Decision Making and Optimization

Master in Data Analytics for Business

Lisbon School of Economics & Management
Universidade de Lisboa

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Integer Linear Programming

Integer Linear Programming Example

TBA Airlines is a small air company, specialized in regional flights. The management is considering an expansion and it has the possibility to buy small or medium size airplanes. Find the best strategy, knowing that at the moment no more than two small airplanes can be bought and that \$100 millions are available to invest. Consider also the values in the following table:

	Small airplane	Medium size airplane
Annual profit per airplane	\$1 million	\$5 millions
Cost per airplane	\$5 millions	\$50 millions

Model of the example

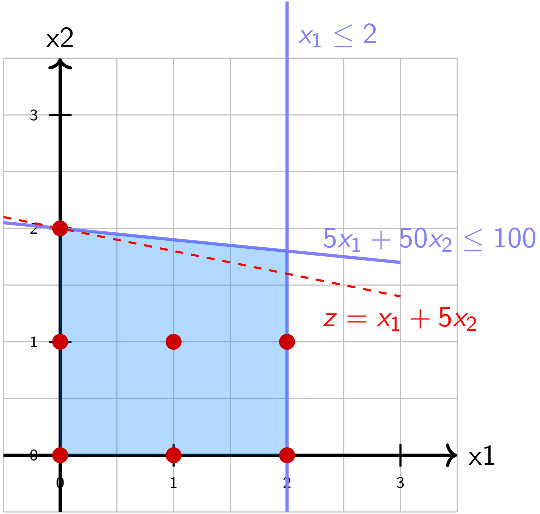
Consider the variables
x_1 - number of small airplanes to buy
x_2 - number of medium size airplanes to buy

The ILP model is

$$\begin{aligned} \max z &= x_1 + 5x_2 \\ \text{s.t. } & 5x_1 + 50x_2 \leq 100 \\ & x_1 \leq 2 \\ & x_1, x_2 \geq 0 \text{ and integer} \end{aligned}$$

Graphical solution

Optimal solution (0,2)
with value 10



Feasible set $= \{(0,0),(0,1),(0,2),(1,0),(2,0),(1,1),(2,1)\}$

Integer Linear Programming (ILP)

(ILP) arises when in the context of Linear Programming the decision variables only make sense if they have integer values, that is if the assumption of divisibility of LP doesn't fit to the problem in hands.

Integer Linear Programming problems where variables only take

– integer values $x_j \in \mathbb{N}, \; j = 1, \ldots, n$
or
– binary values $x_j \in \{0, 1\}, \; j = 1, \ldots, n$

Integer Linear Programming (ILP)

Integer Linear Programming (ILP) model

$$\begin{aligned}
\max \quad & z = \sum_{j=1}^{n} c_j x_j \\
\text{s.t.} \quad & \sum_{j=1}^{n} a_{ij} x_j \leq b_i, \;\; i = 1, \ldots, m \\
& x_j \geq 0 \text{ and integer}, \;\; j = 1, \ldots, n \\
& \text{or} \\
& x_j \in \{0, 1\}, \;\; j = 1, \ldots, n
\end{aligned}$$

Integer Linear Programming

(ILP) arises when to decide on:

- indivisible number decisions (for, example decisions on the number of machines to purchase, the number of people to select for a job)
- "yes-or-no" decisions (for example, the type of projects in which to invest or not to invest)

A ILP is a pure ILP if all the decision variables are required to have integer values;

A mixed ILP is a ILP where only some variables are required to have integer values.

Lower and Upper Bounds

Lower and Upper Bounds

Given an integer linear program (IP)

$$
\begin{aligned}
\max \quad & z = \sum_{j=1}^{n} c_j x_j \\
\text{s.t.} \quad & \sum_{j=1}^{n} a_{ij} x_j \leq b_i, \;\; i = 1, \ldots, m \\
& x_j \geq 0 \text{ and integer}, \;\; j = 1, \ldots, n
\end{aligned}
$$

we can obtain
lower bounds $\underline{z}_\ell$ and upper bounds $\overline{z}_u$ for its optimal value z^*:

$$\underline{z}_\ell \leq z^* \leq \overline{z}_u.$$

How to obtain lower and upper bounds?

Feasible solutions

the **value of any feasible solution z_a** is a

lower bound for a maximization integer linear problem

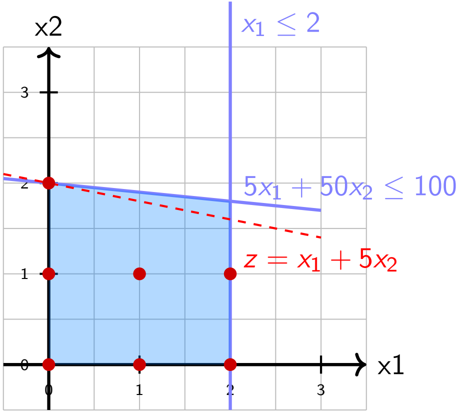
$$z_a \leq z^*$$

upper bound for a minimization integer linear problem

$$z^* \leq z_a$$

Example

Optimal solution (0,2) with value $z^* = 10$



Feasible set $= \{(0,0),(0,1),(0,2),(1,0),(2,0),(1,1),(2,1)\}$

Lower bounds: $z_{(0,0)} = 0$, $z_{(0,1)} = 5$, $z_{(0,2)} = 10$, $z_{(1,0)} = 1$, $z_{(2,0)} = 2$, $z_{(1,1)} = 6$, $z_{(2,1)} = 7$

Relaxation

We obtain a relaxation when we replace the ILP problem with another one for which it is easier to obtain a solution so that we can obtain an approximation to the optimal value of the ILP problem.

Given an ILP we obtain a relaxation when:

(i) the set of feasible solutions is enlarged (obtaining a larger set) and/or

(ii) the objective function is replaced by another one

There are several relaxations that can be obtained, we will use a linear relaxation.

Properties of the Relaxation

If the relaxed problem has no feasible solutions (is impossible), then the ILP has no feasible solutions (is impossible).

If the optimal solution of the relaxed problem is feasible for the ILP problem, then the solution is optimal to the ILP problem.

Relaxation

the **optimal value z_r of a relaxation** is

upper bound for a maximization integer linear problem

$$z^* \leq z_r$$

lower bound for a minimization integer linear problem

$$z_r \leq z^*$$

Linear Relaxation

The linear relaxation of an ILP is the LP obtained by dropping the constraints on the integer values for the variables (the integrality constraints).

we obtain the linear relaxation as follows

$$
\begin{aligned}
\max \quad & z = \sum_{j=1}^{n} c_j x_j \\
\text{s.t.} \quad & \sum_{j=1}^{n} a_{ij} x_j \leq b_i, \; i = 1, \ldots, m \\
& x_j \geq 0 \cancel{\text{ and integer}}, \; j = 1, \ldots, n
\end{aligned}
$$

Example

The linear relaxation of the ILP model is

$$\begin{aligned} \max z &= x_1 + 5x_2 \\ \text{s.t. } & 5x_1 + 50x_2 \leq 100 \\ & x_1 \leq 2 \\ & x_1, x_2 \geq 0 \ \cancel{\text{and integer}} \end{aligned}$$

The optimal solution is $x^*_{LR} = (2, \frac{9}{5})$ with value $z^*_{LR} = 11$

Thus $z^*_{LR} = 11$ is an upper bound

Duality

the **value of a dual feasible solution** z_d of the linear relaxation is

upper bound for a maximization integer linear problem

$$z^* \leq z_d$$

lower bound for a minimization integer linear problem

$$z_d \leq z^*$$

Example

The dual of the linear relaxation of the ILP model is

$$\begin{aligned} \min w &= 100y_1 + 2y_2 \\ \text{s.t. } & 5y_1 + y_2 \geq 1 \\ & 50y_1 \geq 5 \\ & y_1, y_2 \geq 0 \end{aligned}$$

A feasible solution is $y = (1, 1)$ with value $w = 102$

Thus $w = 102$ is an upper bound

another dual feasible solution is $y = (0.1, 1)$ with value $w = 12$

Solving ILP

Solve ILP problems

Methods to (exactly) solve ILP problems

- Branch & Bound
- Cutting Planes
- Branch & Cut
- Branch & Price
- etc.

Approximation methods (give lower and/or upper bounds)

- any exact method that is stopped before completing its search
- Relaxations
- Heuristics
- etc.

Branch & Bound method

The Branch & Bound method

- divides the feasible set and builds an enumeration tree (**branch**)
- removes branches of the tree that corresponds to infeasible situations (**bound**)
- starts by solving the linear relaxation of the initial program
- only solves linear relaxations of the linear problems that are generated by adding constraints to the initial problem

Branch & Bound algorithm - max problem

Step 1: Initialization

- solve the linear relaxation (P_L) of P.L.I.
  If (P_L) is impossible, STOP! (P) is also impossible.
  If its solution is integer, STOP! The optimal solution of (P) has been found.
  Otherwise,
- let $\bar{z}$ be the corresponding optimal value
- let $\underline{z} = -\infty$ or else equal to the value of the o.f. associated with a feasible solution

Branch & Bound algorithm - max problem

Step 2: Branching

- partition the problem from a variable that violates the integrality constraint
- let x_k be the chosen variable with fractional value $\bar{x}_k$
- in one of the problems include the constraints $x_k \leq \lfloor \bar{x}_k \rfloor$
- in the other problem include the constraint $x_k \geq \lfloor \bar{x}_k \rfloor + 1$
- place these problems on the list of problems to be analyzed

Branch & Bound algorithm - max problem

Step 3: Subproblem selection

- if there are no more subproblems to analyze, go to Step 5
- otherwise, select a new problem from the list and go to Step 4

Branch & Bound algorithm - max problem

Step 4: Solve the selected subproblem/Bounding

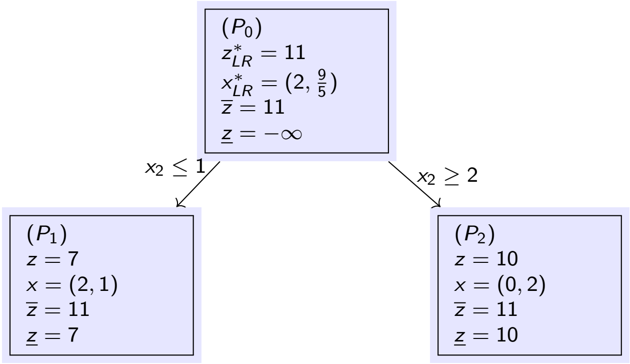
- solve the linear relaxation of the selected problem
- if the linear relation is impossible, abandon the subproblem, cancel the node in the search tree and go to Step 3
- otherwise, let z be the value of the o.f. corresponding to the optimal solution of the linear relaxation
  - if $z < \underline{z}$, abandon this problem, cancel this node, and go to Step 3
  - if $z \geq \underline{z}$ and if in the optimal solution there is at least one integer variable with a fractional value, then go to Step 2
  - if $z \geq \underline{z}$ and if the optimal solution satisfies all the integrality constraints, then cancel the tree node, replace $\underline{z}$ for z and move on to Step 3

Branch & Bound algorithm - max problem

Step 5: Optimality test

- if $\underline{z} = -\infty$ then the problem is impossible and the process ends
- otherwise the optimal solution has been obtained, and the process ends with $z^* = \underline{z}$

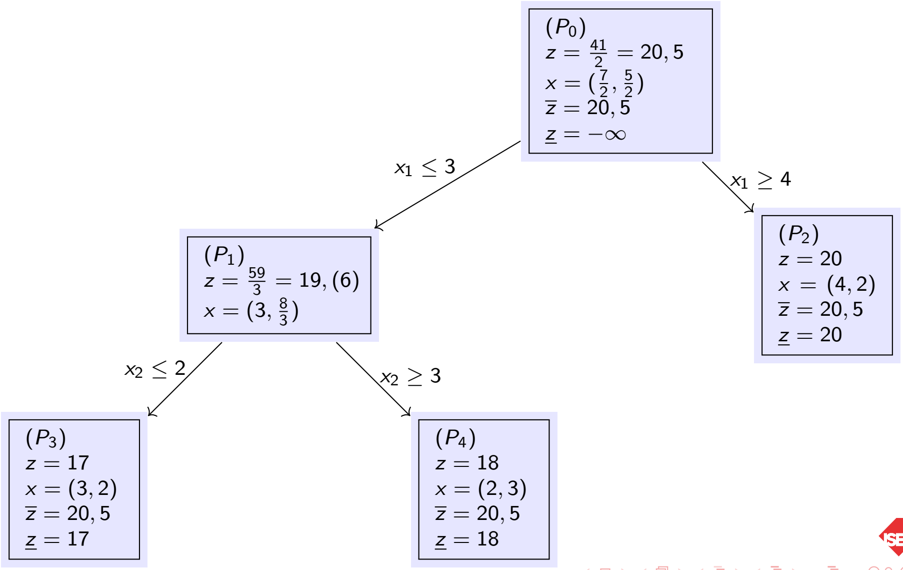
Branch & Bound for the Example



Example 2

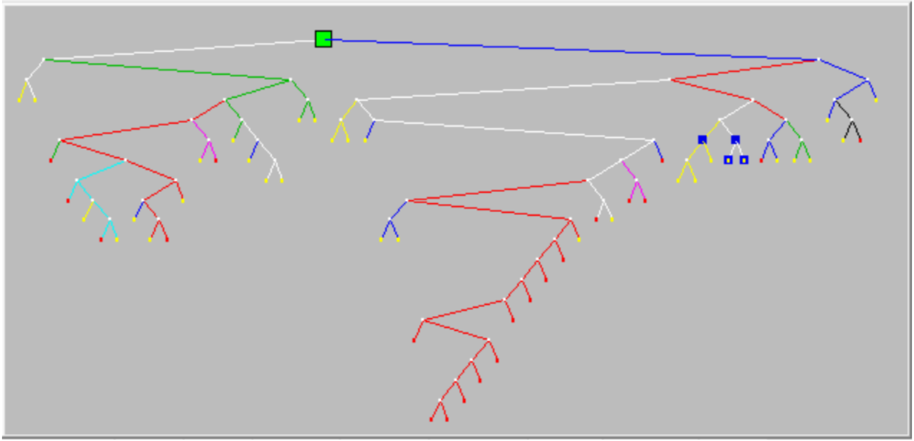
$$
\begin{array}{ll}
\max & z = 3x_1 + 4x_2 \\
\text{s. a:} & -3x_1 + 2x_2 \leq 2 \\
& x_1 + 3x_2 \leq 11 \\
& x_1 + x_2 \leq 6 \\
& x_1, x_2 \geq 0 \text{ and integer}
\end{array}
$$

Example 2

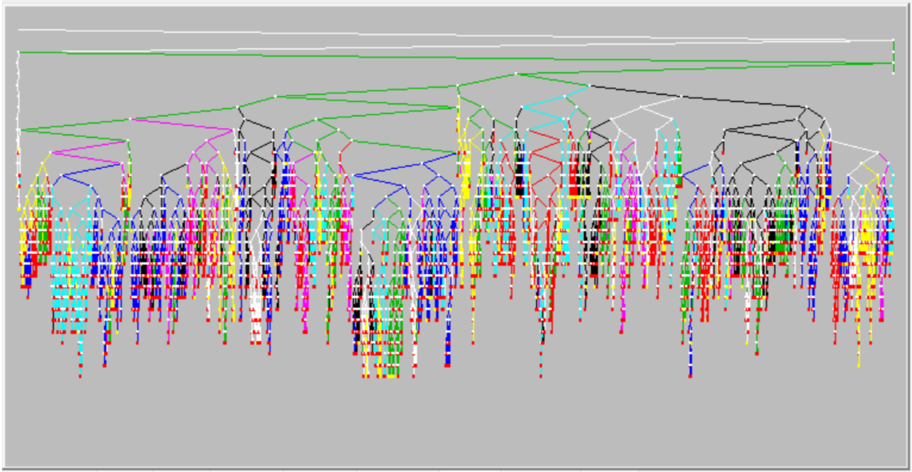


(P_0)
$z = \frac{41}{2} = 20,5$
$x = (\frac{7}{2}, \frac{5}{2})$
$\overline{z} = 20,5$
$\underline{z} = -\infty$
$x_1 \leq 3$
$x_1 \geq 4$
(P_1)
$z = \frac{59}{3} = 19,(6)$
$x = (3, \frac{8}{3})$
(P_2)
$z = 20$
$x = (4,2)$
$\overline{z} = 20,5$
$\underline{z} = 20$
$x_2 \leq 2$
$x_2 \geq 3$
(P_3)
$z = 17$
$x = (3,2)$
$\overline{z} = 20,5$
$\underline{z} = 17$
(P_4)
$z = 18$
$x = (2,3)$
$\overline{z} = 20,5$
$\underline{z} = 18$

Example of a Branch & Bound Tree (Xpress)



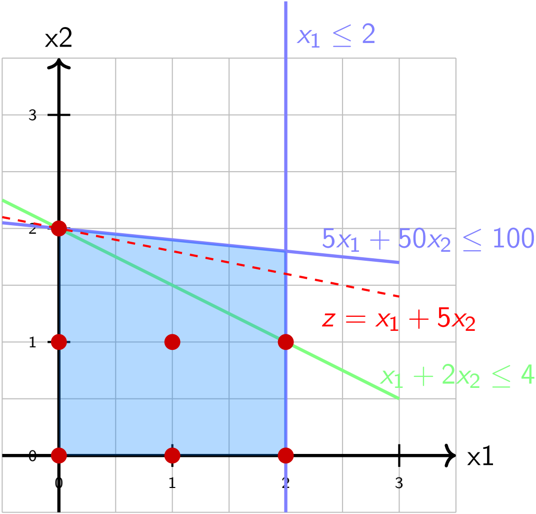
Example of a Branch & Bound Tree (Xpress)



Cutting Planes method

Adds valid inequalities, such as $x_1 + 2x_2 \leq 4$ for our example

Optimal solution (0,2)
with value 10



Software

Small size

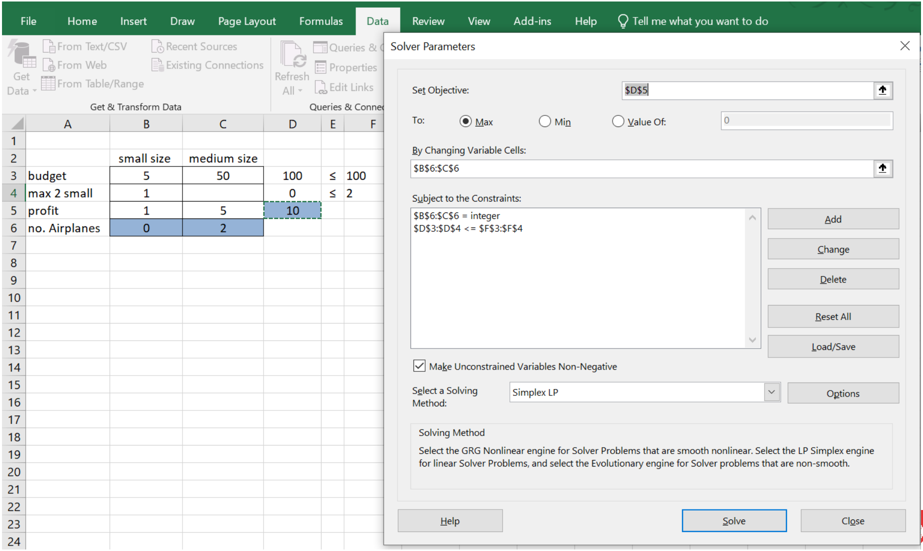
Solver of the Excel
PuLP with Python

Large size

XPress https://www.fico.com/en/products/fico-xpress-optimization
Gurobi https://www.gurobi.com/
CPLEX https://www.ibm.com/analytics/cplex-optimizer

among others
and you may use them with Python, C, C++

Solving using the Solver of the Excel



Heuristics

Heuristics usually produce feasible solutions, thus giving a bound.

A heuristic, or a heuristic technique, is any algorithm or approach that uses a practical method or various shortcuts in order to produce feasible solutions that may not be optimal but are good enough, sufficient, given a limited timeframe or deadline. It is a method that rapidly produces a feasible solution and provides an approximation of the optimal value.

Modelling with binary variables

Integer/Binary Linear Program

– decision variables $x_j \geq 0,\ j = 1, \ldots, n$
– binary variables $y_j \in \{0,1\},\ j = 1, \ldots, nn$

Mixed Integer Linear Programming (ILP) model

$$\begin{aligned} \max \quad & z = \sum_{j=1}^{n} c_j x_j \\ \text{s.t.} \quad & \sum_{j=1}^{n} a_{ij} x_j \leq b_i,\ i = 1, \ldots, m \\ & \text{additional constraints using } x_j \text{ and } y_j \\ & x_j \geq 0,\ j = 1, \ldots, n \\ & y_j \in \{0,1\},\ j = 1, \ldots, nn \end{aligned}$$

Binary decision variables:

$$y_j = \begin{cases} 1, & \text{if } j \text{ is selected}, \\ 0, & \text{otherwise}, \end{cases} \qquad j = 1, \ldots, n,$$

Modelling Fixed Costs

f_j is the fixed cost for considering (selecting/using/producing) activity x_j

$$
\begin{aligned}
\max \quad & z = \sum_{j=1}^{n} c_j x_j - \sum_{j=1}^{n} f_j y_j \\
\text{s.t.} \quad & \sum_{j=1}^{n} a_{ij} x_j \leq b_i, \ \ i = 1, \ldots, m \\
& x_j \leq M \ y_j, \ \ j = 1, \ldots, n \text{ (linking constraints)} \\
& x_j \geq 0, \ \ j = 1, \ldots, n \\
& y_j \in \{0, 1\}, \ \ j = 1, \ldots, n
\end{aligned}
$$

M should be defined so that the value of the variable x_j is bounded only by the functional constraints and not by the linking constraints

Limitations on activities

$$y_j = \begin{cases} 1, & \text{if } j \text{ is selected}, \\ 0, & \text{otherwise}, \end{cases} \qquad j = 1, \ldots, n,$$

- on the maximum number of activities: at most K

$$\begin{array}{ll} \sum_{j=1}^{n} y_j \leq K, & \\ x_j \leq M\ y_j, & j = 1, \ldots, n, \end{array}$$

- incompatible activities: r and s are incompatible

$$\begin{array}{l} y_r + y_s \leq 1, \\ x_r \leq M\ y_r, \\ x_s \leq M\ y_s, \end{array}$$

- complementary activities: s is selected only if r is selected

$$y_s \leq y_r$$

Disjunction of constraints

$$\sum_{j=1}^{n} a_{1j}x_j \leq b_1 + M\,(1 - y_1),$$

$$\sum_{j=1}^{n} a_{2j}x_j \leq b_2 + M\,(1 - y_2),$$

$$y_1 + y_2 = 1$$

$$y_1, y_2 \in \{0, 1\}$$

Example 1

A company plans to expand its business by acquiring 3 new properties from a selection of 5 potential locations, labeled L1, L2, L3, L4, and L5. Each selected property must have an area between 50 ha and 300 ha.
Due to the geographical proximity of locations L1, L2, and L3, the company will select at most two of these three. Additionally, locations L4 and L5 are mutually exclusive – that is, at most one of them can be chosen.
The company has gathered the following information about each location:

	L1	**L2**	**L3**	**L4**	**L5**
Annual expected profit (m.u./ha)	25	30	20	20	25
Annual fixed cost (m.u.)	100	120	110	110	120
Initial investment (m.u./ha)	30	45	20	25	30

The company has a total budget of 18,500 m.u. for the initial investment and aims to maximize its total expected annual profit.

Example 1

Consider the additional constraint

Now consider that if the company chooses locations L4 or L5, it will have an additional 500 u.m. or 300 u.m. respectively to the initial amount.

Example 2

An oil company intends to select 5 out of 10 wells: $P1, P2, \ldots, P10$, to which are associated the cost $c_1, c_2, \ldots, c_{10}$, respectively. According to commitments with the local government, the company must comply with the following restrictions for regional development:

- **r1:** the selection of both P1 and P7 block selection of P8;
- **r2:** the selection of P3 or P4 block selection of P5;
- **r3:** from P5, P6, P7 and P8 at most two can be selected;
- **r4:** the selection of P1 forces selection of P10.

Formulate the problem and solve it assigning costs at your choice.

Combinatorial Problems

Combinatorial Problems

- Knapsack Problem
- Facility Location Problem
- Set Covering Problem
- Travelling Salesperson Problem
- Sequencing Problem