Universidade de Lisboa
Instituto Superior de Economia e Gestão
Msc in Economics and Mathematical Finance

Probability Theory and Stochastic Processes – 1st Semester - 2025/2026

Resit Assessment - 8th of January 2026

Duration: $(120 + \varepsilon)$ minutes, $|\varepsilon| \leq 30$

Version A

Name: ....

Student ID #: ....

Part I

- Complete the following sentences in order to obtain true propositions. The items are independent from each other.
- There is no need to justify your answers.

(a) *(4)* In $\Omega = \{i, s, e, g\}$, consider the set $I = \{\{i, s\}, \{e, g\}\} \subset \mathcal{P}(\Omega)$. If $\mathcal{A}(I)$ is the smallest **algebra** containing I, then

$$\mathcal{A}(I) = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

(b) *(6)* Let $A \in \mathcal{B}(\mathbb{R})$ and $f : A \to \mathbb{N}$ a **bijective** map. Then:

- with respect to the **cardinality**, A is ....
  (use one of the terms: *"finite countable"* / *"infinite countable"* / *"finite uncountable"* / *"infinite uncountable"*)
- the Lebesgue measure of A is ....
- if $A = \mathbb{N} \cup \{-2, -1, 0\}$, then the expression of f could be:

$$f(n) = \ldots\ldots\ldots\ldots$$

(c) *(8)* Let Ω be a finite set. Consider a probability space $(\Omega, \mathcal{F}, \mu)$ where $\mathcal{F}$ is an algebra and μ is a **probability measure**. Let $A, B \subset \Omega$ be two events such that:

$$\mu(A) = \frac{2}{5}, \quad \mu(B) = \frac{3}{10}, \quad \text{and} \quad \mu(A|B) = \frac{1}{6},$$

Then, the following assertions hold:

- $\mu(A \cup B) =$ .... and
- A and B are ....
  (use one of the terms: *"independent"* / *"dependent"*)

(d) *(8)* Consider the measurable space $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ where $\mathcal{B}(\mathbb{R})$ denotes the σ–algebra of the Borelians of $\mathbb{R}$. The letters m, δ_a and μ denote the **Lebesgue**, the **Dirac** centered at $a \in \mathbb{R}$ and the **counting** measure, respectively. With respect to the sets

$$A = [0, 4] \cap \mathbb{Q}^c \quad \text{and} \quad B = \mathbb{Q}\backslash\{0\}$$

we may say that:

1. $m(A) =$ ....
2. $\displaystyle\sum_{n=1}^{\infty} \frac{\delta_{\frac{1}{n}}(B)}{2^n} =$ ....
3. $\mu(B \cap C) = 2$. Then, one possibility for C is ....
4. $(m \times \delta_\pi)(A \times A) =$ ....

(e) *(4)* With respect to the map $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = 4 - x^2$, the **graphical representation** of f^- is:

(f) *(6)* Consider the measurable space $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ where $\mathcal{B}(\mathbb{R})$ denotes the σ–algebra of the Borelians of $\mathbb{R}$. The map $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = e^x$$

is **measurable** because f is .... Indeed, if $\mathcal{U}$ is an open set of $Im(f)$, the set $f^{-1}(\mathcal{U})$ is an .... set of $\mathbb{R}$.

(g) *(6)* Consider the measure space $([0,1], \mathcal{B}([0,1]), m)$ where $\mathcal{B}([0,1])$ denotes the σ–algebra of the Borelians of $[0,1]$ and m is the Lebesgue measure. The function $g : [0,1] \to \{0,2\}$ whose analytical expression is:

$$g(x) = \begin{cases} 0 & \text{if} \quad x \in \mathbb{Q} \cap [0,1] \\ \\ 2 & \text{if} \quad x \in \mathbb{Q}^c \cap [0,1] \end{cases} .$$

is **discontinuous at** $x \in$ .... (*write the maximal set where g is discontinuous*) and integrable with respect to Lebesgue. The map g is **not** integrable in the sense of ....

(h) *(6)* The sequence of maps $f_n(x) = x^{2n}$, $x \in [0,1]$, converges.............................. to the map $f(x) \equiv 0$, $x \in [0,1[$ and $f(1) = 1$. However this convergence is **not uniform** because

....

(i) *(6)* Consider the measure space $(\mathbb{R}, \mathcal{B}(\mathbb{R}), m)$ where $\mathcal{B}(\mathbb{R})$ denotes the σ–algebra of the Borelians of $\mathbb{R}$ and m is the Lebesgue measure. Let $f : [0,1] \to \mathbb{R}$ be:

$$f(x) = \begin{cases} 3x & \text{if } x < 1/2 \\ \\ 3 - 3x & \text{if } x \geq 1/2 \end{cases} .$$

For each $n \in \mathbb{N}$, consider the set ($f^m = f \circ \cdots \circ f$ refers to the composition of maps)

$$\Lambda_n = \{x \in [0,1] : f^m(x) \in [0,1], \forall m \in \{1, \ldots, n\}\}.$$

Then:

1. the set $\Lambda = \bigcap_{n \in \mathbb{N}} \Lambda_n$ is usually called by ....set and $m(\Lambda)=$ ....

2. With respect to the **cardinality** of Λ, it is ....

(j) *(10)* Consider the measure space $(\mathbb{R}, \mathcal{B}(\mathbb{R}), m)$ where $\mathcal{B}(\mathbb{R})$ denotes the σ–algebra of the Borelians of $\mathbb{R}$ and m is the Lebesgue measure. For each $n \in \mathbb{N}$, define the sequence of simple maps $\varphi_n \equiv n\chi_{[0,1/n]} : \mathbb{R}_0^+ \to \mathbb{R}_0^+$. The **graphical representation** of φ_3 is:

In this case, we have:

$$\ldots\ldots\ldots\ldots = \int_{[0,+\infty[} \lim_{n \to +\infty} \varphi_n \, \mathrm{dm} < \lim_{n \to +\infty} \int_{[0,+\infty[} \varphi_n \, \mathrm{dm} = \ldots\ldots\ldots$$

This does not contradict the **Dominated Convergence Theorem** because ....

....

(k) *(6)* Consider the measure space $(\mathbb{R}, \mathcal{B}(\mathbb{R}), m)$ where $\mathcal{B}(\mathbb{R})$ denotes the σ–algebra of the Borelians of $\mathbb{R}$ and m is the Lebesgue measure. Let μ be the **measure**

$$\mu(B) = \int_{B\cap[0,1]} x\, dm(x), \quad B \in \mathcal{B}(\mathbb{R}).$$

Then

$$\begin{aligned} \int_{\mathbb{R}} \sqrt{1+x^2}\, d\mu(x) &= \int_{[0,1]} \text{...} dm(x) \\ &= \int_0^1 x\sqrt{1+x^2} dx \\ &= \text{..............} \end{aligned}$$

(l) *(5)* Consider the measure space $(\mathbb{R}^2, \mathcal{B}(\mathbb{R}^2), m \times \delta_1)$ where $\mathcal{B}(\mathbb{R}^2)$ denotes the σ–algebra of the Borelians of $\mathbb{R}^2$, m is the Lebesgue measure on $\mathbb{R}$ and δ_1 is the Dirac measure centered at 1. Consider the map $f : \mathbb{R}^2 \to \mathbb{R}$ defined by

$$f(x, y) = xy.$$

Then

$$\int_{[0,1]^2} f(x,y)\, dm(x)\, d\delta_1(x) = \text{.................}$$

(m) *(10)* Consider the space $[0, 1]$ endowed with the σ–algebra of borelians $\mathcal{B}([0, 1])$. Define the measure

$$\mu = \frac{1}{2}\delta_0 + \frac{1}{2}m,$$

where δ_0 is the **Dirac** measure centered at 0 and m is the **Lebesgue** measure in $[0, 1]$. Then:

1. $\mu(\left[0, \frac{1}{3}\right])$=.........

2. $\delta_0 \perp m$ because...

3. ....-.................. theorem says that there exists a map f (integrable) such that

$$\delta_0(A) = \int_A f\, \mathrm{d}\mu$$

4. Using the notation of the previous item, we have $f(0) = \dfrac{d\delta_0}{d\mu}(0) =$ ....

(n) *(8)* Consider the measure space $([0,1], \mathcal{B}([0,1]), m)$ where $\mathcal{B}([0,1])$ is a σ-algebra, m is the Lebesgue measure on $[0,1]$. Given $A \in \mathcal{B}([0,1])$ and $c_1 \neq c_2 \in \mathbb{R}$, define the random variable $X : \mathbb{R} \to \mathbb{R}$ as $X = c_1\chi_A + c_2\chi_{A^c}$. The following assertions are true:

1. If $m(A) = p > 0$, then the **distribution** α of X is given by:

$$\alpha = ...\delta_{c_1} + ...\delta_{c_2}$$

   known as the .... distribution.

2. In this case, we may conclude that

$$E(X) = ...$$

(o) *(12)* Consider the following homogeneous Markov chain defined on the **finite** state space $\{1,2,3\}$ with transition probability matrix:

$$\mathbf{T} = \begin{bmatrix} 1/2 & 1/2 & 0 \\ 0 & 1/2 & 1/2 \\ 1/2 & 1/2 & 0 \end{bmatrix}$$

Then:

1. All states of the chain may be classified as...................
   (use one of the terms: *"transient"* / *"recurrent null"* / *"recurrent positive"*)

2. The **stationary distribution** associated to T is (.........,......,........)

3. With respect to the periodicity, all states are .... and the Markov chain is ....

4. The **mean recurrence time** associated to 1, 2 and 3 are equal to ...., ... and ...., respectively.

Part II

- Give your answers in exact form.
- In order to receive credit, you must show all of your work. If you do not indicate the way in which you solve a problem, you may get little or no credit for it, even if your answer is correct.

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1. Consider the measure space $(\mathbb{R}, \mathcal{B}(\mathbb{R}), P)$ where $\mathcal{B}(\mathbb{R})$ denotes the σ–algebra of the Borelians and P is a probability measure. Suppose that $X : \mathbb{R} \to \mathbb{R}$ is random variable and let $\lambda \in \mathbb{R}^+$. Prove that:

$$\forall k \in \mathbb{N}, \quad P(|X| \geq \lambda) \leq \frac{1}{\lambda^k} E(|X|^k).$$

Remark: If necessary, you can denote the set $\{\omega \in \mathbb{R} : |X(\omega)| \geq \lambda\}$ by A.

2. Consider $(f_n)_{n\in\mathbb{N}}$ the sequence of continuous maps

$$f_n(x) = \frac{x}{(x^{2n}+1)^2}, \quad x \in \mathbb{R}_0^+$$

(a) Identify the map $f : \mathbb{R}_0^+ \to \mathbb{R}$ such that $\lim\limits_{n\to+\infty} f_n(x) = f(x)$.

(b) Show that for all $n \in \mathbb{N}$ and $x \in \mathbb{R}_0^+$,

$$|f_n(x)| \leq g(x) = \begin{cases} 1 & \text{if } 0 \leq x < 1 \\ x^{-3} & \text{if } x \geq 1 \end{cases}$$

(c) Compute $\lim\limits_{n\to+\infty} \int_{[0,+\infty[} f_n(x)\,\mathrm{d}m(x)$, where m is the Lebesgue measure of $\mathbb{R}$.

3. Let X and Y be two random variables, both with finite variance $Var(X)$ and $Var(Y)$. Show that:

$$Var(X) \neq Var(Y) \quad \Rightarrow \quad X - Y \text{ and } X + Y \text{ are dependent}$$

4. Let $X : \mathbb{R} \to \mathbb{R}$ be a random variable on a probability space $(\mathbb{R}, \mathcal{F}, P)$ with distribution function

$$F_X(x) = \begin{cases} 0 & \text{if } x < 0 \\ \frac{x^2}{4} & \text{if } 0 \leq x < 2 \\ 1 & \text{if } x \geq 2 \end{cases}$$

Define the new random variables: $Z = \sqrt{X}$ and $W = \chi_{\{\omega \in \mathbb{R} : X(\omega) > 1\}}$. Compute

$$E(Z|W)(\omega) \quad \text{for} \quad \omega \in \{x \in \mathbb{R} : W(x) = 0\}.$$

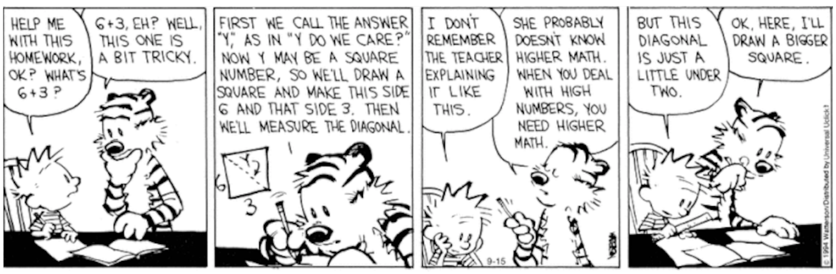
5. Let $(X_n)_n$ be a sequence of **independent** and **identically distributed** random variables on $\mathbb{R}$ such that $E(X_1) = 0$ and $Var(X_1) = \sigma^2 > 0$. Let $\Phi(t)$ be the Taylor expansion of the characteristic map of X_n at $t = 0$.

   (a) Show that $\Phi(t) = 1 - \dfrac{\sigma^2}{2} t^2 + \ldots$ where $\ldots$ represents the sum of monomials of order greater or equal than 3. Compute $\Phi\left(\dfrac{t}{\sqrt{n}}\right)$.

   (b) Compute $\lim_{n \to +\infty} \Phi\left(\dfrac{t}{\sqrt{n}}\right)^n$ and **interpret** the result in the context of *Limit theorems.* Please justify.

Remark: We may assume, without proof, that if $X \sim N(0, \sigma^2)$ then

$$E(e^{itX}) = e^{\frac{-\sigma^2 t^2}{2}}.$$



Credits:

I	II.1	II.2(a)	II.2(b)	II.2(c)	II.3	II.4	II.5(a)	II.5(b)
105	15	10	5	10	15	15	10	15