Lisbon School of Economics & Management
Universidade de Lisboa

Master in Mathematical Finance
Probability Theory and Stochastic Processes
First Semester - 2023/2024

Normal Season - First Midterm Test (MT1) - November 9, 2023 - Time: 1 hour

Justify your answers carefully

The values in [*.*] represent the rating of the corresponding question on a 0-10 scale

1. [**2.0**] Let $\Omega = \{1, 2, 3, 4\}$. Give examples of three different σ-algebras $\mathcal{F}_1, \mathcal{F}_2$, and $\mathcal{F}_3$ of Ω, such that $\mathcal{F}_1 \subset \mathcal{F}_2 \subset \mathcal{F}_3$.

2. [**4.0**] Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space.

(a) Considering X a random variable that can only take one possible value $c \in \mathbb{R}$, describe $\sigma(X)$.

(b) Consider $g : \Omega \to \mathbb{R}$ a $\mathcal{F}$-measurable function. Let $\mathcal{G}$ be a σ-algebra of Ω such that $\mathcal{G} \subset \mathcal{F}$. Is g also $\mathcal{G}$-measurable?

(c) Let $A, B \in \mathcal{F}$ such that $\mathbb{P}(A) = 0$. Compute $\mathbb{P}(B) - \mathbb{P}(A \cup B)$.

3. [**4.0**] Consider $\Omega = [0, +\infty[$ and $f_n : \Omega \to \mathbb{R}$ a sequence of functions such that for each $n \geq 2$,

$$f_n(x) = \frac{1}{1 + x^n}.$$

Justifying every step of your resolution, compute:

(a) $\displaystyle\lim_{n \to +\infty} f_n(x)$, for every $x \in \Omega$.

(b) $\displaystyle\lim_{n \to +\infty} \int_{\Omega} f_n(x)\, dm(x)$, where m is the Lebesgue measure.

(c) $\displaystyle\lim_{n \to +\infty} \int_{[1,+\infty[} f_n(x)\, d\delta_1(x)$, where δ_1 is the Dirac measure at 1.