I - Mathematical Analysis

1. Topology in $\mathbb{R}$

Exercise 1.

Determine the set of minorants, the set of majorants, the supremum, the infimum, the maximum and the minimum (if any) of the following sets, and indicate which of them are bounded sets:

(a) $A = [-1, 1]$;
(b) $B =]-1, 1[$;
(c) $C = [2, 3[\cup[4, 10[$;
(d) $D =]5, 7[\cup\{15\}$;
(e) $E = \{1, 5, 20\}$;
(f) $F = \{4.9, 4.99, 4.999, \dots\}$;
(g) $G = [0, +\infty[$;
(h) $H = \{\frac{1}{n} : n \in \mathbb{N}\}$;
(i) $I = \{\frac{n-1}{n} : n \in \mathbb{N}\}$;
(j) $J = \{(-1)^n \frac{1}{n} : n \in \mathbb{N}\}$;
(k) $K = \{\frac{1}{n} + \frac{1}{m} : n, m \in \mathbb{N}\}$;
(l) $L = \{m + \frac{1}{n} : n, m \in \mathbb{N}\}$.

Exercise 2.

Determine the interior, exterior, boundary, closure, derivative, and set of isolated points of the following sets, and indicate which of them are open, closed, and/or compact sets:

(a) $A = [-1, 1]$;
(b) $B =]-1, 1[$;
(c) $C = [2, 3[\cup[4, 10[$;
(d) $D =]5, 7[\cup\{15\}$;
(e) $E = \{1, 5, 20\}$;
(f) $F = \{4.9, 4.99, 4.999, \dots\}$;
(g) $G = [0, +\infty[$;
(h) $H = \{\frac{1}{n} : n \in \mathbb{N}\}$;
(i) $I = \{\frac{n-1}{n} : n \in \mathbb{N}\}$;
(j) $J = \{(-1)^n \frac{1}{n} : n \in \mathbb{N}\}$;
(k) $K = \{\frac{1}{n} + \frac{1}{m} : n, m \in \mathbb{N}\}$;
(l) $L = \{m + \frac{1}{n} : n, m \in \mathbb{N}\}$;
(m) $M = \mathbb{Z}$;
(n) $N = \mathbb{Q}$;
(o) $O = [0, 1] \setminus \mathbb{Q}$;
(p) $P = [2, 3] \cap \mathbb{Q}$;
(q) $Q = \{x \in \mathbb{R} : x^2 < 49\}$;
(r) $R = \{x \in \mathbb{R} \setminus \mathbb{Q} : x^2 < 49\}$.

2. Induction

Exercise 3.

Prove that if n is a natural number, then

$$1 + 2 + \cdots + n = \frac{n(n+1)}{2}.$$

Exercise 4.

Let S_n be the sum of the first n odd natural numbers. Calculate S_1, S_2, S_3, S_4, and S_5, deduce that

$$S_n = 1 + 3 + 5 + \cdots + (2n-1) = n^2,$$

and prove this last equality using mathematical induction.

Exercise 5.
Using mathematical induction prove that

$$1 + 2 + 2^2 + \cdots + 2^n = 2^{n+1} - 1.$$

Exercise 6.
Prove that for any natural number n,

$$n < 2^n \,.$$

Exercise 7.
Does $2^n < n!$ for any n natural number? Prove that for any natural number $n \geq 4$,

$$2^n < n!\,.$$

3. Sequences and geometrical series

Exercise 8.
Determine whether each of the following sequences is a geometric progression. If so, find the first term, the ratio of the sequence, and determine its monotonicity.

(a) $a_n = -5$

(b) $a_n = \frac{n-1}{n}$

(c) $a_n = 3n$

(d) $a_n = \left(\frac{2}{3}\right)^{n-1}$

(e) $a_n = \left(\frac{2}{3}\right)^{1-n}$

(f) $a_n = \frac{2n}{7}$

(g) $a_n = n^n$

(h) $a_n = 4 \times 3^{1-n}$

(i) $a_n = (n+1)^3$

(j) $a_n = \frac{(n+1)^2}{2}$

(k) $a_n = (-2)^n$

(l) $a_n = n(n-1)$

Exercise 9.
Determine the terms a_3, a_6 and a_8 of the following geometric sequences, knowing that:

(a) $a_1 = 8$ and $a_4 = 320$

(b) $a_9 = 1024$ and $r = 2$

(c) $a_5 = 48$ and $a_6 = 96$

(d) $a_4 = 64$ and $a_7 = -512$

(e) $a_7 = -2$ and $r = -\frac{1}{3}$

(f) $a_7 = -16$ and $r = \sqrt{2}$

(g) $a_3 = -\frac{1}{2}$ and $a_8 = \frac{243}{2}$

(h) $a_4 - a_2 = 8$ and $a_2 + a_3 = 4$

(i) $a_5 = 4a_3$ and $a_3 + a_6 = 63$

Exercise 10.
Compute:

(a) $\dfrac{2}{5} + \dfrac{2}{5^2} + \dfrac{2}{5^3} + \ldots$

(b) $27 + 9 + 3 + 1 + \dfrac{1}{3} + \dfrac{1}{9} + \dfrac{1}{27} \ldots$

(c) $-\dfrac{1}{2} + \dfrac{1}{2^2} - \dfrac{1}{2^3} + \dfrac{1}{2^4} - \ldots$

(d) $\dfrac{1}{\sqrt{2}} + \dfrac{1}{2} + \dfrac{1}{2\sqrt{2}} + \dfrac{1}{4} + \dfrac{1}{4\sqrt{2}} + \ldots$

Exercise 11.

Study the convergence of each of the following geometric series and calculate, when possible, their respective sum.

(a) $\sum_{n=1}^{+\infty} \frac{3}{4^{n+1}}$

(b) $\sum_{n=0}^{+\infty} \frac{7^{n+1}}{5^{n-1}}$

(c) $\sum_{n=0}^{+\infty} \frac{2 \times 6^{n+1}}{9^{n-2}}$

(d) $\sum_{n=0}^{+\infty} 3\cos(n\pi)5^{-n+1}$

(e) $\sum_{n=3}^{+\infty} \frac{7 \times 5^n}{3^{2n+2}}$

(f) $\sum_{n=3}^{+\infty} \frac{2 \times 5^{2n-1}}{3^{4n}}$

(g) $\sum_{n=1}^{+\infty} \frac{\sin^2\left(n\frac{\pi}{2}\right)}{2^n}$

(h) $\sum_{n=1}^{+\infty} \frac{5 \times (-3)^{n-1}}{2^{n+1}}$

Exercise 12.

Express the following rational numbers as the sum of a convergent geometric series and write them in the form $\frac{a}{b}$ with $a, b \in \mathbb{N}$.

(a) $0,77777\ldots$

(b) $0,52525252\ldots$

(c) $0,123123123\ldots$

(d) $0,811111\ldots$

Exercise 13.

Referring to the sum of a convergent geometric series, write the following real numbers in the form a^b with $a \in \mathbb{N}$ and $b \in \mathbb{Q}$.

(a) $\sqrt{2\sqrt{2\sqrt{2\sqrt{2\ldots}}}}$

(b) $\sqrt[5]{7\sqrt[5]{7\sqrt[5]{7\sqrt[5]{7\ldots}}}}$

Exercise 14.

Indicate for which values of $x \in \mathbb{R}$ each of the following geometric series is convergent, and if so, calculate their respective sum.

(a) $\sum_{n=0}^{+\infty} 3x^n$

(b) $\sum_{n=1}^{+\infty} \left(-\frac{x}{2}\right)^n$

(c) $\sum_{n=2}^{+\infty} \left(\frac{3x-4}{5}\right)^n$

3. Domains of real functions of a real variable

Exercise 15.

Determine the domain of the following fucntions:

(a) $f(x) = \sqrt{x+1}$

(b) $g(x) = \ln(1-x)$

(c) $h(x) = \frac{1}{\sqrt{x^2-4}}$

(d) $j(x) = \sqrt{e^{x^2}-1}$

(e) $l(x) = \sqrt{|x-2|-4}$

(f) $m(x) = \ln(\ln x)$

(g) $n(x) = \frac{1}{-1+\ln x}$

(h) $o(x) = \frac{2}{\sqrt{2-|x-1|}}$

4. Inverse trigonometric functions

Exercise 16.
Compute:

(a) $\arcsin\left(\frac{1}{2}\right)$

(b) $\arccos\left(\frac{1}{2}\right)$

(c) $\arctan(1)$

(d) $\arcsin(-1)$

(e) $\arctan(-1)$

(f) $\arccos(-1)$

Exercise 17.
Compute:

(a) $\sin\left(\arcsin\left(\frac{1}{2}\right)\right)$;

(b) $\arcsin\left(\sin\left(\frac{3\pi}{2}\right)\right)$;

(c) $\tan(\arctan(e))$;

(d) $\arccos\left(\cos\left(-\frac{\pi}{3}\right)\right)$;

(e) $\arcsin\left(\sin\left(\frac{7\pi}{6}\right)\right)$;

(f) $\arctan(\tan(e))$.

(g) $\arccos\left(\sin\left(-\frac{\pi}{6}\right)\right)$;

(h) $\arcsin\left(\cos\left(\frac{2\pi}{3}\right)\right)$;

(i) $\arcsin\left(\tan\left(\frac{\pi}{4}\right)\right)$.

Exercise 18.
Determine the domain of the functions defined by the following expressions and present, for each of them, an equivalent expression that does not involve trigonometric functions.

(a) $\cos(\arcsin(x))$;

(b) $\sin(\arccos(x))$;

(c) $\tan(\arccos(x))$;

(d) $\tan(\arcsin(x))$;

(e) $\cos(\arctan(x))$;

(f) $\cos(2\arctan(x))$;

(g) $\sin(2\arccos(x))$;

(h) $\sin^2(\arcsin(x))$;

(i) $\cos^2(\arccos(x))$.

5. Limits and continuity

Exercise 19.
Calculate the following limits:

(a) $\lim_{x\to 0} \sin\frac{1}{x}$;

(b) $\lim_{x\to 0} x\sin\frac{1}{x}$;

(c) $\lim_{x\to +\infty} x\sin\frac{1}{x}$;

(d) $\lim_{x\to 0} \frac{e^{x^2}-1}{x}$;

(e) $\lim_{x\to \pm\infty} x^2-3x+4$;

(f) $\lim_{x\to \pm\infty} \frac{x^2+x-1}{x-3x^2+4}$;

(g) $\lim_{x\to \pm\infty} \frac{4x^3-5x+1}{2x^2-3x+5}$;

(h) $\lim_{x\to \pm\infty} \frac{x(x^2-1)}{x^2(2x+3)}$;

(i) $\lim_{x\to -3} \frac{x^2+3x}{x^2+6x+9}$;

(j) $\lim_{x\to 2} \frac{x^2-4}{x^3-8}$;

(k) $\lim_{x\to 2} \frac{\sqrt{x+2}-\sqrt{3x-2}}{x-2}$;

(l) $\lim_{x\to 1} \frac{\sqrt{x}-1}{x-1}$.

Exercise 20.
Consider the following function defined by

$$h(x) = \begin{cases} 2x + \arccos(x) & , \quad 0 \leq x < 1 \\ 2 & , \quad x = 1 \\ \frac{x+5}{3} & , \quad 1 < x \leq 4 \end{cases}.$$

(a) Prove that h is continuous throughout its domain.

(b) Applying Bolzano's Theorem, show that there exists $c \in]2,4[$ such that $h(c) = c$.

Exercise 21.
Determine the values of a and b such that, at the indicated points, they make the following functions continuous:

(a) $f(x) = \begin{cases} 3x - 7, & x \geq 3 \\ ax + 3, & x < 3 \end{cases}$, at $x = 3$;

(b) $h(x) = \begin{cases} \sin(x), & x \leq 0 \\ ax + b, & x > 0 \end{cases}$, at $x = 0$;

(c) $g(x) = \begin{cases} x + a, & x < -2 \\ 3ax + b, & -2 \leq x \leq 1 \\ ax + 3, & x > 1 \end{cases}$, at $x = -2$ and $x = 1$.

Exercise 22.
Show that the following functions have an extension by continuity to the point $x = 0$ and define this extension:

(a) $f(x) = \frac{1-e^{3x}}{5x}$;

(b) $g(x) = 1 - \frac{\sin(x)}{x}$;

(c) $h(x) = 1 - x\sin\left(\frac{1}{x}\right)$.

6. Differential calculus

Exercise 23.
Knowing that f is a bijective function, that $f(1) = 3$, $f(3) = 2$, $f'(3) = 7$ and $f'(1) = 9$, calculate $(f^{-1})'(3)$.

Exercise 24.
Calculate the equation of the tangent line to the graph of the function f^{-1} at the indicated point, in the following cases:

(a) $f(x) = x^3 + 3$, point $(4, 1)$;

(b) $f(x) = 2x^5 - x^3 + x + 1$, point $(1, 0)$;

(c) $f(x) = 5x^2e^{2x-4}$, point $(20, 2)$.

Exercise 25.
Consider the "hyperbolic sine" function defined in $\mathbb{R}$ by the expression

$$f(x) := \sinh(x) = \frac{e^x - e^{-x}}{2}.$$

(a) Justify that $\sinh : \mathbb{R} \to \mathbb{R}$ is a bijection and obtain an expression for the respective inverse function argsinh.

(b) Obtain an expression for the derivative of argsinh and evaluate $\text{argsinh}'(0)$.

(c) Calculate $\text{argsinh}'(0)$ again using this time the Inverse Function Derivative Theorem.

Exercise 26.

Consider the function $f : x \in \mathbb{R} \setminus \{3\} \to \dfrac{x+2}{x-3}$.

(a) Show that f is invertible and that for all $x \neq 1$, $f^{-1}(x) = \dfrac{3x+2}{x-1}$.

(b) Using the previous part, show that for all $x \neq 1$, $(f^{-1})'(x) = -\dfrac{5}{(x-1)^2}$.

(c) Obtain again the expression of $(f^{-1})'(x)$ using this time the inverse function derivation theorem.

Exercise 27.

Compute:

(a) $\displaystyle\lim_{x\to\frac{\pi}{4}} \frac{e^{\sin(x)} - e^{\cos(x)}}{\sin(x) - \cos(x)}$

(b) $\displaystyle\lim_{x\to 1} \frac{\ln(ex) - 1}{\sin(3x-3)}$

(c) $\displaystyle\lim_{y\to 0} \frac{5y}{\tan(2y)}$

(d) $\displaystyle\lim_{x\to 3} \frac{\ln(2x-5)}{x^2-9}$

(e) $\displaystyle\lim_{x\to 0} \frac{\sin(ax)}{\sin(bx)}$, $a \in \mathbb{R}$, $b \in \mathbb{R} \setminus \{0\}$

(f) $\displaystyle\lim_{t\to 0} \frac{\cos(\frac{\pi}{2} - t)}{\arctan(t)}$

(g) $\displaystyle\lim_{y\to 0} \frac{1-\cos(\alpha y)}{1-\cos(\beta y)}$, $\alpha \in \mathbb{R}$, $\beta \in \mathbb{R} \setminus \{0\}$

(h) $\displaystyle\lim_{r\to 1} \frac{\sqrt[5]{r} - 1}{\sqrt[3]{r} - 1}$

(i) $\displaystyle\lim_{x\to 0} \frac{x\cos(2x)}{\sin(5x)}$

(j) $\displaystyle\lim_{x\to 0^+} \frac{e^{-\frac{1}{x}}}{x}$

(k) $\displaystyle\lim_{x\to 1^+} \left(\frac{x}{x-1} - \frac{1}{\ln(x)}\right)$

(l) $\displaystyle\lim_{x\to 0} \left(\frac{2}{x} - \frac{1}{xe^{3x}}\right)$

(m) $\displaystyle\lim_{y\to +\infty} y(2\arctan(3y) - \pi)$

(n) $\displaystyle\lim_{x\to 0^+} x^{\sqrt{4x}}$

(o) $\displaystyle\lim_{x\to +\infty} \left(1 + \sin\left(\frac{2}{x}\right)\right)^{\frac{x}{4}}$

(p) $\displaystyle\lim_{t\to 0} \left(\frac{\sin(t)}{t}\right)^{\frac{1}{t^2}}$

Exercise 28.

Verify whether Rolle's Theorem applies to:

(a) $f(x) = x^2 - 3x + 2$ in the interval $[1, 2]$;

(b) $g(x) = |x-1|$ in the interval $[0, 2]$.

Exercise 29.

Use Lagrange's Theorem to prove the following inequalities:

(a) $|\sin(x)| \leq |x|$, for all $x \in \mathbb{R}$;

(b) $|\cos(x) - 1| \leq |x|$, for all $x \in \mathbb{R}$;

(c) $e^x > 1 + x$, for all $x > 0$.

Exercise 30.

Consider the following functiuons:

$$f(x) = e^x, \quad g(x) = \frac{1}{\sqrt{x}}, \quad h(x) = \ln(x), \quad \text{and} \quad i(x) = \sqrt{x}.$$

(a) Calculate the Taylor's polynomials of order $n = 1$, $n = 2$ and $n = 3$ of:

(i) f, centered at the point $a = 0$;

(ii) g, centered at the point $a = 1$;

(iii) h, centered at the point $a = 1$;

(iv) i, centered at the point $a = 16$.

(b) Plot (with a calculator or computer) the graph of each of these functions and the graphs of the respective Taylor polynomials of orders 1 and 2.

(c) Use the second-order polynomials obtained in the first part to obtain approximations of

$$\sqrt[10]{e}; \qquad \frac{1}{\sqrt{0,8}}; \qquad \ln(1,1); \qquad \sqrt{17}.$$

Check the quality of these approximations with a calculator.

Exercise 31.

Consider the function f defined by $f(x) = \sqrt{x}$.

(a) Obtain an approximation of $\sqrt{17}$ with the help of the Taylor polynomial of order 1 (the linear approximation) centered at $a = 16$ of the function i.

(b) Write the Taylor's formula of order 1 centered at $a = 16$ for the function i.

(c) Show that the error made with the linear approximation obtained for $\sqrt{17}$ in part (a) does not exceed $\frac{1}{512}$.

Exercise 32.

Consider the functions f and g defined by $f(x) = e^x$ and $h(x) = \ln(x)$.

(a) Explain the Taylor formula of order 2 centered at $a = 0$ for the function f.

(b) Explain the Taylor formula of order 2 centered at $a = 1$ for the function h.

(c) Using the previous parts, increase the error made in Exercise 30 in the second-order approximations of $\sqrt[10]{e}$ and $\ln(1,1)$.

Exercise 33.

Determine the MacLaurin polynomial of order n of the functions defined by the following expressions:

(a) e^{3x} (b) $\ln(1 + 2x)$ (c) $\sinh(2x)$ (d) $\sqrt{x+1}$

Exercise 34.

Determine the critical points of the following functions and, for each of them, find out whether it is a local minimizer, a local maximizer, or a saddle point.

(a) $4x^3 - 8x - 12$

(b) $2x^3 - 3x^2 - 12x + 2$

(c) $x(x-3)^2 + 4$

(d) $x + \frac{1}{x}$

(e) $x^3 - \frac{2}{x}$

(f) $\frac{x}{e^x}$

(g) $\frac{x}{3^x}$

(h) $\frac{x}{1-x^2}$

(i) xe^x

(j) $x\ln(x)$

(k) $e^x(x^2-1)^3$

(l) $e^{2x-1}(x^2-3)^2$

Exercise 35.

Study the concavities and locate the inflection points of the functions defined by the expressions

(a) $5x - x^2$	(d) $(x^2-9)^3$	(g) xe^x	(j) $x - \sin(2x)$
(b) $x^2 - 4x + 2$	(e) $\dfrac{x}{x^2+4}$	(h) e^{-3x^2}	(k) $x \arctan(x)$
(c) $3x - x^5$	(f) $\sqrt{3x}$	(i) $\ln(1-x^2)$	(l) $\lvert 2x^2 + 9x - 5\rvert$

7. Integral calculus

Exercise 36.

Compute the antiderivatives of the functions defined by the following expressions:

(a) $4x^3 + x^2 - x + 1;$

(b) $\sin(3x) - \frac{1}{2}x^5 + \sqrt{x};$

(c) $\frac{1}{x^2} - \frac{1}{\sqrt{x}} - \frac{1}{3}\cos(5x);$

(d) $\frac{1}{\sqrt[3]{x}} - e^{3x} - \frac{2}{1+x^2} + \frac{2}{5}e^{-2x};$

(e) $\tan(x) + \frac{1}{1+x};$

(f) $(2-4x)^5;$

(g) $\sin(x)e^{\cos(x)};$

(h) $x^2e^{x^3} - x\cos(x^2);$

(i) $x^2(x^3+1)^3;$

(j) $\frac{(1+\ln(x))^5}{x};$

(k) $\frac{\arctan x}{1+x^2} + \frac{1}{(1+x^2)\arctan x};$

(l) $\frac{e^x}{1+e^x} + \frac{e^x}{1+e^{2x}};$

(m) $\frac{\cos(x)}{1+\sin(x)} + \frac{\cos(x)}{1+\sin^2(x)};$

(n) $\sin^3(x)\cos(x) + \sin(x)\cos(x);$

(o) $\frac{x+1}{x+3} + \frac{x+5}{x+7};$

(p) $\frac{1}{1+4x^2} + \frac{1}{3+4x^2};$

(q) $\frac{x^2+5x-2}{1+x^2};$

(r) $\frac{x^2-7x+5}{9+x^2};$

(s) $\frac{x}{\sqrt{2-5x^2}} + \frac{1}{\sqrt{2-5x^2}};$

(t) $\frac{x^4}{e^{x^5}} + \frac{e^{\frac{1}{x}}}{x^2}.$

Exercise 37.

Integrate the rational functions defined by the following expressions:

(a) $\frac{x}{x^2-3x+2};$

(b) $\frac{1}{(x^2-1)(x+2)};$

(c) $\frac{x}{x^2+4};$

(d) $\frac{x^2+2}{(x+1)(x^2+1)};$

(e) $\frac{x^3+3}{x^2-1};$

(f) $\frac{x^3+2x^2+3x+4}{x^2+1}.$

Exercise 38.

(a) Show that $\cos(x) = \dfrac{e^{ix}+e^{-ix}}{2}$ and that $\sin(x) = \dfrac{e^{ix}-e^{-ix}}{2i}$.

(b) Deduce from the previous part that $\cos^2(x) = \dfrac{1+\cos(2x)}{2}$ and that $\sin^2(x) = \dfrac{1-\cos(2x)}{2}$. Compute $\displaystyle\int \cos^2(x)dx$ and $\displaystyle\int \sin^2(x)dx$.

(c) Inspired by the previous paragraphs and using Newton's binomial, calculate

$$\int \cos^4(x)dx \text{ and } \int \sin^6(x)dx.$$

Exercise 39.

Calculate the antiderivatives of the following functions using integration by parts:

(a) xe^{2x};

(b) $x^2\sin(2x)$;

(c) $(x^2+x+1)e^{3x}$;

(d) $\arctan(3x)$;

(e) $\ln(x)$;

(f) $\arccos(x)$;

(g) $\arccos(x)+\arcsin(x)$;

(h) $\arctan(x)+\arctan(1/x)$;

(i) $e^x\sin(2x)$;

(j) $e^{2x}\cos(x)$;

(k) $\sqrt{x}\ln(x)$;

(l) $\ln^2(x)$;

(m) $x^2\arctan x$;

(n) $x^3e^{x^2}$;

(o) $x\ln(x)$;

(p) xe^{x-2};

(q) $\sin^2(x)$;

(r) $\cos^2(x)$.

Exercise 40.

Calculate the following antiderivatives using the indicated substitution:

(a) $\displaystyle\int \frac{1}{(1+x^2)^2}\,dx$, $x=\tan(t)$;

(b) $\displaystyle\int \frac{\tan(x)}{1+\cos(x)}\,dx$, $t=\cos(x)$;

(c) $\displaystyle\int \frac{1}{(1-\sin(x))\cos(x)}\,dx$, $t=\sin(x)$;

(d) $\displaystyle\int \frac{e^{2x}}{\sqrt{1+e^x}}\,dx$, $x=\ln(t^2-1)$;

(e) $\displaystyle\int \sqrt{4-x^2}\,dx$, $x=2\cos(t)$;

(f) $\displaystyle\int \frac{t}{\sqrt{t}+\sqrt[3]{t}}\,dt$, with $t=x^\alpha$ ($\alpha\in\mathbb{R}$ to choose).

Exercise 41.

Consider the functions "hyperbolic cosine" and "hyperbolic sine" defined respectively by

$$\cosh(x) = \frac{e^x+e^{-x}}{2} \quad \text{and} \quad \sinh(x) = \frac{e^x-e^{-x}}{2}.$$

(a) Prove that for all $x\in\mathbb{R}$,

(i) $\cosh' = \sinh$ and $\sinh' = \cosh$;

(ii) $\sinh(2x) = 2\cosh(x)\sinh(x)$;

(iii) $\cosh^2(x)-\sinh^2(x) = 1$ (Fundamental Relation of the Hiperbolic Trigonometry).

(b) Referring to the change of variable $t = 2\sinh(x)$, calculate $\displaystyle\int \sqrt{4+t^2}\,dt$.

Exercise 42.

Evaluate the following integrals geometrically. To verify the results, evaluate the integrals analytically as well. The expression $\lfloor x \rfloor$ in parts (c) and (e) represents the function $\lfloor x \rfloor = \max\{n \in \mathbb{Z} : n \leq x\}$.

(a) $\displaystyle\int_0^3 (2x-4)\,dx$

(b) $\displaystyle\int_0^2 (4x-2)\,dx$

(c) $\displaystyle\int_{-2}^3 \lfloor x \rfloor\,dx$

(d) $\displaystyle\int_{-1}^1 t\cos(t)\,dt$

(e) $\displaystyle\int_1^0 \lfloor 3x \rfloor\,dx$

(f) $\displaystyle\int_a^b x\,dx$

Exercise 43.

Compute the following definite integrals:

(a) $\displaystyle\int_0^2 4x^3\,dx$

(b) $\displaystyle\int_2^3 (6x^2-1)\,dx$

(c) $\displaystyle\int_4^5 (4x+3)\,dx$

(d) $\displaystyle\int_1^2 \frac{3}{x^2}\,dx$

(e) $\displaystyle\int_4^9 \sqrt{x}\,dx$

(f) $\displaystyle\int_1^4 \left(3-\frac{1}{\sqrt{x}}\right)dx$

(g) $\displaystyle\int_2^3 -\frac{1}{x^2-2x+1}\,dx$

(h) $\displaystyle\int_2^3 \frac{1}{x^2+x-2}\,dx$

(i) $\displaystyle\int_{-1}^{\sqrt{2}-1} \frac{1}{x^2+2x+3}\,dx$

(j) $\displaystyle\int_1^e \frac{1}{x(1+x^2)}\,dx$

(k) $\displaystyle\int_{-1}^1 \sqrt{1-x^2}\,dx$

(l) $\displaystyle\int_0^1 3(x^4+4x)(x^5+10x^2)^6\,dx$

(m) $\displaystyle\int_\pi^0 \sin(x)\cos^3(x)\,dx$

Exercise 44.

Using the definite integral, calculate the area of the region of the plane defined by:

(a) $y \leq 9-x^2,\ y \geq x^2$

(b) $y \geq x,\ y \leq 2-x^2$

(c) $y \leq 5,\ y \geq -5x+5,\ y \geq \ln(x)$

(d) $y \leq \frac{1}{x},\ 0 < y \leq x,\ x \leq 4$

(e) $y = x^4-4x^2,\ y = \sqrt{4-x^2}$

(f) $y = \sqrt[3]{x},\ y = \sqrt{x}$

(g) $y = \frac{1}{2}x,\ y = x,\ y = x^2$

(h) $x^2 \leq y \leq \frac{1}{x},\ x \geq 0,\ y \leq 2$

(i) $0 \leq y \leq x^2,\ y \leq \frac{1}{x^2}, 0 \leq x \leq 2$

Exercise 45.

Show that if f is continuous in $[a,b]$ and if $\displaystyle\int_a^b f(x)\,dx = 0$, then f has at least one root in $[a,b]$.

Exercise 46.

Let f be a continuous function on $\mathbb{R}$. For $x \neq 0$ the "average of f on the interval $[0;x]$" is defined by

$$M_f(x) = \frac{1}{x}\int_0^x f(t)\,dt.$$

(a) Indicate the value of $\lim\limits_{x\to 0} M_f(x)$.

(b) Prove that M_f is constant if and only if f is constant.

(c) Prove that the codomain of M_f is contained in that of f.

Exercise 47.

Compute the following limits:

$(a)\ \lim_{x\to 0} \dfrac{\int_0^x \ln(t^2+1)\,dt}{x^3}$; $\qquad (b)\ \lim_{x\to 0} \dfrac{\int_0^{x^2} \cos(t^2)\,dt}{x^2}$.

Exercise 48.

Determine the domain, monotonic intervals and local extrema of the following functions:

$(a)\ G(x) = \int_x^0 \sqrt{1+t^2}\,dt$; $\qquad (b)\ H(x) = \int_0^{x^2} e^{-t^2}\,dt$; $\qquad (c)\ I(x) = \int_0^{2x} \ln(t+2)\,dt$.

Exercise 49.

Let, for $x > 0$, $F(x) = \int_1^x \frac{1}{t} e^{\frac{t^2+1}{t}}\,dt$. Prove that $F\left(\frac{1}{x}\right) = -F(x)$.

Exercise 50.

For $t \in [0;1]$ consider $x = \arctan(t)$.

(a) Prove that $\cos(x) = \dfrac{1}{\sqrt{1+t^2}}$ and that $\sin(x) = \dfrac{t}{\sqrt{1+t^2}}$.

(b) Let $I = \int_0^{\frac{\pi}{4}} \dfrac{\tan(x)}{(\tan(x)+1)(1+\sin^2(x))}\,dx$. Using substitution $x = \arctan(t)$, prove that

$$I = \int_0^1 \frac{t}{(1+2t^2)(t+1)}\,dt.$$

(c) Compute I.

Exercise 51.

Compute the following improper integrals (or show that they are divergent):

$(a)\ \int_{-\infty}^{-1} \dfrac{1}{x^3}\,dx \qquad (c)\ \int_0^{+\infty} \dfrac{\arctan(x)}{x^2+1}\,dx \qquad (e)\ \int_3^5 \dfrac{1}{x-3}\,dx$

$(b)\ \int_0^4 \dfrac{1}{\sqrt{x}}\,dx \qquad (d)\ \int_0^1 \ln(x)\,dx \qquad (f)\ \int_0^1 \dfrac{1}{\sqrt{1-x^2}}\,dx$

II - Linear Algebra

1. Vectors

Exercise 52.
Let $u, v, w \in \mathbb{R}^n$, $\mathbf{0} = (0, \ldots, 0)$ the null vector of $\mathbb{R}^n$, and let $\alpha, \beta \in \mathbb{R}$. Prove the following properties:

(a) $(u + v) + w = u + (v + w)$;

(b) $u + v = v + u$;

(c) $\mathbf{0} + v = v$;

(d) $v + (-v) = \mathbf{0}$;

(e) $\alpha(u + v) = \alpha u + \alpha v$;

(f) $(\alpha + \beta)u = \alpha u + \beta u$;

(g) $\alpha(\beta u) = (\alpha\beta)u$;

(h) $1u = u$.

Exercise 53.
Let $u = (-1, 3, 4), v = (2, 1, -1), w = (-2, -1, 3) \in \mathbb{R}^3$. Compute:

(a) $||u||$;

(b) $||v||$;

(c) $||u + v||$;

(d) $|| - u||$;

(e) $u \cdot (v + w)$;

(f) $d(u, v)$.

Exercise 54.
Prove that:

(a) $||u|| \geq 0, \forall u \in \mathbb{R}^n$;

(b) $||v|| = 0$ if and only if $v = \mathbf{0}$, with $\mathbf{0} = (0, \ldots, 0)$ the null vector of $\mathbb{R}^n$;

(c) $||\alpha u|| = |\alpha| \, ||u||, \forall u \in \mathbb{R}^n, \forall \alpha \in \mathbb{R}$;

Exercise 55.
Consider the vector $v = (2, 4, -3)$. Determine if v is or is not a linear combination of the following vectors:

(a) $a = (1, 2, 0)$, $b = (0, 0, 1)$;

(b) $a = (1, 1, 0)$, $b = (0, 1, 1)$;

(c) $a = (2, 0, 0)$, $b = (1, 3, 0)$, $c = (1, 1, 1)$;

(d) $a = (1, 1, 1)$, $b = (2, 0, 2)$, $c = (3, 1, 3)$.

Exercise 56.
Find out whether the following families of vectors are linearly independent:

(a) $\underline{a} = \{(-2, 4), (1, -2)\}$;

(b) $\underline{b} = \{(2, 1), (-6, -3)\}$;

(c) $\underline{c} = \{(2, 3), (3, 2)\}$;

(d) $\underline{d} = \{(0, 0, 3), (0, 1, 1), (2, 0, 1\}$;

(e) $\underline{e} = \{(2, 0, 0, 0), (1, 3, 0, 0), (4, 2, 1, 0), (2, 1, 2, 1)\}$;

(f) $\underline{f} = \{(1, 1, 1, 1), (0, -2, 3, 5), (3, 3, 3, 3), (1, 2, 2, -7)\}$.

Exercise 57.
Let $\underline{u} = \{u_1, u_2, \ldots, u_p\}$ be a family of p non-null vectors of $\mathbb{R}^n$ that are pairwise orthogonal.

(a) Prove that the family $\underline{u}$ is linearly independent.

(b) Application: show that the family $\underline{u} = \{(1, 1, 1, 1), (1, -1, 1, -1), (-3, 0, 3, 0)\}$ is linearly independent.

Exercise 58.

Discuss, depending on the parameter $\lambda \in \mathbb{R}$, the linear independence of the following families of vectors:

(a) $\underline{u} = \{(3, -1), (1, 1 + \lambda)\}$;

(b) $\underline{u} = \{(1, -2), (\lambda, -1)\}$;

(c) $\underline{v} = \{(1, 2, 3), (-2, \lambda, -6)\}$.

Exercise 59.

Consider a family $\underline{u} = \{u_1, u_2, u_3\}$ of linearly independent vectors of $\mathbb{R}^n$. Prove that the family formed by the vectors $v_1 = 2u_1$, $v_2 = u_1 + u_2$ and $v_3 = u_1 + 3u_3$ is also linearly independent.

Exercise 60.

Prove that a set of k vectors of $\mathbb{R}^n$, with $k \geq 2$, is linearly dependent if and only if one of the vectors is a linear combination of the others.

Exercise 61.

Prove that if the zero vector belongs to a set of vectors of $\mathbb{R}^n$, then the set is linearly dependent.

2. Matrices

Exercise 62.

Consider the matrices

$$A = \begin{bmatrix} -3 & -1 & 0 \\ 2 & 1 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 3 & 5 \\ -1 & 2 & 2 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 2 \\ 3 & 1 \\ 0 & -2 \end{bmatrix} \text{ and } D = \begin{bmatrix} 4 & -3 \\ 2 & -1 \\ 0 & 3 \end{bmatrix}.$$

Indicate which of the following operations are defined and compute, if applicable, the resulting matrix:

(a) $3A$

(b) $A + B$

(c) $B + C$

(d) $4C - 2D$

(e) AB

(f) $(CD)^T$

(g) $(2A)(5C)$

(h) A^2

(i) $(AC)^2$

(j) $(2A - BD)$

(k) $A^T A$

(l) $BC - CB$

(m) AC.

(n) CA.

Exercise 63.

Determine the values of the real parameter α for which the following matrix is symmetric:

$$\begin{bmatrix} \alpha & \alpha^2 - 1 & -3 \\ \alpha + 1 & 2 & \alpha^2 + 4 \\ -3 & 4\alpha & -1 \end{bmatrix}.$$

Exercise 64.

Let $A \in \mathcal{M}_{n \times m}(\mathbb{R})$. Indicate the dimensions of the following matrices:

(a) A^T

(b) AA^T

(c) $A^T A$.

Exercise 65.

Let A, B, and C be square matrices of the same dimension. Indicate which of the following propositions are necessarily true, and provide a counterexample to the others.

(a) $A = B \Rightarrow AC = BC$.

(b) $AC = BC \Rightarrow A = B$.

(c) $AB = BA$.

(d) $AB = 0 \Rightarrow A = 0 \vee B = 0$.

(e) $A + C = B + C \Rightarrow A = B$.

(f) $A^2 = I \Rightarrow A = \pm I$.

Exercise 66.

Let A be a square matrix such that $A^T = 4A$. Show that A is the zero matrix.

Exercise 67.

Let $(L_i)_{1\leq i\leq n}$ and $(C_j)_{1\leq j\leq m}$ be the families of row vectors and column vectors of a given matrix $A \in \mathcal{M}_{n\times m}(\mathbb{R})$, respectively. Show that if these families are both linearly independent, A is a square matrix.

Exercise 68.

Determine the rank of the following matrices:

(a) $\begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}$

(b) $\begin{bmatrix} 3 & 6 & 0 \\ -6 & -12 & 1 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & -6 & 0 & 0 \\ -6 & 0 & 9 & 1 \\ 0 & 1 & -1 & 4 \end{bmatrix}$

(d) $\begin{bmatrix} 1 & 1 & -1 \\ 2 & 0 & 2 \\ -3 & 1 & -4 \end{bmatrix}$

(e) $\begin{bmatrix} 2 & 4 & 1 & 1 \\ -4 & 8 & 0 & 0 \\ -2 & 12 & 1 & 1 \\ 0 & 0 & 0 & 3 \end{bmatrix}$

(f) $\begin{bmatrix} 1 & 2 & -2 & 3 \\ 2 & 4 & -2 & 6 \\ 3 & 6 & -6 & 19 \end{bmatrix}$

(g) $\begin{bmatrix} 1 & 1 & 1 \\ 2 & 0 & 4 \\ 3 & 2 & 4 \\ 0 & 5 & -5 \end{bmatrix}$

(h) $\begin{bmatrix} 1 & -3 & 5 \\ 4 & -12 & 20 \\ 3 & -9 & 15 \\ 2 & -6 & 10 \end{bmatrix}$

(i) $\begin{bmatrix} 1 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix}$

Exercise 69.

Discuss the rank of the following matrices in terms of the parameter $t \in \mathbb{R}$:

(a) $\begin{bmatrix} t & 0 & t^2-2 \\ 0 & 1 & 1 \\ -1 & t & t-1 \end{bmatrix}$

(b) $\begin{bmatrix} t+3 & 5 & 6 \\ -1 & t-3 & -6 \\ 1 & 1 & t+4 \end{bmatrix}$

(c) $\begin{bmatrix} t & 1 & 2 \\ 1 & 1 & 1 \\ -1 & 1 & 1-t \end{bmatrix}$

Exercise 70. Considere, para $a, b \in \mathbb{R}$, as matrizes

$$A = \frac{1}{8}\begin{bmatrix} 16 & -8 & -8 \\ a & 2 & b \\ 1 & 1 & -1 \end{bmatrix} \text{ e } B = \begin{bmatrix} 1 & 2 & 4 \\ 0 & 1 & 6 \\ 1 & 3 & 2 \end{bmatrix}.$$

Sabendo que $B = A^{-1}$, determine os valores de a e de b.

Exercise 71.
Prove that the following matrices are invertible and compute their corresponding inverse matrix:

(a) $\begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix}$ (b) $\begin{bmatrix} 2 & 4 \\ 1 & 3 \end{bmatrix}$ (c) $\begin{bmatrix} 1 & 1 & -1 \\ 2 & 0 & 2 \\ -3 & 1 & -4 \end{bmatrix}$ (d) $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & -1 & 1 \end{bmatrix}$

(e) $\begin{bmatrix} 1 & -1 & 1 \\ 1 & 0 & 1 \\ 2 & 1 & 1 \end{bmatrix}$ (f) $\begin{bmatrix} 1 & 3 & 3 \\ 1 & 3 & 4 \\ 1 & 4 & 3 \end{bmatrix}$ (g) $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{bmatrix}$ (h) $\begin{bmatrix} 2 & 3 & 4 \\ 4 & 3 & 1 \\ 1 & 2 & 4 \end{bmatrix}$

Exercise 72.
Prove that a diagonal matrix is invertible if and only if no element of its main diagonal is zero.

Exercise 73.
Let A be a square matrix such that $5AA^T + 2A + 5I = 0$. Show that A is invertible, and derive its inverse in terms of A.

Exercise 74.
Let A and B be invertible matrices of order n.

(a) Prove that $B^{-1}A^{-1}$ is the inverse matrix of AB.

(b) Solve the matrix equation $XA^{-1} + (AB)^{-1} = A$ in order of X.

Exercise 75.
For each matrix A, perform the indicated calculation, deduce that A is invertible and express A^{-1}.

(a) $A = \begin{bmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix}$, A^2+A. (b) $A = \begin{bmatrix} 1 & 0 & 2 \\ 0 & -1 & 1 \\ 1 & -2 & 0 \end{bmatrix}$, $A^3 - A$. (c) $A = \begin{bmatrix} 3 & 1 \\ 0 & 3 \end{bmatrix}$, $A^2 - 6A + 9I$.

3. Determinants

Exercise 76. Consider the matrices $A = \begin{bmatrix} 3 & 2 \\ 4 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} 4 & 2 \\ 1 & 4 \end{bmatrix}$.

(a) Determine A^{-1} and B^{-1}.

(b) Compute the determinant of the matrices A, B, A^T, B^T, $2A$, $3B$, AB, BA, A^{-1}, B^{-1} and $A^{-1}B$, identifying the properties that the calculations illustrate.

Exercise 77.
Let A be a square matrix of order 4 with $\det(A) = 2$. Calculate:

(a) $\det(A^2)$ (b) $\det(3A)$ (c) $\det(-A^{-1})$

(d) $\det(2A^T)$ (e) $\det(AA^TA^{-1})$ (f) $\det\left(\frac{1}{2}A^4\right)$.

Exercise 78.

Consider a square matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ invertible. Prove that

$$A^{-1} = \frac{1}{\det(A)} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

Exercise 79.

Consider the matrix $C = \begin{bmatrix} 1 & 2 & 3 & 4 \\ -a & a & a & a \\ b & b & b & 0 \\ 5 & 6 & 7 & 8 \end{bmatrix}$, with a and b are two real parameters.

It is known that $\det(C) = 5$. Calculate the determinant of the matrix $D = \begin{bmatrix} 5 & 6 & 7 & 8 \\ -2a & 2a & 2a & 2a \\ 7b & 7b & 7b & 0 \\ 1 & 2 & 3 & 4 \end{bmatrix}$.

Exercise 80.

Compute the determinants of the following matrices:

(a) $\begin{bmatrix} 1 & 2 & 2 \\ 1 & 0 & 1 \\ 3 & 0 & 5 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & 1 & -1 \\ 2 & 0 & 2 \\ -3 & 1 & -4 \end{bmatrix}$ (c) $\begin{bmatrix} 1 & 1 & 1 \\ 2 & 3 & 2 \\ 5 & 1 & 7 \end{bmatrix}$ (d) $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$

(e) $\begin{bmatrix} 1 & 1 & 3 & 1 \\ 4 & 5 & 6 & 1 \\ 0 & 0 & 3 & 0 \\ 7 & 8 & 9 & 1 \end{bmatrix}$ (f) $\begin{bmatrix} 10 & 0 & -5 & 15 \\ -2 & 7 & 3 & 0 \\ 8 & 14 & 0 & 2 \\ 0 & -21 & 1 & -1 \end{bmatrix}$ (g) $\begin{bmatrix} -1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 \\ 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & -1 \end{bmatrix}$

Exercise 81.

Calculate the determinants of the following matrices in terms of the real parameters a, b, and c:

(a) $\begin{bmatrix} a & b & c \\ c & a & b \\ b & c & a \end{bmatrix}$ (b) $\begin{bmatrix} 1 & a & b+c \\ 1 & b & c+a \\ 1 & c & a+b \end{bmatrix}$ (c) $\begin{bmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{bmatrix}$

(d) $\begin{bmatrix} a & a & b & 0 \\ a & a & 0 & b \\ b & 0 & a & a \\ 0 & b & a & a \end{bmatrix}$ (e) $\begin{bmatrix} a+b & a & a & a \\ a & a+b & a & a \\ a & a & a+b & a \\ a & a & a & a+b \end{bmatrix}$

Exercise 82.

Determine for which values of the real parameter λ the following matrices are invertible:

(a) $\begin{bmatrix} 1 & 1 & \lambda \\ \lambda & 1 & 1 \\ 1 & \lambda & 1 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & 2 & 1 \\ 1 & 1 & 2 \\ \lambda & \lambda & \lambda \end{bmatrix}$ (c) $\begin{bmatrix} 1 & 0 & \lambda \\ 0 & \lambda & 0 \\ \lambda & 0 & 1 \end{bmatrix}$ (d) $\begin{bmatrix} 1 & 2 & \lambda \\ 1 & 1 & 0 \\ 2 & 3 & 1 \end{bmatrix}$

4. Systems of linear equations

Exercise 83.

For each of the following system of linear equations:

(1) Explain the corresponding augmented matrix and reduce it to a row echelon form matrix;

(2) Classify the system and determine, if applicable, its number of degrees of freedom;

(3) Explicitly solve the system.

(a) $\begin{cases} -2x - 3y + z = 3 \\ 4x + 6y - 2z = 1 \end{cases}$ (b) $\begin{cases} x - y + z = 0 \\ x + 2y - z = 0 \\ 2x + y + 3z = 0 \end{cases}$ (c) $\begin{cases} x + y + z = 1 \\ x - y + z = 0 \\ y + 2z = 1 \\ 2x + y + 4z = 2 \end{cases}$

(d) $\begin{cases} x - y + 2z + w = 1 \\ 2x + y - z + 3w = 3 \\ x + 5y - 8z + w = 1 \\ 4x + 5y - 7z + 7w = 7 \end{cases}$ (e) $\begin{cases} x + y + z + w = 0 \\ x + 3y + 2z + 4w = 0 \\ 2x + y - w = 1 \end{cases}$ (f) $\begin{cases} x + y - z + w = 2 \\ 2x - y + z - 3w = 1. \end{cases}$

Exercise 84.

Classify the following systems according to the real parameters a and b:

(a) $\begin{cases} x + y + z = 3 \\ x - y + z = 1 \\ 2x - 2y + az = 2 \end{cases}$ (b) $\begin{cases} x + y = a \\ x + 2y = a^2 \\ x + 3y = a^3 \end{cases}$ (c) $\begin{cases} ax + z = 2 \\ x + y + z = 2 \\ 3x + y + 3z = 6 \end{cases}$

(d) $\begin{cases} x + y + z = 1 \\ z - y + 2z = a \\ 2x + bz = 2 \end{cases}$ (e) $\begin{cases} 2x + y = b \\ 3x + 2y + z = 0 \\ x + ay + z = 2 \end{cases}$ (f) $\begin{cases} by + az = 1 \\ y + az = 0 \\ x + by = 0 \end{cases}$

Exercise 85.

Prove that the system of linear equations $\begin{cases} x + 2y - z = -5 \\ 2x - y + z = 6 \\ x - y - 3z = -3. \end{cases}$ satisfies the conditions for Cramer's Rule.

Exercise 86.

Prove that the system of linear equations $\begin{cases} x + y = 3 \\ x + z = 2 \\ y + z = 1. \end{cases}$ satisfies the conditions for Cramer's Rule and determine the value of the unknown z of the solution.

Exercise 87.

Show that, regardless of the values of the real parameters a, b and c, the system of linear equations $\begin{cases} 3x + y = a \\ x - y + 2z = b \\ 2x + 3y - z = c \end{cases}$ consistent system with a unique solution and compute the corresponding solution in two ways:

(a) Using the Cramer's Rule;

(b) Starting by computing the inverse matrix of $A = \begin{bmatrix} 3 & 1 & 0 \\ 1 & -1 & 2 \\ 2 & 3 & -1 \end{bmatrix}$.