Lisbon School of Economics & Management
Universidade de Lisboa

Universidade de Lisboa
Instituto Superior de Economia e Gestão
Doctorate Degree (PhD) in Economics

Advanced Mathematical Economics – 1st Semester - 2025/2026

Resit Exam – 09th of January 2026

Duration: 2 hours

Version A

Name: ....

Student ID #: ....

- Give your answers in exact form.
- In order to receive credit, you must show all of your work. If you do not indicate the way in which you solve a problem, you may get little or no credit for it, even if your answer is correct.
- Please answer Groups I and II on separate sheets.

Group I

1. Let $\alpha \in \mathbb{R}$. Consider the following IVP of **second order** (y is a function of x):

$$\begin{cases} y'' + \alpha y' + 6y = 12x + 4 \\ y(0) = -1 \\ y'(0) = 1 \end{cases}$$

for which $y_p(x) = 2x - 1,\ x \in \mathbb{R},$ is a particular solution.

Show that $\alpha = 5$ and solve the problem.

2. Consider the linear system of ODEs in $\mathbb{R}^2$ given by (x and y depend on $t \in \mathbb{R}$):

$$\begin{cases} \dot{x} = x + 2y \\ \dot{y} = -2x + y \end{cases}$$

(a) Write the **general solution** of the system.

(b) Sketch the **phase portrait** of the system.

(c) In the phase portrait of (b), locate the **unique** solution such that $x(0) = 0$ and $y(0) = 3$. What is the limit of this solution when $t \to -\infty$?

3. (Logistic law by Verhulst) Let p be a C^∞ map such that (p depends on $t \in \mathbb{R}_0^+$):

$$\begin{cases} p' = 10p - p^2 \\ p(0) = a \end{cases}$$

(a) Write all possible values of $a \in \mathbb{R}_0^+$ for which the solution of the IVP is **monotonically increasing**.

(b) Solve **explicitly** the IVP for $a = 3$.

4. (Lotka-Volterra system) Consider the system of ODEs in $\mathbb{R}^2$ given by (x and y depend on $t \in \mathbb{R}$):

$$\begin{cases} \dot{x} = 2x - xy \\ \dot{y} = -4y + 8xy \end{cases}$$

(a) Find the equilibria of the system.

(b) Classify the origin $(0, 0)$ according to its **Lyapunov stability**.

(c) Sketch the phase portrait of the system in a small neighbourhood of $(0, 0)$. Justify.

Group II

5. Let X and Y be two random variables with zero mean, uncorrelated, and with the same variance $\sigma^2 > 0$. Consider the stochastic process $(Z_t : t \in \mathbb{Z})$ defined by:

$$Z_t = f(t) \cdot X + g(t) \cdot Y, \quad t \in \mathbb{Z},$$

where f and g are deterministic functions.

(a) Find expressions for f and g so that the process $(Z_t : t \in \mathbb{Z})$ has constant variance but is not necessarily weakly stationary.

(b) Specify f and g such that $(Z_t : t \in \mathbb{Z})$ is weakly stationary.

6. Solve the stochastic differential equation

$$dX_t = \kappa(\theta - X_t)\, dt + \sigma\sqrt{X_t}\, dW_t, \qquad X(0) = X_0 > 0,$$

where $\kappa, \theta, \sigma > 0$ and $(W_t)_{t\geq 0}$ is a standard Wiener process. Apply the change of variable $Y(t) = \sqrt{X_t}$.

7. Consider the stochastic differential equation (SDE)

$$dX(t) = RX(t)\, dt + \sigma X(t)\, dW(t), \qquad X(0) = X_0 > 0,$$

where $R \in \mathbb{R}$, $\sigma \in \mathbb{R}^+$, $W(t)$ is a standard Wiener process, and its solution is given by

$$X(t) = X_0 \exp\left(\left(R - \frac{\sigma^2}{2}\right) t + \sigma W(t)\right).$$

(a) Determine the distribution of $X(t)$ and calculate the mean and variance of $X(t)$.

(b) Now consider $R = 0.378$, $\sigma^2 = 0.149$, and $X_0 = 27.595$. Compute:

$$(i)\, P(30 \leq X(1) \leq 45), \quad (ii)\, E(X(1)), \quad \text{and} \quad (iii)\, Var(X(1)).$$

Credits Group I

1	2(a)	2(b)	2(c)	3(a)	3(b)	4(a)	4(b)	4(c)
1.5	1.5	1	1	1	1.5	1	0.5	1

Credits Group II

5(a)	5(b)	6	7(a)	7(b)
1.5	1.5	3.0	2.0	2.0