Universidade de Lisboa
Instituto Superior de Economia e Gestão
Msc in Economics, Mathematical Finance,
Monetary and Financial Economics

Mathematical Economics – 1st Semester - 2024/2025

Regular Assessment - 17th of December 2024

Duration: $(120 + \varepsilon)$ minutes, $|\varepsilon| \leq 30$

Version A

Name: ....

Student ID #: ....

Part I

- Complete the following sentences in order to obtain true propositions. The items are independent from each other.
- There is no need to justify your answers.

(a) *(4)* The (maximal) domain of $f : D_f \to \mathbb{R}$ defined by

$$f(x,y) = \frac{\sqrt{x^2 + (y-1)^2}}{\ln x^2}.$$

is the set

$D_f =$ ....

(simplify the expression)

(b) *(7)* With respect to the set

$$\Omega = \left\{(x,y) \in \mathbb{R}^2 : x \in [1,2[\,\wedge\, 0 \leq y \leq e^{-x}\right\}$$

we may conclude that the point (......, ...) $\in \partial\Omega \backslash \Omega$ and

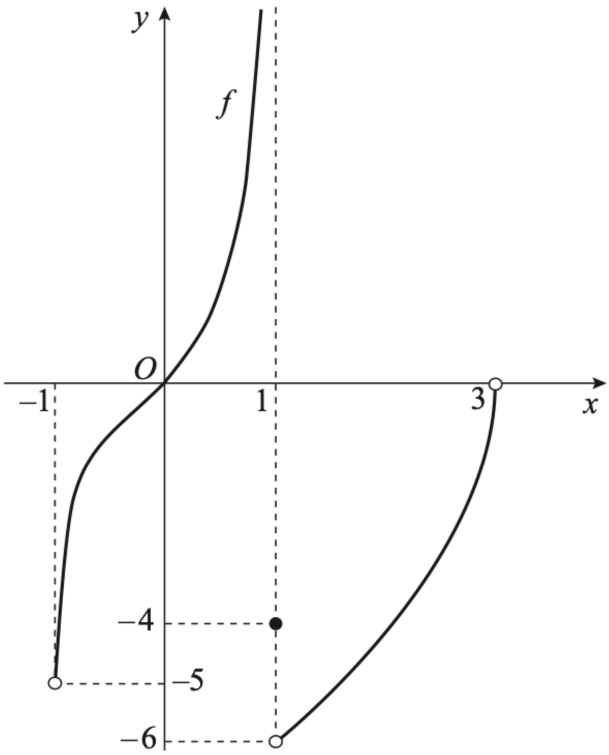
$int(\Omega) =$ ....

Since Ω is **not** ...., then Ω is **not** compact.

(c) *(5)* The graph of $f :]-1, 3[\to \mathbb{R}$ is below. We know that

$$\lim_{x \to 1^-} f(x) = +\infty, \qquad f(1) = -4$$

and $(x_n)_{n \in \mathbb{N}}$ is a sequence in $]1, 3[$ such that $\lim_{n \in \mathbb{N}} x_n = 1$.



Then

$$\lim_{n \to +\infty} f(x_n) = \text{..........}$$

(d) *(5)* With respect to a map $f : \mathbb{R}^2 \to \mathbb{R}$, one knows that $\nabla f(x, y) = (4x^3y^2; 2x^4y)$. If $f(x, y)$ does not have constant terms, then

$$f(x, y) = \text{..}$$

(e) *(4)* The map $f : \mathbb{R} \to \mathbb{R}$ is continuous, $f(-2) = 4$ and $f(1) = 8$. The Intermediate Value Theorem ensures that the equation $f(x) =$ .... has at least one solution in $]-2, 1[$. Moreover, if f is .... then the solution is unique.

(f) *(8)* The graphical representation of the correspondence $H\colon [0,2] \rightrightarrows [0,2]$ defined by

$$H(x) = \begin{cases} [\sqrt{x}, 2 - \frac{x}{2}] & x \leq 1 \\ \{2\} & x > 1 \end{cases}$$

is:

The set of fixed points of H is explicitly given by: ....

(g) *(5)* With respect to the map given by $f(x,y) = 2x - y - x^2 - y^2$, one knows that $\nabla f(1, -1/2) = (0,0)$. Since

$$H_f(x,y) = \begin{pmatrix} \cdots\cdots & 0 \\ \cdots\cdots & \cdots\cdots \end{pmatrix},$$

then f is stricly concave in $\mathbb{R}^2$ and thus $f(1,-1/2)$ is a .... maximum of f.

(h) *(4)* Consider the following **maximisation** problem

$$\text{maximize} \quad x^2 - y, \qquad \text{subject to} \quad x^2 + y^2 \leq 1.$$

The Karush-Kuhn-Tucker conditions are:

$$\begin{cases} 2x - 2\lambda x = 0 \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots = 0 \\ \lambda(1 - x^2 - y^2) = \cdots\cdots \\ x^2 + y^2 \leq 1 \end{cases}$$

(i) *(8)* The .... law (associated to a given population of size p that depends on the time $t \geq 0$) states that

$$p' = ap - bp^2, \qquad a \gg b \in \mathbb{R}^+.$$

The phase portrait of the differential equation is:

If $\varphi(t)$ is the solution of the differential equation such that $\varphi(0) = \dfrac{a}{2b}$, then

$$\lim_{t \to +\infty} \varphi(t) = \ldots\ldots\ldots$$

(j) *(4)* Assuming that y depends on $x \in \mathbb{R}$, any solution of the differential equation $y' = \dfrac{1}{y^4 + 1}$ is monotonically.......................

(k) *(8)* Assuming that y depends on x, the **graph** of the solution of the IVP

$$\begin{cases} y'' + 4y = 0 \\ y'(0) = 0 \\ y(0) = 4 \end{cases}$$

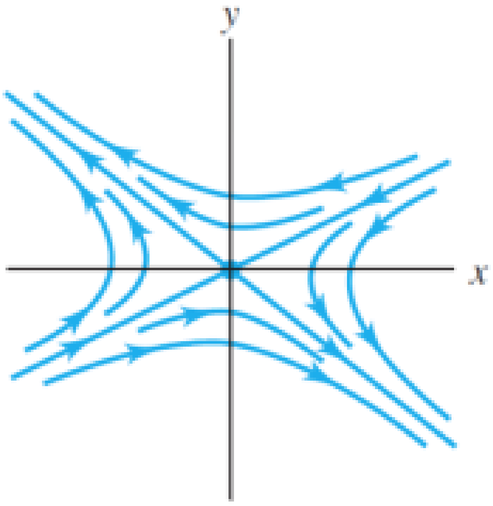
is

(l) *(8)* Assuming that x and y depend on t, the linearisation of

$$(*) \begin{cases} \dot{x} = 2x - xy^{2024} \\ \dot{y} = 3y - yx^{2024} \end{cases} \qquad \text{around } (0,0) \qquad \text{is} \qquad (**) \begin{cases} \dot{x} = \ldots\ldots\ldots\ldots \\ \dot{y} = \ldots\ldots\ldots\ldots \end{cases}$$

With respect to the Lyapunov's stability, we may conclude that (0,0) is .... Furthermore, ...-.........................Theorem says that, there exists a small neighbourhood of $(0,0)$ where the dynamics of (*) and (**) are "qualitatively" the same (topologically conjugated).

(m) *(5)* The phase portrait of $\dot{X} = AX$, where $A \in M_{2\times 2}(\mathbb{R})$ and $X = (x, y) \in \mathbb{R}^2$, is given by:



Hence the following inequality is valid: $\det(A)$...........

(n) *(10)* Consider the following problem of optimal control where $x : [0, 10] \to \mathbb{R}$ is the *state*, $u : [0, 10] \to \mathbb{R}$ is the *control* and t is the independent variable:

$$\min_{u(t)\in\mathbb{R}} \int_0^{10} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\; dt, \quad x'(t) = u(t), \quad x(0) = 1 \quad \text{and} \quad x(10) \text{ free.}$$

We may turn the previous problem into the following maximization problem:

$$\max_{u(t)\in\mathbb{R}} \int_0^{10} -\frac{x(t)^2}{2} - \frac{u(t)^2}{2}\, dt, \quad x'(t) = u(t), \quad x(0) = 1 \quad \text{and} \quad x(10) \text{ free.}$$

Then the Hamiltonian is given by (*specify the formulas to the case under consideration*):

$$H(t, x, u, p) = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

The Pontryagin maximum principle says that the optimal control $u^\star$ should satisfy the equality .... The Hamiltonian differential equations are given by:

$$\begin{cases} \dot{x} = \ldots\ldots\ldots\ldots\ldots \\ \dot{p} = \ldots\ldots\ldots\ldots\ldots \end{cases}$$

Part II

- Give your answers in exact form. For example, $\frac{\pi}{3}$ is an exact number while 1.047 is a decimal approximation for the same number.
- In order to receive credit, you must show all of your work. If you do not indicate the way in which you solve a problem, you may get little or no credit for it, even if your answer is correct.

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1. Consider the map $f : [1, +\infty[\to \mathbb{R}$ defined by

$$f(x) = \sqrt{x} + \frac{x}{4}$$

   (a) Show that f satisfies the hypotheses of the Banach fixed point Theorem.

   (b) Find the fixed point of f and compute $\lim_{n\to+\infty} f^n(2024)$.
   (Remark: $f^n = f \circ f \circ ... \circ f$ denotes the composition map).

2. Consider the map f defined in $D = \{(x, y) \in \mathbb{R}^2 : y > 0\}$ as

$$f(x, y) = xy - x^2 \ln y$$

   (a) Find, if they exist, the local extrema of f.

   (b) The map f has a global minimum when restricted to $M = \{(x, y) \in D : xy = 1\}$. Find it.

3. Consider the following IVP (y is a function of x):

$$\begin{cases} xy' + y = xe^x \\ y(1) = 1 \end{cases}$$

   Write the solution $y(x)$ of the IVP, identifying its maximal domain.

4. Consider the linear system of ODEs in $\mathbb{R}^2$ given by (x and y depend on t):

$$\begin{cases} \dot{x} = x + y \\ \dot{y} = -9x + y \end{cases}$$

   (a) Write the general form of the solution.

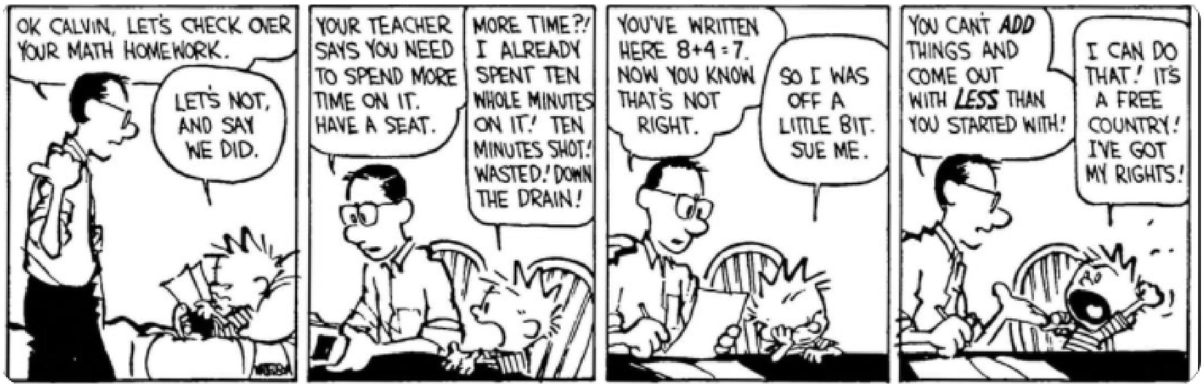
   (b) Sketch, in the phase portrait, the **unique** solution such that $x(0) = 0, y(0) = 3$.

5. Consider the following Problem on *Calculus of Variations*, where $x : [0,1] \to \mathbb{R}$ is a smooth function on t:

$$\max_{x(t)\in\mathbb{R}} \int_0^1 [1 - x(t)^2 - \dot{x}(t)^2]\, dt, \quad \text{with} \quad x(0) = 1 \quad \text{and} \quad x(1) \in \mathbb{R}.$$

(a) Write the Euler-Lagrange equation and the transversality condition applied to the case under consideration.

(b) Find the solution of the problem.



Credits:

I	II.1(a)	II.1(b)	II.2(a)	II.2(b)	II.3	II.4(a)	II.4(b)	II.5(a)	II.5(b)
85	10	10	15	15	15	15	10	10	15