STATISTICAL LABORATORY

Lisbon School of Economics & Management
Universidade de Lisboa

Applied Mathematics for Economics and Management
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https://doity.com.br/estatistica-aplicada-a-nutricao

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PROGRAM

1. Fundamental Concepts of Statistics
2. Exploratory Data Analysis
3. Organizing and Summarizing Data
4. Association and Relationships Between Variables
5. Index Numbers
6. Time Series Analysis

LECTURE 9: LINK INDICES AND CHAIN INDICES

LINK INDEX (YEAR-TO-YEAR INDEX)

Purpose

- To compare **each period with the one immediately before it**.
- This index measures **short-term changes** and avoids having one fixed base year.

Formula:

$$I_{t|t-1} = \frac{p_t}{p_{t-1}} \times 100$$

(or the same using quantities)

Where:

- p_t = price in period t
- p_{t-1} = price in the previous period

Example:

Suppose the price of a product was:

- Year 1: 4.00
- Year 2: 5.00

$$I_{2|1} = \frac{5.00}{4.00} \times 100 = 125$$

→ **Prices increased by 25% from Year 1 to Year 2.**

CHAIN INDEX (CHAINED INDEX)

Purpose

- To measure the **change from a base year (0) to any later year (t)** by multiplying the **sequence of link indices.**
- Useful when **changing product baskets** over time.

Main Formula:

$$\boxed{I_{t|0}^{chain} = I_{t|t-1} \times I_{t-1|t-2} \times \cdots \times I_{1|0}}$$

$$I_{t|0}^{chain} = I_{1|0} \times \frac{I_{2|1}}{100} \times \cdots \times \frac{I_{t|t-1}}{100}$$

Alternative Notation for Chain Indices

$$I_{1|0}^{\text{chain}} = I_{1|0}$$

$$I_{2|0}^{\text{chain}} = I_{2|1} \times I_{1|0}$$

$$I_{3|0}^{\text{chain}} = I_{3|2} \times I_{2|1} \times I_{1|0}$$

$$\vdots$$

$$I_{t|0}^{\text{chain}} = I_{t|t-1} \times I_{t-1|t-2} \times \cdots \times I_{1|0}$$

Note:
The link (moving-base) indices are not expressed as percentages, but as relative decimals.

Example (continuing previous data):

If:

- $I_{1|0} = 110$
- $I_{2|1} = 125$

Then:

$$\boxed{I_{2|0}^{chain} = 110 \times \frac{125}{100} = 137.5}$$

Note:
The link (moving-base) indices are expressed as percentages.

→ From the base year (0) to Year 2, prices increased by 37.5%.

FIXED-BASE AND MOVING-BASE INDICES

Fixed-base Index

- Compares **each period to a single base period** (usually Year 0).
- Used to measure **long-term trends** relative to a constant reference.

$$I^{fixed}_{t|0} = \frac{x_t}{x_0} \times 100$$

Moving-base Index

- Compares **each period to a changing base**.
- Useful for analysing **short-term fluctuations**.

$$I^{moving}_{t|t-1} = \frac{x_t}{x_{t-1}} \times 100$$

Types of Moving-base Indices

Type	Definition	Purpose
Link Index *(Índice Elos)*	The base is the **immediately previous period** $(t-1)$.	Captures **period-to-period variation**.
Homologous Index *(Índice Homólogo)*	The base is the **same period in the previous year** $(t-12)$.	Used to **control for seasonal patterns**.

CHAINED INDEX: FIXED-BASE OR MOVING-BASE?

In **Silvestre (2007)**, the distinction is made as follows:

- **Chained indices** are obtained from **link relatives** (i.e., **moving-base indices**, where each period is compared to the previous one).
- Then, these link indices are **multiplied in sequence** to show the accumulated change since a fixed initial period.
- Therefore, **even though the calculation uses moving-base indices**, the final result is expressed **relative to a fixed initial period**. So, a **chained index** is considered a **fixed-base index**, because it shows the evolution from the initial period t_0.

Index type	Comparison	Note
Fixed base	Always with t_0	e.g., Laspeyres, Paasche
Moving base (link relatives)	Each period with $t-1$	e.g., link index
Chained index	Calculated from link relatives	Final result relative to t_0 → considered **fixed-base**

LECTURE 9: BASE CHANGE AND RECONCILIATION

CHANGING THE BASE OF AN INDEX

$$i_{t|0} = \frac{x_t}{x_0} = \frac{x_t}{x_0} \cdot \frac{x_b}{x_b} = \frac{x_t}{x_b} \cdot \frac{x_b}{x_0} = i_{t|b} \cdot i_{b|0}$$

$$i_{t|b} = \frac{i_{t|0}}{i_{b|0}} \cdot 100$$

Explanation (Notes):

- $i_{t|0}$ → index of period t with original base 0
- $i_{t|b}$ → index of period t with new base b
- $i_{b|0}$ → index of period b with original base 0
- This formula shows that the index with the original base can be expressed as the **product of the index with the new base and the index of the new base relative to the original base**.

Usage:

- Allows rescaling of index series without recalculating all values from scratch.

CHANGING THE BASE OF AN INDEX: EXAMPLE

Example: Suppose we have a price series:

Period (t)	Price x_t	Index $x_{t\|0} = (x_t / x_0) \times 100$
0	100	100
1	120	120
2	150	150

We want to **change the base** from period 0 to period 1 ($b = 1$).

Compute $i_{t|b}$ using the formula:

$$\boxed{i_{t|b} = \frac{i_{t|0}}{i_{b|0}} \cdot 100}$$

- $i_{1|1} = \frac{120}{120} \cdot 100 = 100$
- $i_{2|1} = \frac{150}{120} \cdot 100 = 125$

Interpretation:
- Now, period 1 becomes the base (100).
- Period 2 shows a 25% increase relative to period 1.

CHANGING THE BASE OF AN INDEX: EXAMPLE

Example: Suppose we have a price series:

Period (t)	Price x_t	Index $x_{t\|0} = (x_t / x_0) \times 100$ (Base 0)	Index $x_{t\|1} = (x_t / x_1) \times 100$ (Base 1)
0	100	100	83.33
1	120	120	100
2	150	150	125

- $i_{t|0}$ = Index with original base (period 0 = 100)
- $i_{t|1}$ = Index with new base (period 1 = 100)
- Period 0 relative to new base 1: $i_{0|1} = \frac{i_{0|0}}{i_{1|0}} \cdot 100 = 83.33$
- Period 2 relative to new base 1: $i_{2|1} = \frac{i_{2|0}}{i_{1|0}} \cdot 100 = 125$

Interpretation:

- Now all periods are measured relative to period 1. Period 0 is lower than 100, indicating it was below the new base.

RECONCILIATION OF INDICES: BASE UNIFORMIZATION

Base uniformization consists in expressing all index series on the **same reference base**, so that they can be compared.

This may occur in two situations:

1. **The indices already share the same base,** but we want to change to a *new* common base.
2. **The indices have different bases,** so the first step is to convert them to a *common* base.

RECONCILIATION OF INDICES: BASE UNIFORMIZATION

We consider two index series:

- **Index A** defined for periods $0, 1, \ldots, b$, with **base 0**
- **Index B** defined for periods $b, b+1, \ldots, T$, with **base b**

The goal is to obtain **one single index series** over the whole period.

Index A	$I^A_{0\|0}$	$I^A_{1\|0}$	$I^A_{2\|0}$	$\cdots$	$I^A_{b\|0}$				
Index B					$I^B_{b\|b}$	$I^B_{b+1\|b}$	$I^B_{b+2\|b}$	$\cdots$	$I^B_{T\|b}$

Situation 1 — Final Series Expressed in Base b

$$\boxed{I^C_{t|b} = \begin{cases} \dfrac{I^A_{t|0}}{I^A_{b|0}} \times 100, & t = 0, 1, \ldots, b \\ I^B_{t|b}, & t = b+1, \ldots, T \end{cases}}$$

- **Interpretation:**

  Convert **Index A** to base b; keep **Index B** as it already uses base b.

Important Note on "×100" and "÷100"

If indices are recorded as **index numbers**, where the base = **100**,
→ **Keep the "×100" and "÷100"** in the formulas.
If indices are recorded as **simple ratios** (base = **1**, not 100),
→ **Remove the "×100" and "÷100"** from both formulas.

RECONCILIATION OF INDICES: BASE UNIFORMIZATION

We consider two index series:

- **Index A** defined for periods $0, 1, \ldots, b$, with **base 0**
- **Index B** defined for periods $b, b+1, \ldots, T$, with **base b**

The goal is to obtain **one single index series** over the whole period.

Index A	$I^A_{0\|0}$	$I^A_{1\|0}$	$I^A_{2\|0}$	$\cdots$	$I^A_{b\|0}$				
Index B					$I^B_{b\|b}$	$I^B_{b+1\|b}$	$I^B_{b+2\|b}$	$\cdots$	$I^B_{T\|b}$

Situation 2 — Final Series Expressed in Base 0

$$\boxed{I^C_{t|0} = \begin{cases} I^A_{t|0}, & t = 0, 1, \ldots, b \\ I^B_{t|b} \cdot \dfrac{I^A_{b|0}}{100}, & t = b+1, \ldots, T \end{cases}}$$

- **Interpretation:**

  Keep **Index A** as it is; convert **Index B** from base b to base 0.

Important Note on "×100" and "÷100"

If indices are recorded as **index numbers**, where the base = **100**,
→ **Keep the "×100" and "÷100"** in the formulas.

If indices are recorded as **simple ratios** (base = **1**, not 100),
→ **Remove the "×100" and "÷100"** from both formulas.

WHY DIVISION IN CASE 1 AND MULTIPLICATION IN CASE 2?

Case 1 — Converting from Base 0 to Base b (Division)

$$\boxed{I^{A}_{t|b} = \frac{I^{A}_{t|0}}{I^{A}_{b|0}} \times 100}$$

We **divide** because we need to **remove** the cumulative change from period 0 to period b.

Interpretation:

"To express period t relative to b, we cancel the growth up to b."

→ Changing to a *later* base (0 → b) requires dividing.

Case 2 — Converting from Base b to Base 0 (Multiplication)

$$\boxed{I^{B}_{t|0} = I^{B}_{t|b} \times \frac{I^{A}_{b|0}}{100}}$$

We **multiply** because we need to **add** the cumulative change from period 0 to period b.

Interpretation:

"To express period t relative to 0, we combine the change from 0→b and b→t."

→ Changing to an *earlier* base (b → 0) requires multiplying.

WHY DIVISION IN CASE 1 AND MULTIPLICATION IN CASE 2?

Direction of Base Change	Operation	Why
From an **older** base to a **more recent** base	$\boxed{\textbf{Divide}}$	Removes the cumulative variation up to b
From a **more recent** base to an **older** base	$\boxed{\textbf{Multiply}}$	Adds the cumulative variation up to b

RECONCILIATION OF INDICES: BASE UNIFORMIZATION - EXAMPLE

Suppose we have two index series:

- **Index A** (base year = 0):

$$I^A_{0|0} = 100, \quad I^A_{1|0} = 110, \quad I^A_{2|0} = 120, \quad I^A_{3|0} = 150$$

- **Index B** (base year = 3):

$$I^B_{3|3} = 100, \quad I^B_{4|3} = 105, \quad I^B_{5|3} = 115$$

We want a single reconciled series over $t = 0, \ldots, 5$.

RECONCILIATION OF INDICES: BASE UNIFORMIZATION - EXAMPLE

Case A — Final series expressed with base = 3

Formula

$$I^C_{t|3} = \begin{cases} \dfrac{I^A_{t|0}}{I^A_{3|0}} \times 100, & t = 0,1,2,3 \\ I^B_{t|3}, & t = 4,5 \end{cases}$$

Reason: converting from an *older* base (0) to a *later* base (3) requires **dividing** by the index at the new base (and re-scaling to 100).

To change to a later base (older → later): divide by the index at the new base (and ×100 if indices use base-100).

Reason: converting from an *older* base (0) to a *later* base (3) requires **dividing** by the index at the new base (and re-scaling to 100).

Calculation (since $I^A_{3|0} = 150$)

- $I^C_{0|3} = 100/150 \times 100 = 66.67$
- $I^C_{1|3} = 110/150 \times 100 = 73.33$
- $I^C_{2|3} = 120/150 \times 100 = 80.00$
- $I^C_{3|3} = 150/150 \times 100 = 100.00$
- $I^C_{4|3} = 105$ (from B)
- $I^C_{5|3} = 115$ (from B)

Year(t)	0	1	2	3	4	5
$I_{t\|3}C$	66.67	73.33	80	100	105	115

RECONCILIATION OF INDICES: BASE UNIFORMIZATION - EXAMPLE

Case B — Final series expressed with base = 0

Formula

$$I^C_{t|0} = \begin{cases} I^A_{t|0}, & t = 0, 1, 2, 3 \\ I^B_{t|b} \times \dfrac{I^A_{b|0}}{100}, & t = 4, 5 \end{cases}$$

where $b = 3$.

Reason: converting from a *later* base (3) back to an *earlier* base (0) requires **multiplying** by the base index $I^A_{3|0}$ (and adjusting by 100).

To change to an earlier base (later → older): multiply the later-base index by the index of the reference period expressed in the older base (and ÷100 if indices use base-100).

Calculation (since $I^A_{3|0} = 150$)

- For $t = 0, 1, 2, 3$ keep A: 100, 110, 120, 150
- $I^C_{4|0} = 105 \times 150/100 = 157.50$
- $I^C_{5|0} = 115 \times 150/100 = 172.50$

Year(t)	0	1	2	3	4	5
$I_{t|3}C$	100	110	120	150	157.50	172.50

LECTURE 9:TIMES SERIES

TIMES SERIES: INTRODUCTION

Data observed over time

- measurements or values recorded at successive time points.

Observations are collected in time order

- the sequence matters; order affects analysis.

Useful for studying trends, cycles, and temporal patterns

- e.g., stock prices, temperature, monthly sales.

Graphical representation

- often visualized in a **time plot (chronogram)** to detect patterns.

Different from cross-sectional data

- cross-sectional data are collected **at a single point in time** for different individuals, firms, or regions, while chronological data track **one variable over time**.

Note:

- **Time is the essential factor** in chronological data, unlike cross-sectional data where time does not vary.

OBJECTIVES OF STUDYING TIME SERIES

Description

- To summarize and illustrate the behavior of a variable over time.

Explanation

- To identify the causes behind observed patterns and fluctuations.

Forecasting

- To predict future values based on past trends.

Control

- To monitor and manage processes by detecting deviations from expected behavior.

DEFINITION OF A TIME SERIES

- Let x be a variable observed over time.
- Suppose we collect n observations:

$$\boxed{x_1, x_2, x_3, \ldots, x_n}$$

- **Abbreviated notation:** $\{x_t\}_{t=1}^{n}$
- **Graphical representation:** displayed in a **time plot (chronogram)**

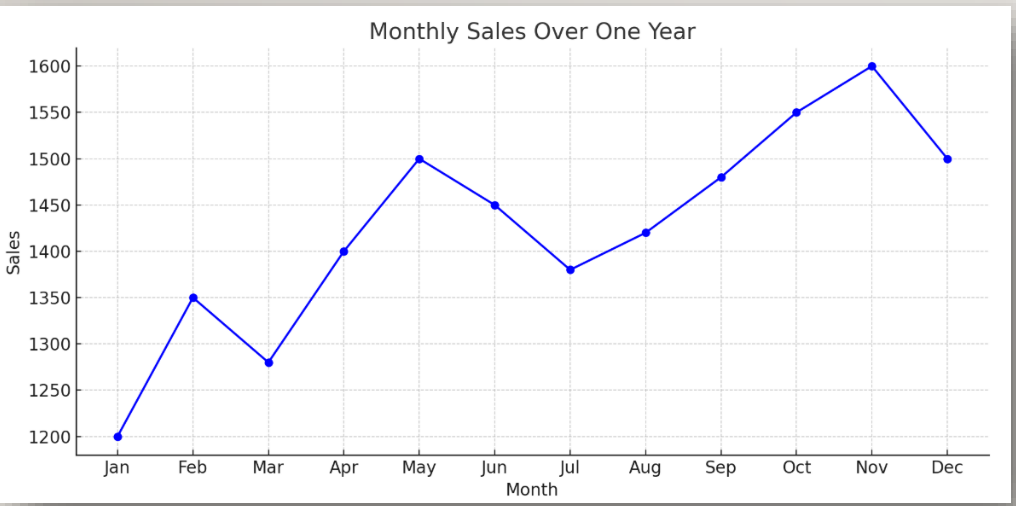
TIME SERIES: EXAMPLE

Example:

- Monthly sales of a company over one year:

$$x_1 = 1200,\ x_2 = 1350,\ x_3 = 1280, \ldots,\ x_{12} = 1500$$

Chronogram



Interpretation:

- Sales **increased from January to February**, then **decreased from February to March**.
- After that, sales **rose steadily until May**, followed by a **decline until June**.
- From June onwards, sales **increased again**, reaching the **highest value in November**.
- Finally, sales **declined in December**.

Summary: The monthly sales show a **fluctuating trend** with **two main growth periods** and a **peak in November**, indicating possible **seasonal effects or promotional impacts**.

COMPONENTS AND MODELS OF A TIME SERIES

- **Additive Model:**

$$\boxed{x_t = T_t + s_t + c_t + e_t}$$

- **Multiplicative Model:**

$$\boxed{x_t = T_t \cdot s_t \cdot c_t \cdot e_t}$$

- **Mixed Models (Silvestre):**

1. **Type 1:**

$$\boxed{x_t = (T_t + c_t) \cdot s_t + e_t}$$

2. **Type 2:**

$$\boxed{x_t = T_t \cdot c_t \cdot s_t + e_t}$$

Components:

- Trend (T_t)
- Seasonality (s_t)
- Cyclical or Oscillatory Movements (c_t)
- Residual or Irregular Component (e_t)

Notes:

•**Trend (T_t):**
The long-term movement or general direction of the time series over a period of time. Shows whether the series is increasing, decreasing, or stable.

•**Seasonality (s_t):**
Regular, repeating patterns within a fixed period, such as months, quarters, or days. Often linked to calendar or weather effects.

•**Cyclical or Oscillatory Movements (c_t):**
Medium- to long-term fluctuations that occur irregularly, often related to economic or business cycles.

•**Residual or Irregular Component (e_t):**
Random or unpredictable variations that are not explained by trend, seasonality, or cyclical movements.

COMPONENTS AND MODELS OF A TIME SERIES

Components:

- **Trend (T_t)**
- **Seasonality (s_t)**
- **Cyclical or Oscillatory Movements (c_t)**
- **Residual or Irregular Component (e_t)**

- **Additive Model:**

  Assumes that components add together to form the series: $x_t = T_t + s_t + c_t + e_t$
- **Multiplicative Model:**

  Assumes that components multiply together: $x_t = T_t \cdot s_t \cdot c_t \cdot e_t$
- **Mixed Models (Silvestre):**
  1. **Type 1:** $x_t = (T_t + c_t) \cdot s_t + e_t$ → trend and cycle add, then multiplied by seasonality, plus residual
  2. **Type 2:** $x_t = T_t \cdot c_t \cdot s_t + e_t$ → trend, cycle, and seasonality multiply, plus residual

COMPONENTS OF A TIME SERIES: TREND

A **time series** is a variable observed over time (monthly, yearly, daily, etc.).

Example: *Monthly sales of a store.*

A time series can be decomposed into **four main components**:

1) Trend (T_t)

The **long-term movement** of the series.

Key question:

> "Is the series **generally** increasing, decreasing, or remaining stable over time?"

- If sales grow year after year → **Upward trend**
- If they decrease → **Downward trend**

Example:

Year 1: 100 → Year 2: 120 → Year 3: 145 → Year 4: 160

Overall direction = **increasing** → upward trend.

COMPONENTS OF A TIME SERIES: SEASONALITY

2) Seasonality (S_t)

A **regular and repeating pattern** within a fixed period (usually within each year).

Key question:

> "Does the series show a pattern that repeats at the **same times** each year?"

Examples:

- Higher sales every **Christmas**
- More restaurant demand **every summer**
- Higher electricity usage **every winter**

Mental image:

Small repeating waves **always occurring in the same months**.

COMPONENTS OF A TIME SERIES: CYCLE OR OSCILLATORY MOVEMENTS

3) Cycle (C_t)

Medium-to-long-term fluctuations that **do not follow a fixed calendar pattern**.

Key question:

> "Are there expansions and contractions that happen, but not at regular intervals?"

Examples:

- Economic expansion → recession → recovery → expansion
- Business cycles that last several years

Difference between Seasonality and Cycle

Seasonality	Cycle
Repeats on a **fixed schedule** (e.g., every December)	**No fixed period**; timing varies
Short period (within the year)	Long period (several years)

COMPONENTS OF A TIME SERIES: IRREGULAR OR RESIDUAL COMPONENT

4) Irregular or Residual Component (E_t)

The **unpredictable** part of the series; random variation.

Includes:

- Unexpected events (strikes, storms, pandemics)
- Measurement errors
- Accidents and non-systematic shocks

Mental image:

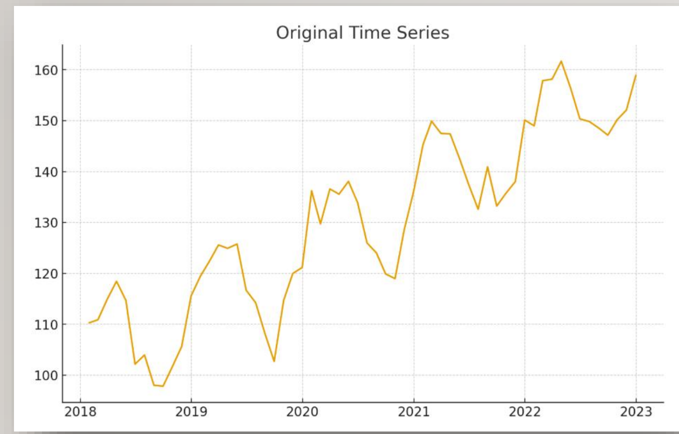
Noise that cannot be explained by Trend, Seasonality, or Cycle.

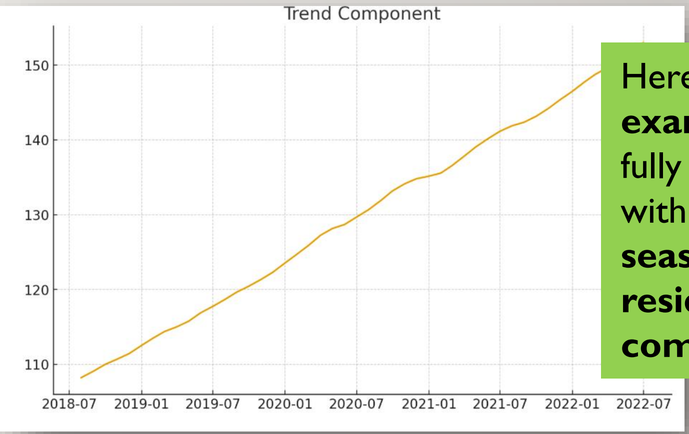
COMPONENTS OF A TIME SERIES

Summary Table

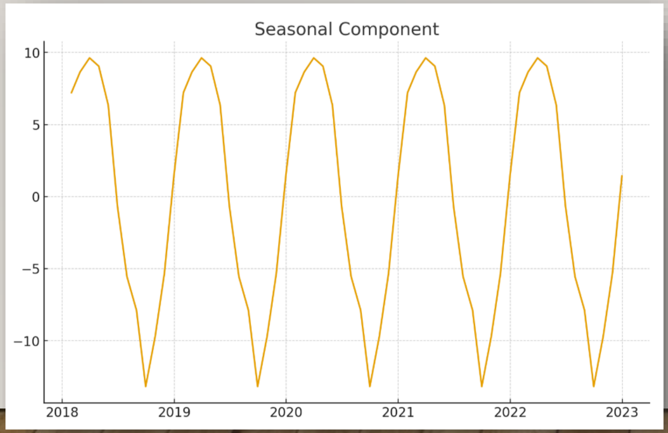
Component	Meaning	Example
Trend (T_t)	Long-term direction	Sales gradually increase over years
Seasonality (S_t)	Repeating pattern within each year	Christmas demand spike every year
Cycle (C_t)	Long-term economic fluctuations	Periods of recession and recovery
Residual (E_t)	Unpredictable/random variation	Strikes, pandemic, shocks

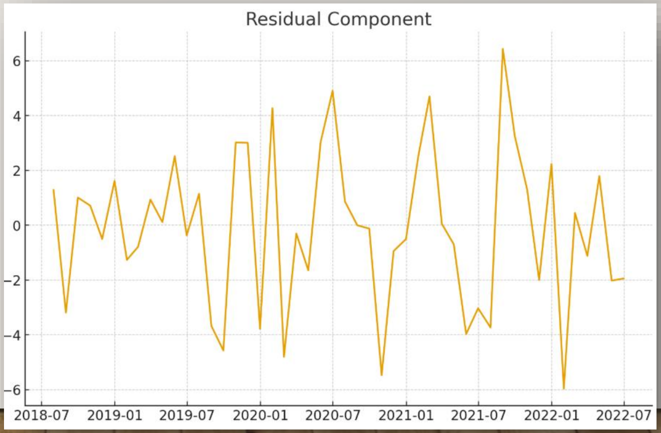
COMPONENTS AND MODELS OF A TIME SERIES: EXAMPLE





Here is a **real example**, already fully decomposed with **trend, seasonality, and residual components**.





COMPONENTS AND MODELS OF A TIME SERIES: EXAMPLE

Original Series

This shows the actual observed values over time.

We can already see:

- A general **upward trend**
- A **repeating seasonal pattern** (waves)

Trend Component (T_t)

A **smooth long-term movement.**

→ Here, the trend steadily increases from 100 to about 160.

Seasonal Component (S_t)

A **repeating pattern** within each year.

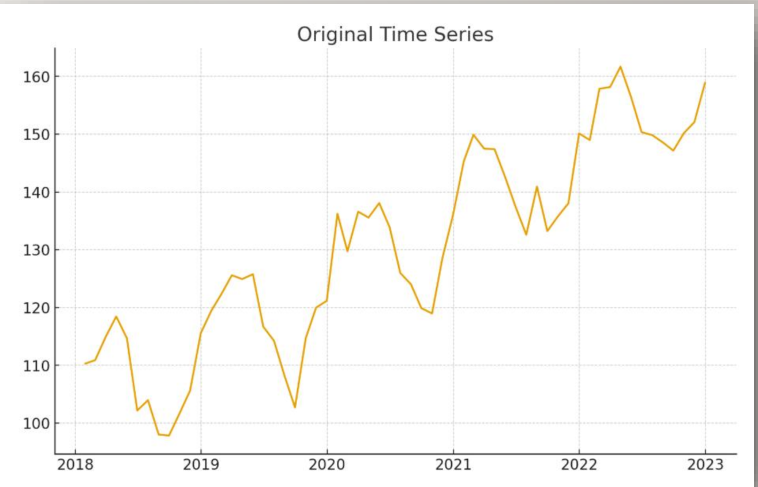
→ Notice that the pattern repeats every **12 months**, always with the same shape

Residual Component (E_t)

This is what remains after removing **Trend** and **Seasonality**:

$$E_t = X_t - T_t - S_t$$

→ It represents random shocks and unpredictable variation.



We **generated a monthly time series** with:

- an upward **trend**
- a yearly **seasonal pattern**
- some **random noise**

and then applied **seasonal decomposition** (additive model).

Note:
In this series, no cyclical component is considered, since economic cycles require long time spans and do not follow a fixed periodic pattern. Therefore, the decomposition is simplified to: $X_t = T_t + S_t + E_t$

IDENTIFYING THE APPROPRIATE TIME SERIES MODEL

Feature Observed in the Time Plot	Type of Model	Interpretation
The seasonal fluctuations have **constant amplitude** over time	**Additive Model** $X_t = T_t + S_t + C_t + E_t$	Seasonal variations do **not** depend on the level of the series.
The seasonal fluctuations **increase or decrease proportionally** to the level of the series	**Multiplicative Model** $X_t = T_t \cdot S_t \cdot C_t \cdot E_t$	Seasonal variations are **relative** to the level of the series.
The seasonal fluctuations **increase slightly**, but **not fully proportionally** to the level	**Mixed Model** $X_t = (T_t + C_t) \cdot S_t + E_t$	Part of the variation is constant, and part depends on the level of the series.

Key Guiding Rule

- **If the seasonal amplitude** stays constant → *Additive*
- **If the seasonal amplitude** grows with the trend → *Multiplicative*
- **If the amplitude grows slightly but not proportionally** → *Mixed*

THANKS!

Questions?