(\theta_j)\ p(a_i, \theta_j)$$

Example

		demand				$EV(a_i) =$
		θ_1	θ_2	θ_3	θ_4	$\sum_{j=1}^{n} h(\theta_j)p(a_i, \theta_j)$
Size built	a_1	5	10	18	25	17.9
	a_2	8	7	12	23	15
	a_3	21	18	12	21	17.7
	a_4	30	22	19	15	19.1
	$h_\theta(\theta_j)$	0.1	0.2	0.3	0.4	

Decision Tree

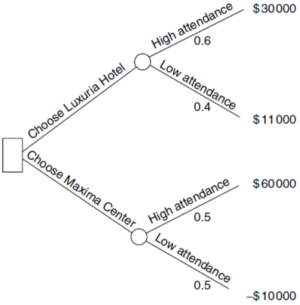
The primary benefit of a decision tree is that it provides an illustration (or picture) of the decision-making process. This makes it easier to correctly compute the necessary expected values and to understand the process of making the decision.



$EV(a_1) = 17.9$ $EV(a_2) = 15$ $EV(a_3) = 17.7$ $EV(a_4) = 19.1$

Decision tree: example

A committee is organizing an event and has to choose between two venues: the Luxuria Hotel and the Maxima Center.
They estimate the potential profit at these locations on the basis of two scenarios: high attendance and low attendance.
If the Luxuria Hotel is chosen, there is a 60% chance that the attendance will be high, resulting in a profit of $30 000 (after expenses). However, there is a 40% chance that the attendance will be low, in which case the profit will be only $11 000. If the Maxima Center is chosen, there is a 50% chance of high attendance, resulting in a profit of $60 000, and a 50% chance of low attendance, resulting in a loss of $10 000.



Expected Value of Perfect Information

Suppose there is an opportunity to conduct a research study that would provide information that could be used to improve the probability estimates for the states of nature. To determine the potential value of this information, we first assume that the study could provide perfect information about the states of nature; that is, we assume that it could determine with certainty which state of nature will occur before a decision is made. To take advantage of this perfect information, we will develop a decision strategy that should be followed once it is known which state of nature will occur. A decision strategy is simply a decision rule that specifies the decision alternative to be selected after new information becomes available.

Example

		demand				$EV(a_i) =$
		θ_1	θ_2	θ_3	θ_4	$\sum_{j=1}^{n} h(\theta_j)p(a_i, \theta_j)$
Size built	a_1	5	10	18	25	17.9
	a_2	8	7	12	23	15
	a_3	21	18	12	21	17.7
	a_4	30	22	19	15	19.1
	$h_\theta(\theta_j)$	0.1	0.2	0.3	0.4	

Suppose Nature State is known:

- θ_1 the best action would be a_1 with cost 5
- θ_2 the best action would be a_2 with cost 7
- θ_3 the best action would be a_2 or a_3 with cost 12
- θ_4 the best action would be a_4 with cost 15

thus the expected return would be
$0.1 \times 5 + 0.2 \times 7 + 0.3 \times 12 + 0.4 \times 15 = 11.5$

Expected Value of Perfect Information

Expected value with perfect information

$$EVwPI = 11.5$$

Expected value without perfect information

$$EVwoPI = 15$$

Expected Value of Perfect Information

$$EVPI = |EVwPI \;-\; EVwoPI| = 15 - 11.5 = 3.5$$

Expected Value of Perfect Information

A research study will not provide "perfect" information. However, if the research study is a good one, the information gathered could be worth a significant portion of the EVPI.

Given the EVPI value, one might seriously consider a study as a way to get more information about the states of nature.

Example

A businessman wants to decide where to invest 100 000 €. He has to choose one of three alternative projects, P1, P2 or P3. The return on the projects depends on the performance of the economy, which may stagnate or improve. The following table shows the data

	economy	
	stagnates	improves
P_1	7%	5%
P_2	−10%	14%
P_3	6%	6%

Actions:

- a_1 invest in P_1
- a_2 invest in P_2
- a_3 invest in P_3

$A = \{a_1, a_2, a_3\}$

Nature states:

- θ_1 economy stagnates
- θ_2 economy improves

$\Theta = \{\theta_1, \theta_2\}$

Payoffs matrix:

	θ_1	θ_2
a_1	7000	5000
a_2	−10000	14000
a_3	6000	6000

Example

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$A = \{a_1, a_2, a_3\}$

Nature states:

θ_1 economy stagnates
θ_2 economy improves

$\Theta = \{\theta_1, \theta_2\}$

Payoffs matrix:

	θ_1	θ_2
a_1	7000	5000
a_2	−10000	14000
a_3	6000	6000

Example

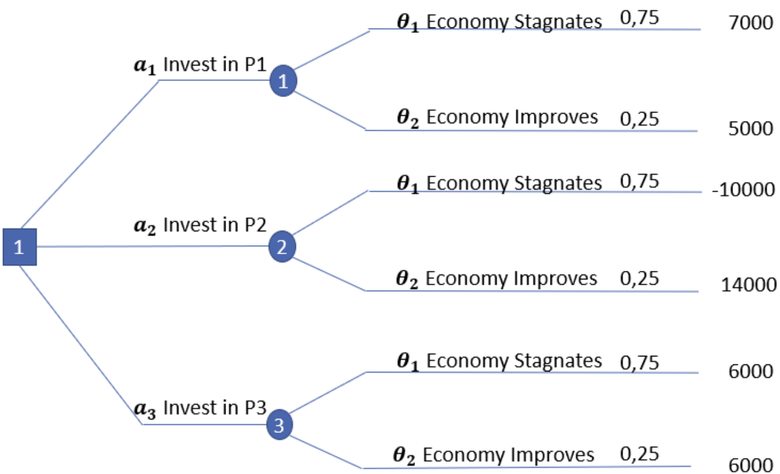
The probability that the economy will stagnate is 3 times higher than the probability that it will improve.

	θ_1	θ_2	$EV(a_i)$
a_1	7000	5000	$0.75 \times 7000 + 0.25 \times 5000 = 6500$
a_2	−10000	14000	$0.75 \times (-10000) + 0.25 \times 14000 = -4000$
a_3	6000	6000	$0.75 \times 6000 + 0.25 \times 6000 = 6000$
$h(\theta_j)$	$\frac{3}{4}$	$\frac{1}{4}$	

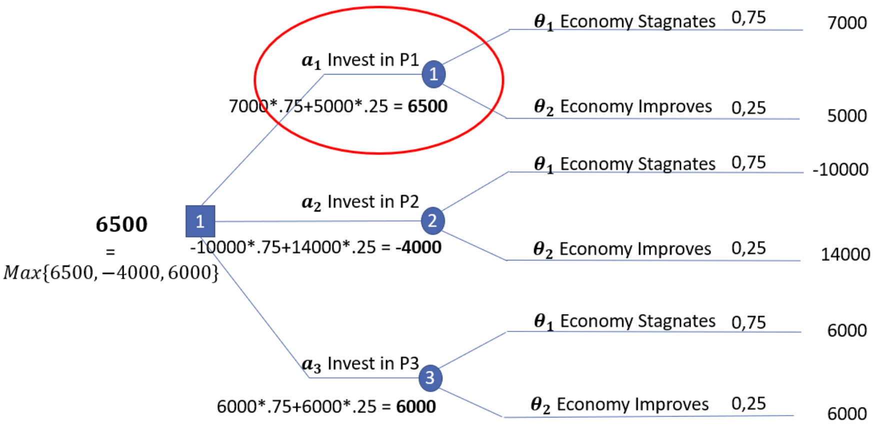
Maximum Likelihood:
$\max_{\theta_j \in \Theta} h(\theta_j) = \frac{3}{4}$ for θ_1 thus $\max_{a_i \in A} p(a_i, \theta_1) = 7000$ select a_1

Bayes criteria:
$\max_{a_i \in A} EV(a_i) = 6500$ select a_1

Example: decision tree



Example: decision tree



Example: EVPI

$EVwoPI = 6500$

if Nature state is known (in advance) to be:

θ_1 the best action would be a_1 with a return of 7000

θ_2 the best action would be a_2 with a return of 14000

thus

$EVwPI = 0.75 \times 7000 + 0.25 \times 14000 = 8750$

Maximum value the decision maker is willing to pay to remove uncertainty

$$EVPI = |EVwPI - EVwoPI| = 8750 - 6500 = 2250$$

Risk analysis and Sensitivity analysis

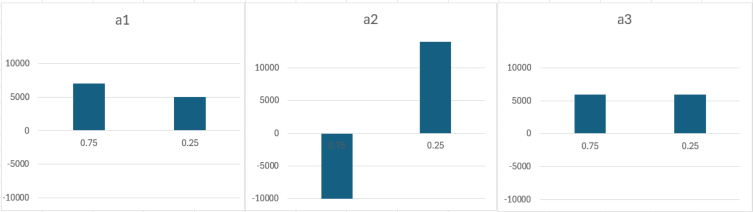
Risk analysis and Sensitivity analysis

Risk analysis helps the decision maker recognize the difference between the expected value of a decision alternative and the payoff/cost that may actually occur.

Sensitivity analysis also helps the decision maker by describing how changes in the state-of-nature probabilities and/or changes in the payoffs/costs affect the recommended decision alternative.

Risk analysis

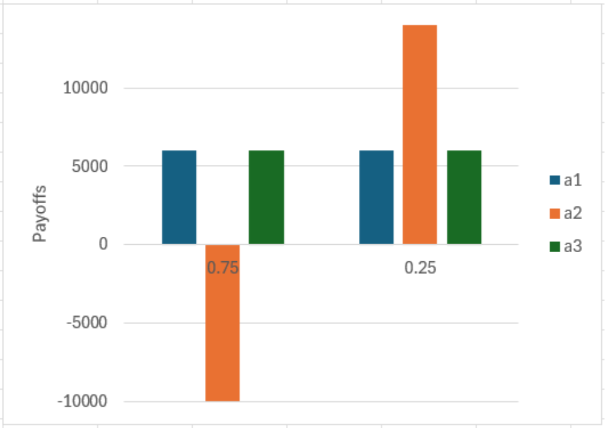
The risk profile for a decision alternative shows the possible payoffs (costs) along with their associated probabilities.



Alternative a_1 may be considered less risky than alternative a_2.

Risk analysis

The risk profile for a decision alternative shows the possible payoffs (costs) along with their associated probabilities.



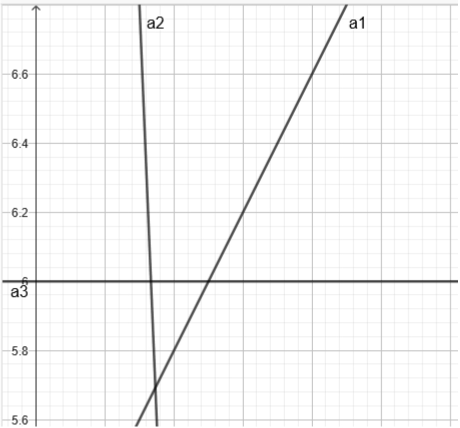
Sensitivity analysis: changes in the probabilities

	θ_1	θ_2	$EV(a_i)$
a_1	7000	5000	$p7000 + (1-p)5000 = 5000 + 2000p$
a_2	-10000	14000	$p(-10000) + (1-p)14000 = 14000 - 24000p$
a_3	6000	6000	$p6000 + (1-p)6000 = 6000$
$h(\theta_j)$	p	$1-p$	

Crossover Point is the point where the expected value lines of alternative intersect and the decision shifts from one alternative to the other.

$5 + 2p = 6 \Leftrightarrow p = 1/2 \Longrightarrow (1/2, 6)$
$5 + 2p = 14 - 24p \Leftrightarrow p = 9/26 \Longrightarrow (9/26, 130/26)$
$14 - 24p = 6 \Leftrightarrow p = 9/26 \Longrightarrow (1/3, 6)$

Sensitivity analysis: changes in the probabilities



- alternative a_2 provides the largest expected value for $0 \leq p \leq 1/3$
- alternative a_3 provides the largest expected value for $1/3 \leq p \leq 1/2$
- alternative a_1 provides the largest expected value for $1/2 \leq p \leq 1$

Sensitivity analysis: changes in the cost/payoff

	θ_1	θ_2	$EV(a_i)$
a_1	7000	5000	6500
a_2	-10000	14000	-4000
a_3	6000	6000	6000
$h(\theta_j)$	$\frac{3}{4}$	$\frac{1}{4}$	

According to Bayes' criterion, the best alternative, the one that $\max_i\{EV(a_i)\}$, is a_1 as

$$EV(a_1) = 6500 \geq EV(a_3) = 6000 \geq EV(a_2) = -4000$$

and the second best alternative is a_3 with $EV(a_3) = 6000$
Thus, alternative decision a_1 will remain optimal, as long as

$$EV(a_1) \geq 6000$$

Sensitivity analysis: changes in the cost/payoff

Let
a^* be the best alternative, according to some criterion,

$\bar{a}$ the second best alternative with expected value $EV(\bar{a})$ (for payoffs $EV(\bar{a}) < EV(a^*)$, for costs $EV(\bar{a}) > EV(a^*)$),

$p_j = p(a^*, \theta_j)$ the payoff/cost of alternative a^* for the nature state θ_j, thus $EV(a^*) = \sum_{j=1}^{n} h(\theta_j) p_j$

Hence, alternative a^* remains the best (optimal) as long as
$EV(a^*) > EV(\bar{a})$ for payoffs matrix and
$EV(a^*) < EV(\bar{a})$ for costs matrix

analyse the behaviour of $EV(a^*)$ for changes in a single value p_k

Sensitivity analysis: changes in the cost/payoff

how can a single value p_k change? $EV(a^*) = h(\theta_k)p_k + \sum_{j=1,j\neq k}^{n} h(\theta_j)p_j$
for payoffs matrix, alternative a^* remains the best (optimal) as long as

$$EV(a^*) = h(\theta_k)p_k + \sum_{j=1,j\neq k}^{n} h(\theta_j)p_j > EV(\bar{a})$$

$$\iff p_k > \frac{1}{h(\theta_k)}(EV(\bar{a}) - \sum_{j=1,j\neq k}^{n} h(\theta_j)p_j)$$

for costs matrix, alternative a^* remains the best (optimal) as long as

$$EV(a^*) = h(\theta_k)p_k + \sum_{j=1,j\neq k}^{n} h(\theta_j)p_j < EV(\bar{a})$$

$$\iff p_k < \frac{1}{h(\theta_k)}(EV(\bar{a}) - \sum_{j=1,j\neq k}^{n} h(\theta_j)p_j)$$

Example: changes in the cost/payoff

how can value $p_{11} = p(a^*, \theta_1)$ change?

$$EV(a^*) = 0.75 \times p_{11} + 0.25 \times 5000 > 6000 = EV(\bar{a})$$

$$\iff p_{11} > \frac{1}{0.75}(6000 - 0.25 \times 5000) = 6333,3(3)$$

how can value $p_{12} = p(a^*, \theta_2)$ change?

$$EV(a^*) = 0.75 \times 7000 + 0.25 \times p_{12} > 6000 = EV(\bar{a})$$

$$\iff p_{12} > \frac{1}{0.25}(6000 - 0.75 \times 7000) = 3000$$

Example: changes in the cost/payoff

how can value $p_{31} = p(a_3, \theta_1)$ change?

$$EV(a_3) = 0.75 \times p_{31} + 0.25 \times 6000 < 6500 = EV(a^*)$$

$$\iff p_{31} < \frac{1}{0.75}(6500 - 0.25 \times 6000) = 6666,6(6)$$

how can value $p_{32} = p(a_3, \theta_2)$ change?

$$EV(a_3) = 0.75 \times 6000 + 0.25 \times p_{32} < 6500 = EV(a^*)$$

$$\iff p_{32} < \frac{1}{0.25}(6500 - 0.75 \times 6000) = 8000$$

Decision Analysis with Sample Information

Decision Analysis with Sample Information

We have used preliminary or *a priori* probabilities for the states of nature, that are the best probability values available at that time.
However, to make the best possible decision, the decision maker may want to seek additional information about the states of nature.

This new information can be used to revise or update the prior probabilities so that the final decision is based on more accurate probabilities for the states of nature.

Most often, additional information is obtained through experiments designed to provide sample information about the states of nature.

These revised probabilities are called posterior probabilities.

Posterior Probabilities

consider S to be the distribution associated with the study, which is intended to gather additional information about the states of nature and let

$S = s_j$ be the result provided by the study favorable to each nature state θ_j, $j = 1, \ldots, m$

the posterior probabilities are

$$P(\theta = \theta_j \mid S = s_k), \; j, k = 1, \ldots, m$$

Revise some properties of probabilities:

- $P(A \cup B) = P(A) + P(B) - P(A \cap B)$
- $P(B) = P(B \cap A) + P(B \cap \bar{A})$
- $P(A|B) = \frac{P(A \cap B)}{P(B)}$
- $P(A \cap B) = P(A|B)P(B) = P(B|A)P(A)$
- $P(A|B) = \frac{P(B|A)P(A)}{P(B)}$
- $P(A|B) + P(\bar{A}|B) = 1$

De Morgan's Laws:

- $\overline{A \cap B} = \bar{A} \cup \bar{B} \Leftrightarrow P(\overline{A \cap B}) = P(\bar{A} \cup \bar{B})$
- $\overline{A \cup B} = \bar{A} \cap \bar{B} \Leftrightarrow P(\overline{A \cup B}) = P(\bar{A} \cap \bar{B})$

Example: Posterior Probabilities

An expert always correctly predicted stagnation, but in 40% of the cases where there was improvement, the expert predicted stagnation.

consider S to be the distribution associated with the study provided by the expert

$S = s_1$ the expert predicts stagnation

$S = s_2$ the expert predicts improvement

thus we have

$$P(S = s_1 | \theta = \theta_j) = \begin{cases} 1 & \theta = \theta_1 \\ 0.4 & \theta = \theta_2 \end{cases}$$

$$P(S = s_2 | \theta = \theta_j) = \begin{cases} 0 & \theta = \theta_1 \\ 0.6 & \theta = \theta_2 \end{cases}$$

remember the *a priori* probabilities:

$$h(\theta_1) = P(\theta = \theta_1) = 3/4; \qquad h(\theta_2) = P(\theta = \theta_2) = 1/4$$

Example: Posterior Probabilities

we can obtain the Marginal distribution of S

$$P(S = s) = \sum_{j=1}^{m} P(S = s \mid \theta = \theta_j) P(\theta = \theta_j)$$

thus

$$P(S = s_1) = 0.85 \qquad P(S = s_2) = 0.15$$

the posterior probabilities are

$$P(\theta = \theta_j \mid S = s_j)$$

that can be obtained with

$$P(\theta = \theta_j \mid S = s_j) = \frac{P(S = s_j \wedge \theta = \theta_j)}{P(S = s_j)}$$

Example: Posterior Probabilities

$$P(\theta = \theta_1 \mid S = s_1) = \frac{15}{17}$$

$$P(\theta = \theta_2 \mid S = s_1) = \frac{2}{17}$$

$$P(\theta = \theta_1 \mid S = s_2) = 0$$

$$P(\theta = \theta_2 \mid S = s_2) = 1$$

Example: Posterior Probabilities

When $S = s_1$ the Expected Value of each alternative are:

	θ_1	θ_2	$EV(a_i)$
a_1	7000	5000	6764.71
a_2	−10000	14000	−7176.47
a_3	6000	6000	6000.00
	$\frac{15}{17}$	$\frac{2}{17}$	

When $S = s_2$ the Expected Value of each alternative are:

	θ_1	θ_2	$EV(a_i)$
a_1	7000	5000	5000
a_2	−10000	14000	14000
a_3	6000	6000	6000
	0	1	

Expected Value of the Experience (EVE)

When $S = s_1$ the Bayes criteria selects

$$\max_{a_i}\{6764.71; -7176.47; 6000.00\} = 6764.71$$

When $S = s_2$ the Bayes criteria selects

$$\max_{a_i}\{5000, 14000.6000\} = 14000$$

Thus

$$EVE = 0.85 \times 6764.71 \; + \; 0.15 \times 14000 = 7850$$

Expected Value of Sample Information

Expected Value without Sample Information

$$EVwoSI = 6500$$

Expected Value with Sample Information

$$EVwSI = 7850$$

Expected Value of Sample Information

$$EVSI = |EVwoSI - EVwSI| = 7850 - 6500 = 1350$$

Efficiency of Sample Information

$$E = \frac{EVSI}{EVPI} \times 100$$

Low efficiency ratings for sample information may lead the decision maker to look for other types of information. However, high efficiency ratings indicate that the sample information is almost as good as perfect information and that additional sources of information would not yield significantly better results.

For the example:

$$E = \frac{1350}{2250} \times 100 = 60\%$$