Proof that Profit Maximization Implies Cost Minimization

Setup

Let the production function be $f : \mathbb{R}^n_+ \to \mathbb{R}_+$, increasing and concave. Let $p > 0$ denote the output price and $w \geq 0$ the vector of n input prices. Let p and w be exogenously determined.

Profit maximization

$$\max_x \; \pi(x) = pf(x) - wx.$$

Let x^* be the optimal choice, with output $y^* = f(x^*)$. Then, for each input i, the n FOCs must hold:

$$p\frac{\partial f(x^*)}{\partial x_i} = w_i, \tag{PM}$$

Cost minimization

For a given output level y^*, the firm minimizes the cost of producing y^*:

$$\min_x \; wx \quad \text{s.t.} \quad f(x) = y^*.$$

The Lagrangian is

$$\mathcal{L}(x, \lambda) = wx - \lambda\big(f(x) - y^*\big).$$

The $n + 1$ FOCs are:

$$\begin{aligned} w_i &= \lambda \frac{\partial f(x^c)}{\partial x_i}, \\ f(x^c) &= y^* \end{aligned} \tag{CM}$$

We also assume that $\lambda \geq 0$ (Intuitively: Recall that we need both gradients to point in the same direction, slide 13 Ch 4).

Proof that PM $\Rightarrow$ CM using FOCs

Now that we have set up both problems, we can start with the actual proof. First, we show that $(x^*, \lambda = p)$ satisfies the FOCs of the CM problem.

From the profit-maximization FOCs (PM):

$$p\frac{\partial f(x^*)}{\partial x_i} = w_i.$$

Setting $\lambda = p$ and $x = x^*$ satisfies all CM conditions:

$$w_i = \lambda \frac{\partial f(x^*)}{\partial x_i}, \quad f(x^*) = y^*, \quad \lambda = p > 0.$$

Hence, $(x^*, \lambda = p)$ is a solution of the cost-minimization problem.

Since $V(q)$ is convex, then there is a unique solution for the minimum. Hence, x^* must minimize cost for output y^* and $x^* = x^c$.

It follows,

$$\boxed{\text{If } x^* \text{ maximizes profit, then it also minimizes cost for } y^* = f(x^*).}$$

Remember: cost minimization doees not imply profit maximization!

Economic interpretation:

At the profit-maximizing output, the firm must produce that output at the minimum possible cost. Otherwise, it could reduce cost and raise profit—contradicting profit maximization.

What we also get from this proof:

$$MC = \lambda = p. \tag{1}$$

So, at the profit-maximizing output, price equals marginal cost. That's the efficiency condition for a **competitive firm**. Indeed, note that the proof goes through only if firms are **price takers**.

Also intuition for equation (1) follows from:

$$\underbrace{p\frac{\partial f(x^*)}{\partial x_i}}_{\text{Value of the marginal product of input } i} = \underbrace{w_i}_{\text{Marginal cost of additional input } i} \tag{PM}$$

$$\underbrace{\lambda\frac{\partial f(x^c)}{\partial x_i}}_{\text{Cost of the marginal product of input } i} = \underbrace{w_i}_{\text{Marginal cost of additional input } i} \tag{CM}$$

PM: the firm hire inputs until their value of marginal product equals their price.

CM: the firm chooses inputs so that the cost of producing the marginal product equals the price of the input.