Assume the economy has 3 periods: $t = 0, 1, 2$. The state variable can take 3 different values in periods 1 and 2: $s_t = a, b,$ or c. There are 3 different assets. Asset 1 is a one period riskless bond that pays 1 euro in all states of period 2 and all states of period 1. The price of the one period riskless bond is 1 in all states of nature. The other 2 assets (Asset 2 and Asset 3) pay dividends $\mathbf{d}(s^2)$ in period 2 and have prices in periods 0 and 1 denoted by $\mathbf{p}(s^0)$ and $\mathbf{p}(s^1)$. The dividends are: $\mathbf{d}(s_0, a, a) = \begin{bmatrix} 13 \\ 10 \end{bmatrix}$, $\mathbf{d}(s_0, a, b) = \begin{bmatrix} 9 \\ 10 \end{bmatrix}$, $\mathbf{d}(s_0, a, c) = \begin{bmatrix} 12 \\ 14 \end{bmatrix}$, $\mathbf{d}(s_0, b, a) = \begin{bmatrix} 12 \\ 14 \end{bmatrix}$, $\mathbf{d}(s_0, b, b) = \begin{bmatrix} 12 \\ 12 \end{bmatrix}$, $\mathbf{d}(s_0, b, c) = \begin{bmatrix} 8 \\ 15 \end{bmatrix}$, $\mathbf{d}(s_0, c, a) = \begin{bmatrix} 8 \\ 15 \end{bmatrix}$, $\mathbf{d}(s_0, c, b) = \begin{bmatrix} 9 \\ 15 \end{bmatrix}$, $\mathbf{d}(s_0, c, c) = \begin{bmatrix} 11 \\ 9 \end{bmatrix}$. The prices are: $\mathbf{p}(s_0) = \begin{bmatrix} 10 \\ 13.5 \end{bmatrix}$, $\mathbf{p}(s_0, a) = \begin{bmatrix} 12 \\ 12 \end{bmatrix}$, $\mathbf{p}(s_0, b) = \begin{bmatrix} 10 \\ 14 \end{bmatrix}$, $\mathbf{p}(s_0, c) = \begin{bmatrix} 9 \\ 14 \end{bmatrix}$.

(a) Define lack of arbitrage for this economy.

Solution:

There is an arbitrage opportunity if there is a portfolio of assets $\{Z_i\}_{i=1}^{3}$ with a non-positive cost

$$\sum_{i=1}^{3} p(Z_i) \leq 0$$

where $p(Z_i)$ is the price of asset i which pays a non-negative amount in all states of nature and a strictly positive amount in at least one state of nature

$$\sum_{i=1}^{3} Z_{i,k} R_{i,k} \geq 0, \text{ for all states } k$$

$$\text{and}$$

$$\sum_{i=1}^{3} Z_{i,k} R_{i,k} > 0, \text{ for at least one } k$$

where $R_{i,k}$ is the rate of return of asset i in state k

(b) Are there arbitrage opportunities in this economy? It is enough to explain how one would prove this.

Solution:

There are no arbitrage opportunities if there is a strictly positive discount factor, m. Given the payoff x (price plus dividend) and the price $p(x)$ need to verify the equality below for all nodes of the tree.

$$\begin{aligned} p(x) &= E(mx) = \textstyle\sum_k \pi_k m_k x_k \\ &= \frac{1}{R^f} \textstyle\sum_k \pi_k^* x_k = \frac{1}{R^f} E^*(x) \end{aligned}$$

Where E^* denotes that the expectation is taken with respect to the artificial probability π_k^* (see lecture 3). For instance for the first node:

$$\begin{bmatrix} 1 \\ 10 \\ 13.5 \end{bmatrix} = \pi_1^* \begin{bmatrix} 1 \\ 12 \\ 12 \end{bmatrix} + \pi_2^* \begin{bmatrix} 1 \\ 10 \\ 14 \end{bmatrix} + \pi_3^* \begin{bmatrix} 1 \\ 9 \\ 14 \end{bmatrix}$$

Have to solve this equation and verify if all π_k^* are strictly positive.

(c) Are markets complete? Here too it is enough to explain how one would prove this.

Solution:

Markets are complete if the discount factor is unique. It is enough to verify if the determinant of the 3 payoffs in the period after the states s_0, (s_0, a), (s_0, b), and (s_0, c) are non-zero. For instance for the state (s_0, c) the determinant is

$$\begin{vmatrix} 1 & 1 & 1 \\ 8 & 9 & 11 \\ 15 & 15 & 9 \end{vmatrix}$$

(d) What are Arrow-Debreu securities?

Solution:

Are assets with payoffs equal to one unit of the good in one state of nature and zero in all the remaining states.

(e) Determine the price of all Arrow securities for state $s^t = (s_0, b)$.

Solution:

The price of the Arrow security in state (s_0, b) that pays one unit in state (s_0, b, a) is:

$$y_1 \cdot 1 + y_2 \cdot 10 + y_3 \cdot 14$$

where (y_1, y_2, y_3) is such that

$$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = y_1 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + y_2 \begin{bmatrix} 12 \\ 12 \\ 8 \end{bmatrix} + y_3 \begin{bmatrix} 14 \\ 12 \\ 15 \end{bmatrix}$$

(f) Assume there is an European Call Option for asset 2 with exercise price $X = 11$ and maturity date $t = 2$. What is the price of the option?

Solution:

The price of the option in state (s_0, b):

$$\max \left\{ \begin{bmatrix} 12 \\ 12 \\ 8 \end{bmatrix} - \begin{bmatrix} 11 \\ 11 \\ 11 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \right\} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}.$$

Once we know the prices of the Arrow securities in (s_0, b), then the price of the option is equal to the sum of the prices of 2 Arrow securities, the one that pays in state (s_0, b, a) and the one that pays in state (s_0, b, b).