Planning short-term financing

Define the following decision variables

- $\mathbf{x_i}$ amount financed through the credit line of up to 100 000 euros per month at an interest rate of 1% per month, for $i = 1, \ldots, 5$, as it is available every month and the loan is paid the following month, so the last month (month 6) will not be used so that all the loans are paid off within the period under study;
- $\mathbf{y_i}$ amount financed through the 3-month commercial paper at a total interest rate of 2% for the 3-month period, we have $i = 1, 2$ as this financing option is only available the first two months, loan will be paid on the third month;
- $\mathbf{z_i}$ the excess funds will be invested at an interest rate of 0.1% per month.

A proposed model for Project 1 is the following

$$
\begin{aligned}
\max \quad & z_6 && (1)\\
\text{s. to} \quad & && \\
& x_1 + y_1 \geq z_1 + 100, && (2)\\
& x_2 + y_2 - 1.01x_1 + 1.001z_1 \geq z_2 + 200, && (3)\\
& x_3 - 1.01x_2 + 1.001z_2 \geq z_3 - 100, && (4)\\
& x_4 - 1.02y_1 - 1.01x_3 + 1.001z_3 \geq z_4 + 150, && (5)\\
& x_5 - 1.02y_2 - 1.01x_4 + 1.001z_4 \geq z_5 - 550, && (6)\\
& - 1.01x_5 + 1.001z_5 \geq z_6 + 100, && (7)\\
& x_i \leq 100, \quad i = 1, \ldots, 5, && (8)\\
& x_i \geq 0, \quad i = 1, \ldots, 5, && (9)\\
& y_i \geq 0, \quad i = 1, 2, && (10)\\
& z_i \geq 0, \quad i = 1, \ldots, 6. && (11)
\end{aligned}
$$

The objective function (1) aims at maximizing the "excess funds" available at the end of the 6^{th} month.

Each inequality (2)–(7) shows the flow balance of each month $i = 1, \ldots, 6$, as the objective function aims at maximizing the "excess funds", these inequalities can be of $\geq$, equality is also a correct way of writing them. On the right hand side of each inequality we have the "excess funds" z_i plus the given cash flow. On the left side of each inequality we have the amount financed either through the credit line x_i or throught the commercial paper y_i and the corresponding payments (with a minus sign) where the multiplied factor is 1 plus the interest rate (since the amount lent plus interest is returned). We also have the return of the amount invested in the previous month with interests.

The inequalities (8) define the upper limit for these variables as the credit line is available of up to 100 000 euros per month.

The inequalities (9), (10) and (11) define the value of the decision variables as in $\mathbb{R}$ and nonnegative.

To write the dual model we need to define the dual decision variables

- w_i associated to each constraint (2)–(7), $i = 1, \ldots, 6$;
- u_i associated to each constraint (8), $i = 1, \ldots, 5$.

The dual is

$$\begin{aligned}
\min \quad & 100w_1 + 200w_2 - 100w_3 + 150w_4 - 550w_5 + 100w_6 + \sum_{i=1}^{5} 100u_i && (12)\\
\text{s. to} \quad & && \\
& w_i \;\; - 1.01w_{i+1} + u_i \quad \geq 0, \qquad i = 1, \ldots, 5 && (13)\\
& w_1 \;\; - 1.02w_4 \geq 0, && (14)\\
& w_2 \;\; - 1.02w_5 \geq 0, && (15)\\
& - w_i + 1.001w_{i+1} \geq 0, \qquad i = 1, \ldots, 5 && (16)\\
& - w_6 \geq 1, && (17)\\
& w_i \leq 0, \qquad i = 1, \ldots, 6, && (18)\\
& u_i \geq 0, \qquad i = 1, \ldots, 5. && (19)
\end{aligned}$$

which is equivalent to

$$\begin{aligned}
\min \quad & 100w_1 + 200w_2 - 100w_3 + 150w_4 - 550w_5 + 100w_6 + \sum_{i=1}^{5} 100u_i && (20)\\
\text{s. to} \quad & && \\
& w_1 \;\; - 1.01w_2 + u_1 \;\; \geq 0, && (21)\\
& w_2 \;\; - 1.01w_3 + u_2 \;\; \geq 0, && (22)\\
& w_3 \;\; - 1.01w_4 + u_3 \;\; \geq 0, && (23)\\
& w_4 \;\; - 1.01w_5 + u_4 \;\; \geq 0, && (24)\\
& w_5 \;\; - 1.01w_6 + u_5 \;\; \geq 0, && (25)\\
& w_1 \;\; - 1.02w_4 \;\; \geq 0, && (26)\\
& w_2 \;\; - 1.02w_5 \;\; \geq 0, && (27)\\
& - w_1 + 1.001w_2 \;\; \geq 0, && (28)\\
& - w_2 + 1.001w_3 \;\; \geq 0, && (29)\\
& - w_3 + 1.001w_4 \;\; \geq 0, && (30)\\
& - w_4 + 1.001w_5 \;\; \geq 0, && (31)\\
& - w_5 + 1.001w_6 \;\; \geq 0, && (32)\\
& - w_6 \geq 1, && (33)\\
& w_i \leq 0, \qquad i = 1, \ldots, 6, && (34)\\
& u_i \geq 0, \qquad i = 1, \ldots, 5. && (35)
\end{aligned}$$

Each constraint (13) (or equivalently constraints (21)-(25)) is associated to each variable x_i, $i = 1, \ldots, 5$.
Each constraint (14) and (15) (or equivalently constraints (26) and (27)) is associated to each variable y_i, $i = 1, 2$.

Each constraint (16) (or equivalently constraints (28)-(32)) is associated to each variable w_i, $i = 1, \dots, 5$.

Constraint (17) (equivalently (33)) is associated to variable z_6.

All variables in the primal are non-negative (≥ 0), so in standard form, so all associated constraints in the dual are also in standard form, since the dual is a minimisation problem, they are $\geq$ constraints.

Primal constraints (2)–(7) are $\geq$ constraints, and since the primal is a maximisation problem they are not in standard form, so the corresponding variables in the dual, w_i, are not in standard form either, so they are nonpositive (≤ 0).

Primal constraints (8) are $\leq$ constraints, and since the primal is a maximisation problem they are in standard form, so the corresponding variables in the dual, u_i, are in standard form, so they are nonnegative (≥ 0).