Universidade de Lisboa
Instituto Superior de Economia e Gestão
BsC in Economics, Finance and Management

<u>Mathematics II</u> – 1st Semester - 2024/2025

Regular Assessment - 6th of December 2024

Duration: $(120 + \varepsilon)$ minutes, $|\varepsilon| \leq 30$

Version A

Name: ....

Student ID #: ....

Part I

- Complete the following sentences in order to obtain true propositions. The items are independent from each other.
- There is no need to justify your answers.

(a) *(4)* If $A : \mathbb{R}^3 \to \mathbb{R}^3$ is a linear map such that

$$A(1,0,0) = (5,0,0), \quad A(0,1,0) = (0,0,0) \quad \text{and} \quad A(0,0,1) = (0,0,0),$$

then the eigenvalues of A are .... and their geometric multiplicities are ...., respectively.

(b) *(5)* Consider the map $f : D_f \to \mathbb{R}$ defined by $f(x,y) = \dfrac{\sqrt{x^2 + (y-1)^2}}{\ln x^2}$.
The set D_f may be explicitly written as:

$$D_f = \text{...}$$

(c) *(9)* The **symmetric** matrix associated to the quadratic form in $\mathbb{R}^2$:

$$Q(x,y) = (5x - 3y)^2$$

is

$$\begin{pmatrix} ..... & ..... \\ ..... & ..... \end{pmatrix}$$

and Q may be classified as .... defined.

Since Q is positively homogeneous of degree ...., Euler's identity says that:

$$x\frac{\partial Q}{\partial x}(x,y) + y\frac{\partial Q}{\partial y}(x,y) = ...$$

(d) *(7)* With respect to the set

$$\Omega = \left\{(x,y) \in \mathbb{R}^2 : x \in [1,2[\wedge \frac{1}{x} \leq y\right\}$$

we may conclude that the point (...., ...) $\in \partial\Omega \backslash \Omega$ and

$int(\Omega) =$ ....

Since Ω is unbounded, then Ω is **not** ....

(e) *(6)* The map $f(x,y) = e^{y+x^2}$ has a (global) maximum and a (global) minimum when restricted to the set

$$M = \left\{(x,y) \in \mathbb{R}^2 : ....\right\}$$

This is a consequence of ....'s Theorem because M is .... and f is continuous.

(f) *(6)* If $f(x,y) = \dfrac{x^2 y}{x^2 - y^2}$ where $x \neq \pm y$, then

$$......... = \lim_{(x,y)\to(0,0), y=x^2+x} f(x,y) \neq \lim_{(x,y)\to(0,0), y=2x} f(x,y) = ...,$$

which means that f

....

(g) *(4)* If $u_n = \left(\left(1+\frac{1}{2n}\right)^n; \frac{1-n\pi}{2n}\right)$, $n \in \mathbb{N}$, and $f(x,y) = -2x + \cos y$, then

$$\lim_{n\to+\infty} f(u_n) = ...$$

(h) *(4)* With respect to the continuous map $f : \mathbb{R}^2 \to \mathbb{R}$, its directional derivatives along the non-null vector (v_1, v_2) at $(1,2)$ are given by

$$D_{(v_1,v_2)}f(1,2) = \frac{v_1^2}{3+v_2^2}.$$

Hence we may conclude that:

$$\frac{\partial f}{\partial x}(1,2) = ... \quad \text{and} \quad \frac{\partial f}{\partial y}(1,2) = ...$$

(i) *(6)* With respect to a smooth map $f : \mathbb{R}^2 \to \mathbb{R}$, one knows that

$$\nabla f(x,y) = (-2x\sin(x^2+y); -\sin(x^2+y) + 3y^2).$$

If $f(x,y)$ does not have constant terms, then

$$f(x,y) = ...$$

(j) *(3)* The **differential of order 2** of the map $f(x,y) = e^{-3y}$ at $(0,0)$ is given by

$$D_2 f(0,0)(h_1,h_2) = ...h_1^2 + ...h_1h_2 + ...h_2^2$$

(k) *(6)* If $f(x,y) = xy$, $x(t) = t^2$ and $y(t) = 1 + t^3$, by the *Chain rule*, we get:

$$\frac{df}{dt}(t) = ...$$

(there is no need to simplify the expression).

(l) *(4)* With respect to the map given by $f(x,y) = 2x - y - x^2 - y^2$, one knows that $\nabla f(1, -1/2) = (0,0)$. Since

$$H_f(x,y) = \begin{pmatrix} \cdots\cdots & 0 \\ \cdots\cdots & \cdots\cdots \end{pmatrix},$$

then f is stricly concave in $\mathbb{R}^2$ and thus $f(1,-1/2)$ is a .... maximum of f.

(m) *(4)* The following equality holds:

$$\int_1^2 \int_1^{x^2} e^{x+y} \text{ dy dx} = \int_{\cdots\cdots}^{\cdots\cdots} \int_{\cdots\cdots}^{\cdots\cdots} e^{x+y} \text{ dx dy}$$

(n) *(4)* The .... law (associated to a given population of size p that depends on the time $t \geq 0$) states that

$$p' = kp, \qquad k \in \mathbb{R}.$$

If $p(0) = 20$ and $k = 3$, then the **solution** of the previous differential equation is given **explicitly** by

...., where $t \in \mathbb{R}_0^+$.

(o) *(3)* Assuming that y depends on $x \in \mathbb{R}$, any solution of the differential equation $y' = \dfrac{1}{y^4+1}$ is monotonically ....

(p) *(5)* The graph of the solution of the IVP $\begin{cases} y'' + 4y = 0 \\ y'(0) = 0 \\ y(0) = 4 \end{cases}$ is

Part II

- Give your answers in exact form. For example, $\frac{\pi}{3}$ is an exact number while 1.047 is a decimal approximation for the same number.
- In order to receive credit, you must show all of your work. If you do not indicate the way in which you solve a problem, you may get little or no credit for it, even if your answer is correct.

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1. For $\alpha \in \mathbb{R}\backslash\{-4, 4\}$, consider the following matrix $\mathbf{A} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \alpha & 4 \\ 0 & 4 & \alpha \end{bmatrix}$.

   (a) Classify the quadratic form $Q(X) = X^T\mathbf{A}X$, $X \in \mathbb{R}^3$, as function of α.

   (b) Find the value of α for which $(0, 1, 1)$ is an eigenvector of $\mathbf{A}$ associated to 3.

2. Consider the map $f(x, y) = \begin{cases} \dfrac{x^2(x-y)}{\sqrt{x^2+y^2}} & \text{if } y > x \\ 0 & \text{if } y \leq x \end{cases}$.

   Show that:

   (a) f is continuous in $\mathbb{R}^2$.

   (b) f has a global maximum.

3. Consider the map f defined in $D = \{(x, y) \in \mathbb{R}^2 : y > 0\}$ as

   $$f(x, y) = xy - x^2 \ln y$$

   (a) Find, if they exist, the local extrema of f.

   (b) The map f has a global minimum when restricted to $M = \{(x, y) \in D : xy = 1\}$. Find it.

4. Consider the set $\Omega = \{(x, y) \in \mathbb{R}^2 : x^2 \leq y \leq -3x \wedge x \leq -1\}$.

   (a) Sketch the set Ω in the cartesian plane (x, y).

   (b) Compute

   $$\iint_\Omega \frac{e^{\frac{y}{x}}}{x} \text{ dx dy}.$$

5. Consider the following IVP (y is a function of x):

$$\begin{cases} xy' + y = xe^x \\ y(1) = 1 \end{cases}$$

Write the solution $y(x)$ of the IVP, identifying its maximal domain.

Credits:

I	II.1(a)	II.1(b)	II.2(a)	II.2(b)	II.3(a)	II.3(b)	II.4(a)	II.4(b)	II.5
80	15	10	15	10	15	15	10	15	15