Instituto Superior de Economia e Gestão

Decision Making and Optimization

Topics for the Exam Prototype 2023-2024

December 2023 **Duration: 2h**

Notes: -Justify all answers and present the calculations carried out.
-Answer all questions using methodologies taught in Decision Making and Optimization classes.

-Here are some solutions and resolution topics.

Name: ______________________ **No.** __________

1. A company produces two products, Product A and Product B. The production of each product requires time on two machines, Machine 1 and Machine 2. The company has a total of 100 hours of time available on Machine 1 and 200 hours of time available on Machine 2. The production of Product A requires 2 hours on Machine 1 and 4 hours on Machine 2. The production of Product B requires 3 hours on Machine 1 and 1 hour on Machine 2. The profit per unit of Product A is 10 and the profit per unit of Product B is 15. The company wants to maximize its profit.

   (a) Propose a linear programing formulation to this problem.

   Define and describe the variables, the objective function and the constraints.

   Let x be the number of units of Product A produced and y be the number of units of Product B produced.

   The objective function is to maximize the profit: $Z = 10x + 15y$.

   The constraints are:

   time constraint on Machine 1: $2x + 3y \leq 100$,

   time constraint on Machine 2: $4x + y \leq 200$,

   non-negativity constraints: $x \geq 0, y \geq 0$.

   The complete model is

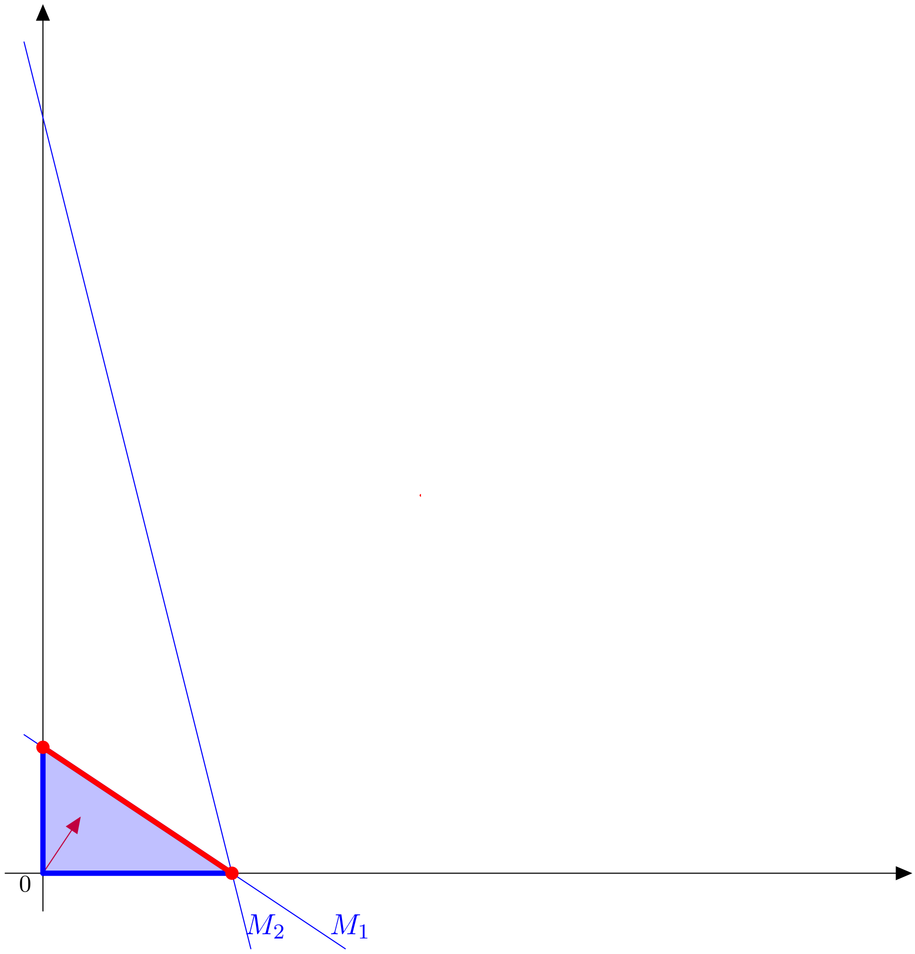
$$\begin{aligned} \max \quad & Z = 10x + 15y \\ \text{s.t.} \quad & 2x + 3y \leq 100, \\ & 4x + y \leq 200, \\ & x \geq 0, y \geq 0. \end{aligned}$$

   The first constraint states that the total time spent on Machine 1 cannot exceed 100 hours. The second constraint states that the total time spent on Machine 2 cannot exceed 200 hours. The non-negativity constraint states that the number of units produced of each product cannot be negative.

   (b) Use the graphical method to solve the problem.

   The problem has several optimal alternative solutions $x^* = \alpha(0,\ 33 + \frac{1}{3}) + (1 - \alpha)(50, 0) = \alpha(0, \frac{100}{3}) + (1 - \alpha)(50, 0)$, with $\alpha \in [0, 1]$, all with value $z = 500$.

   See below a scketch of the graphical resolution.



(c) Interpret the solution obtained.

To achieve a maximum profit of $z = 500$, there are several alternative solutions, the ones in the segment line displayed in red, between the two extreme points $(0, \frac{100}{3})$ and $(50, 0)$. Here are described the optimal solutions that correspond to extreme points.

One alternative optimal solution is $x^* = (0, 33 + \frac{1}{3}) = (0, \frac{100}{3})$ which means that the optimal plan of the company is to produce $\frac{100}{3}$ units of Product B and not produce Product A. This plan utilizes the full time of Machine 1, 100 hours, and $\frac{100}{3}$ hours of the 200 hours available from Machine 2.

Another alternative optimal solution is $x^* = (50, 0)$ which means that the optimal plan of the company is to produce 50 units of Product A and not produce Product B. This plan utilizes the full time of Machine 1, 100 hours, and the full time, 200 hours, of Machine 2.

2. A non-profit organization collects donations of medicine (M), food (F), and clothing (C). In a few days, there will be transportation to an institution, and there is a need to organize the loading. The selection team, whether selecting only medicines, clothes or food, has the capacity to process 20 tonnes. Selecting one tonne of medication takes three times longer than selecting one tonne of clothes or food. The packaging team can process 10 tonnes of donations. Packaging medication and food takes three times and twice as long as packaging clothes (in tonnes), respectively. This shipment

must contain at least 2 tonnes of medication. The following LP problem was formulated to decide which goods to prepare for the next shipment:

$$
\begin{array}{lrcrcrcl}
\max & \multicolumn{7}{l}{Z = 2x_M + 2x_F + x_C} \\
\text{s.t.} & 3x_M & + & x_F & + & x_C & \leq & 20 \\
 & 3x_M & + & 2x_F & + & x_C & \leq & 10 \\
 & x_M & & & & & \geq & 2 \\
 & x_M, & & x_F, & & x_C & \geq 0 &
\end{array}
$$

Where x_j represents the tons of donation of type j ($j = M, F, C$) to prepare for shipment. The objective function translates the total utility considering that one ton of medicine has the same utility as one ton of food, which in turn is twice that of clothes. If necessary, refer to the Solver/Excel reports available in the next page to answer the following questions.

(a) Write and interpret the solution of the problem, including the value of the objective function and the slack (or auxiliary) variables.

$x^* = (2, 2, 0, 12, 0, 0)$ with value $z^* = 8$

(b) Write the dual.

$$
\begin{aligned}
\min \quad & W = 20y_1 + 10y_2 + 2y_3 \\
\text{s.t.} \quad & 3y_1 + 3y_2 + y_3 \geq 2, \\
& y_1 + 2y_2 \geq 2, \\
& y_1 + y_2 \geq 1, \\
& y_1, y_2 \geq 0, \\
& y_3 \leq 0.
\end{aligned}
$$

(c) Interpret the meaning of the first ~~dual~~ slack variable.

The first dual variable y_1 is associated to the first constraint of the primal which is related with the capacity of the selection process.

It's optimal value is 12. It means that the associated primal constraint is binding (is satisfied as an equality). When the actual basis is mantained, it represents the amount of the increase value, in the objective function, when the RHS (right hand side) value $b_1 = 20$ (the capacity of the selection process) of the associated constraint increases one unit.

(d) Determine the optimal solution of the dual (decision and slack variables) by complementarity relations.

The complementarity relations are

$$
\begin{aligned}
(20 - (3x_M + x_F + x_C))y_1 &= 0, \\
(10 - (3x_M + 2x_F + x_C))y_2 &= 0, \\
(x_M - 2)y_3 &= 0, \\
x_M(3y_1 + 3y_2 + y_3 - 2) &= 0, \\
x_F(y_1 + 2y_2 - 2) &= 0, \\
x_C(y_1 + y_2 - 1) &= 0.
\end{aligned}
$$

The primal solution is $x^* = (2, 2, 0, 12, 0, 0)$ thus the previous relations become

$$\begin{aligned} 12y_1 &= 0, \\ 0y_2 &= 0, \\ 0y_3 &= 0, \\ 2(3y_1 + 3y_2 + y_3 - 2) &= 0, \\ 2(y_1 + 2y_2 - 2) &= 0, \\ 0(y_1 + y_2 - 1) &= 0. \end{aligned}$$

Hence

$$\begin{aligned} &y_1 = 0, \\ &\text{nothing can be concluded about } y_2, \\ &\text{nothing can be concluded about } y_3, \\ &(3y_1 + 3y_2 + y_3 - 2) = 0, \\ &(y_1 + 2y_2 - 2) = 0, \\ &\text{nothing can be concluded.} \end{aligned}$$

Therefore

$$\begin{aligned} &y_1 = 0, \\ &3y_1 + 3y_2 + y_3 - 2 = 0, \\ &y_1 + 2y_2 - 2 = 0. \end{aligned}$$

is equivalent to

$$\begin{aligned} &y_1 = 0, \\ &3y - 2 + y_3 - 2 = 0, \\ &2y_2 - 2 = 0. \end{aligned}$$

and gives

$$\begin{aligned} &y_1 = 0, \\ &y_3 = -1, \\ &y_2 = 1. \end{aligned}$$

The decision variables of the dual are $y^* = (0, 1, -1)$ and by replacing in the constraints we obtain the value of the slack variables $y_4 = y_5 = y_6 = 0$. We have $y^* = (0, 1, -1, 0, 0, 0)$.

Note: If you have not solved (2d) and need these values in the following questions, assume $y_1 = 0, y_2 = 2, y_3 = -2$.

(e) Determine the change in total utility if it were not mandatory to transport medications.

The variable y_3 is the dual variable associated with constraint 3. It has a value of $y_3 = -1$. This means that decreasing the RHS of this constraint by 1 will increase the total utility by 1, while maintaining the current basis. The Sensitivity Report shows that the allowable decrease of b_3 is 2. Therefore, decreasing b_3 will increase the objective function value. To achieve the greatest increase in the objective function value, we must increase b_3 by the maximum amount possible, which is 2. Each unit increase in b_3 results in a corresponding increase of 1 in the objective function, resulting in a total increase of 2 units. The total utility will increase by 2 units, resulting in a total of 10.

(f) What would be the new optimal solution and optimal value if the utility of a ton of medicines decreased to 1 per ton?

A ton of medicines has a utility of $c_M = 2$. According to the Sensitivity Report, the allowable decrease is infinite. Therefore, the optimal solution remains unchanged, $x^* = (2, 2, 0, 12, 0, 0)$, while maintaining the current basis, but the value of the objective function decreases by 2 for each unit the utility decreases, resulting in a new value of 6.

Microsoft Excel 16.0 Answer Report

Objective Cell (Max)

Cell	Name	Original Value	Final Value
\$E\$3	max	0	8

Variable Cells

Cell	Name	Original Value	Final Value	Integer
\$B\$9	variable xM	0	2	Contin
\$C\$9	variable xF	0	2	Contin
\$D\$9	variable xC	0	0	Contin

Constraints

Cell	Name	Cell Value	Formula	Status	Slack
\$E\$5	selection capacity	8	\$E\$5<=\$G\$5	Not Binding	12
\$E\$6	packaging capacity	10	\$E\$6<=\$G\$6	Binding	0
\$E\$7	minimum meds	2	\$E\$7>=\$G\$7	Binding	0

Microsoft Excel 16.0 Sensitivity Report

Variable Cells

Cell	Name	Final Value	Reduced Cost	Objective Coefficient	Allowable Increase	Allowable Decrease
B9	variable xM	2	0	2	1	1E+30
C9	variable xF	2	0	2	1E+30	0
D9	variable xC	0	0	1	0	1E+30

Constraints

Cell	Name	Final Value	Shadow Price	Constraint R.H. Side	Allowable Increase	Allowable Decrease
E5	selection capacity	8	0	20	1E+30	12
E6	packaging capacity	10	1	10	24	4
E7	minimum meds	2	-1	2	1.333333333	2

3. Three available volunteers with the ability to perform any of three urgent tasks have been identified. The time needed to perform the tasks (T1, T2 and T3) by the volunteers (V1, V2 and V3) is shown in the following table:

	T1	T2	T3
V1	1	3	4
V2	2	1	3
V3	1	4	4

The aim is to assign a task to each of the volunteers in order to consume as little time as possible in carrying out all the tasks.

The decision variables are binary. Volunteer i performs task j when the value is 1, and 0 otherwise, $\forall i \in \{1,2,3\}, \forall j \in \{1,2,3\}$.

$$
\begin{aligned}
\min \quad & z = x_{11} + 3x_{12} + 4x_{13} + 2x_{21} + x_{22} + 3x_{23} + x_{31} + 4x_{32} + 4x_{33} \\
\text{s.t.} \quad & x_{11} + x_{12} + x_{13} = 1, \\
& x_{21} + x_{22} + x_{23} = 1, \\
& x_{31} + x_{32} + x_{33} = 1, \\
& \\
& x_{11} + x_{21} + x_{31} = 1, \\
& x_{12} + x_{22} + x_{32} = 1, \\
& x_{13} + x_{23} + x_{33} = 1, \\
& \\
& x_{ij} \in \{0,1\}, \quad \forall i \in \{1,2,3\}, \forall j \in \{1,2,3\}.
\end{aligned}
$$

(a) Identify, among the problems studied, one that could be used to solve this problem.

Assignment problem

(b) Propose a feasible solution to the problem and indicate the time that corresponds to it.

$x^* = (0, 0, 1, 0, 1, 0, 1, 0, 0)$ with value $z^* = 6$,

Volunteer V1 performs task T3. Volunteer V2 performs task T2. Volunteer V3 performs task T1. The total time required is 6.

(c) Suppose that a fourth volunteer, V4, is identified who can only perform two of the tasks: task T2 with time 3 and task T3 with time 2. Write a formulation in ILP to include this volunteer and ensure that:

- V4 is assigned one of the tasks it can perform;
- volunteer V2 can only take on task T2 as long as V3 takes on T1.

The decision variables are binary. Volunteer i performs task j when the value is 1, and 0 otherwise, $\forall i \in \{1, 2, 3, 4\}, \forall j \in \{1, 2, 3\}$.

$$\begin{aligned}
\min \quad & z = x_{11} + 3x_{12} + 4x_{13} + 2x_{21} + x_{22} + 3x_{23} + x_{31} + 4x_{32} + 4x_{33} + Mx_{41} + 3x_{42} + 2x_{43} \\
\text{s.t.} \quad & x_{11} + x_{12} + x_{13} \leq 1, \\
& x_{21} + x_{22} + x_{23} \leq 1, \\
& x_{31} + x_{32} + x_{33} \leq 1, \\
& x_{42} + x_{43} = 1, \\
& x_{11} + x_{21} + x_{31} = 1, \\
& x_{12} + x_{22} + x_{32} + x_{42} = 1, \\
& x_{13} + x_{23} + x_{33} + x_{43} = 1, \\
& x_{22} \geq x_{31}, \\
& x_{ij} \geq 0, \quad \forall i \in \{1, 2, 3, 4\}, \forall j \in \{1, 2, 3\}.
\end{aligned}$$

For instance, consider $M = 10$. Alternatively, avoid using variable x_{41}.

4. A farmer must determine whether to plant corn or wheat. If he plants corn, and the weather is warm, he earns \$8 000; if he plants corn, and the weather is cold, he earns \$5 000. If he plants wheat, and the weather is warm, he earns \$7 000; if he plants wheat, and the weather is cold, he earns \$6 500. In the past, 40 percent of all years have been warm and 60 percent have been cold.

(a) Build the payoff matrix.

Let the set of **alternative actions** be $A = \{C, W\}$ with $C \leftarrow$ plant corn, and $W \leftarrow$ plant wheat. Let the set of $n = 2$ **nature states** be $\Theta = \{\theta_w, \theta_c\}$ with $\theta_w \leftarrow$ the weather is warm, and $\theta_c \leftarrow$ the weather is cold. The payoff matrix with values $p(a, \theta)$ is

	θ_w	θ_c
C	8000	5000
W	7000	6500

(b) Determine Laplace action and explain its meaning.

The Laplace action is such that $\max_{a \in A} \left\{ \frac{1}{n} \sum_{\theta \in \Theta} p(a, \theta) \right\}$.

	θ_w	θ_c	$\frac{1}{n}\sum_{\theta\in\Theta} p(a,\theta)$
C	8000	5000	$\frac{1}{2}(8000+5000) = 6500$
W	7000	6500	$\frac{1}{2}(7000+6500) = 6750$

Thus $\max_{a\in A}\left\{\frac{1}{n}\sum_{\theta\in\Theta} p(a,\theta)\right\} = \max_{a\in A}\{6500, 6750\} = 6750$, therefore the Laplace action should be W, plant wheat.

(c) What action does Bayes' procedure recommend.

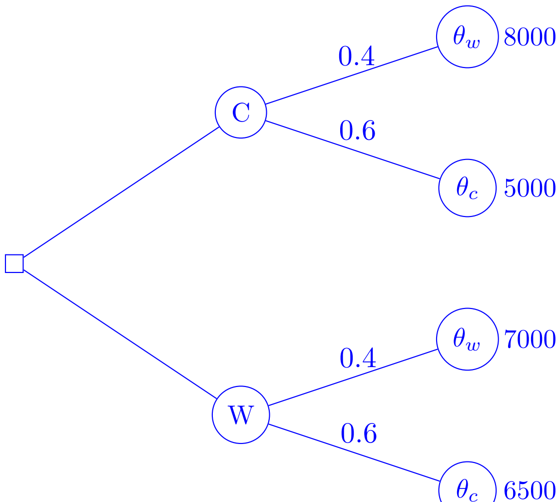
The *a priori* probability $h_\theta(\bar{\theta})$ of each nature state is $h_\theta(\theta_w) = P(\theta = \theta_w) = 0.4$, and $h_\theta(\theta_c) = P(\theta = \theta_c) = 0.6$.

The Bayes' action is such that $\max_{a\in A}\left\{\sum_{\bar{\theta}\in\Theta} h_\theta(\bar{\theta})\ p(a,\bar{\theta})\right\}$.

	θ_w	θ_c	$\sum_{\bar{\theta}\in\Theta} h_\theta(\bar{\theta})\ p(a,\bar{\theta})$
C	8000	5000	$0.4\times 8000 + 0.6\times 5000 = 6200$
W	7000	6500	$0.4\times 7000 + 0.6\times 6500 = 6700$
$h_\theta(\bar{\theta})$	0.4	0.6	

Thus $\max_{a\in A}\left\{\sum_{\bar{\theta}\in\Theta} h_\theta(\bar{\theta})\ p(a,\bar{\theta})\right\} = \max_{a\in A}\{6200, 6700\} = 6700$, therefore the Bayes's procedure recomends action W, plant wheat.

(d) Build a decision tree.



(e) Compute the expected value of perfect information (EVPI).

	θ_w	θ_c	$\sum_{\bar{\theta}\in\Theta} h_\theta(\bar{\theta})\ p(a,\bar{\theta})$
C	8000	5000	$0.4\times 8000 + 0.6\times 5000 = 6200$
W	7000	6500	$0.4\times 7000 + 0.6\times 6500 = 6700$
$h_\theta(\bar{\theta})$	0.4	0.6	

The expected return without any additional information (corresponds to the Bayes's criteria value) is 6700.

The expected return using a "perfect" prediction and weighting by the *a priory* probability is $0.4\times 8000 + 0.6\times 6500 = 7100$.

Hence $\quad \text{EVPI} = 7100 - 6700 = 400$.