STATISTICS I

Bachelor's degrees in Economics and Finance

2^{nd} Year/2^{nd} Semester

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Pratical Class Nº 8

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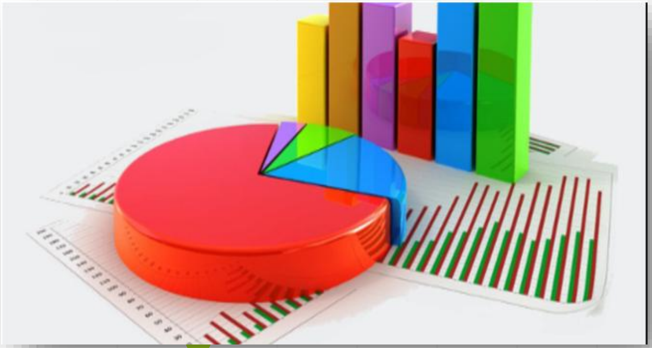

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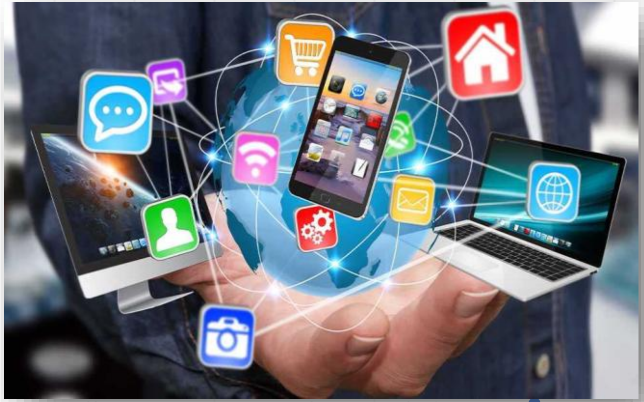

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1. Basic Probability Theory.
2. Univariate random variables.
3. Expected Values.
4. Multivariate random variables (random vectors).
5. Expected Values of Functions of Random Vectors.
6. Special Random Variables and Repeated Sampling Distributions.

Bibliography:

- Miller & Miller, John E. , Freund's Mathematical Statistics with applications , 8th Edition, Pearson Education, [MM], 2013
- P. Newbold, W. Carlson, B. Thorne, , Statistics for Business and Economics , 8th Edition, Pearson Education, [N], 2012

Two-Dimensional Discrete Random Variables: Exercises

1

Expectations and Parameters for two Dimensional Random Variables

Chapter 5

1. If X and Y have the joint probability distribution

$$f(x, y) = 1/4, \text{ for all } (x, y) \in \{(-3, -5), (-1, -1), (1, 1), (3, 5)\}.$$

Compute the $E(Z^2)$, where $Z = XY - 2X$.

Exercise 1

$f(x,y) = 1/4$, for all $(x,y) \in \underbrace{\{(-3,-5),(-1,-1),(1,1),(3,5)\}}_{D_{X,Y} = \{(x,y):\, x \in D_X \wedge y \in D_Y \wedge f_{X,Y}(x,y) > 0\}}$.

$$E[g(X,Y)] = \sum_{(x,y)\in D_{X,Y}} g(x,y)\, f_{X,Y}(x,y)$$

$$E(Z^2) = E[(\underbrace{XY - 2X}_{g(X,Y)})^2] = \sum_{(x,y)\in D_{X,Y}} (xy - 2x)^2 f_{X,Y}(x,y) =$$

$$= \frac{((-3)(-5) - 2(-3))^2 + ((-1)(-1) - 2(-1))^2 + (1\times 1 - 2\times 1)^2 + (3\times 5 - 2\times 3)^2}{4} =$$

$$= \frac{532}{4} = 133$$

3. If X and Y have the joint probability distribution

X/Y	-1	0	1
0	0	1/6	1/12
1	1/4	0	1/2

Show that

(a) $cov(X, Y) = 0$;

(b) the two random variables are not independent.

Exercise 3 a)

X and Y have the joint probability distribution:

X/Y	-1	0	1	$f_X(x)$
0	0	1/6	1/12	3/12
1	1/4	0	1/2	3/4
$f_Y(y)$	$\frac{1}{4}$	$\frac{1}{6}$	$\frac{7}{12}$	

a)

$$E(X) = \sum_{x \in D_X} x \, f_X(x) = 0 \times \frac{3}{12} + 1 \times \frac{3}{4} = \frac{3}{4}$$

$$E(Y) = \sum_{y \in D_Y} y \, f_Y(y) = -1 \times \frac{1}{4} + 0 \times \frac{1}{6} + 1 \times \frac{7}{12} = -\frac{1}{4} + \frac{7}{12} = -\frac{3}{12} + \frac{7}{12} = \frac{4}{12} = \frac{2}{6} = \frac{1}{3}$$

$$E(XY) = \sum_{(x,y) \in D_{X,Y}} \sum x y \, f_{X,Y}(x,y) = 0 \times 0 \times \frac{1}{6} + 0 \times 1 \times \frac{1}{12} + 1 \times (-1) \times \frac{1}{4} + 1 \times 1 \times \frac{1}{2} =$$

$$= -\frac{1}{4} + \frac{1}{2} = -\frac{1}{4} + \frac{2}{4} = \frac{1}{4}$$

$$\operatorname{cov}(X,Y) = E(XY) - E(X)E(Y) = \frac{1}{4} - \frac{1}{3} \times \frac{3}{4} = \frac{1}{4} - \frac{1}{4} = 0, \quad QED$$

Exercise 3 b)

b)

$f_{X,Y}(0,-1) = 0 \neq f_X(0)\, f_X(-1) = \frac{3}{12} \times \frac{1}{4} \Rightarrow X \not\perp Y$, QED

5. Let X_1,X_2, and X_3 be independent random variables with means 4, 9, and 3 and the variances 3, 7, and 5.

   (a) Find the means and the variances of $Y = 2X_1 - 3X_2 + 4X_3$ and $Z = X_1 + 2X_2 - X_3$.

   (b) Repeat (a) and (b) dropping the assumption of independence and using instead the information that $cov(X_1, X_2) = 1$, $cov(X_2, X_3) = -2$, and $cov(X_1, X_3) = -3$.

Exercise 5 a)

$E(X_1) = 4 \qquad Var(X_1) = 3$
$E(X_2) = 9 \qquad Var(X_2) = 7$
$E(X_3) = 3 \qquad Var(X_3) = 5$

X_1, X_2 and X_3 are independent from each other

Note 1: $E\left(\sum_{i=1}^{m} a_i E(X_i)\right) = \sum_{i=1}^{m} a_i E(X_i)$

Note 2: If $X_1, \ldots, X_n$ are random variables and $a_1, \ldots, a_n$ are constants and $Y = \sum_{i=1}^{n} a_i X_i$, then

$$Var(Y) = \sum_{i=1}^{n} a_i^2 Var(X_i) + 2 \underbrace{\sum_{i=1}^{n} \sum_{j=1, j<i}^{n} a_i a_j Cov(X_i, X_j)}_{=0,\text{ if } X_i, X_j \text{ are independent}}.$$

With $m = 3$ we have: $Var(Y) = \sum_{i=1}^{3} a_i^2 Var(X_i) + 2 \sum_{i=1}^{3} \sum_{j=1, j<i}^{3} a_i a_j cov(X_i, X_j)$

a) Since $m = 3$ and $X_i \perp X_j$ $(i, j = 1, 2, 3;\ i \neq j)$ we have

$$Var\left(\sum_{i=1}^{3} a_i X_i\right) = \sum_{i=1}^{3} a_i^2 Var(X_i)$$

$Y = 2X_1 - 3X_2 + 4X_3$

$E(Y) = E(2X_1 - 3X_2 + 4X_3) = 2E(X_1) - 3E(X_2) + 4E(X_3) =$

$= 2 \times 4 - 3 \times 9 + 4 \times 3 = 8 - 27 + 12 = -7$

$Var(Y) = Var(2X_1 - 3X_2 + 4X_3) = 2^2 Var(X_1) + (-3)^2 Var(X_2) + 4^2 Var(X_3) =$

$= 4 \times 3 + 9 \times 7 + 16 \times 5 = 155$

Exercise 5 a)

$$Z = X_1 + 2X_2 - X_3$$

$$E(Z) = E(X_1 + 2X_2 - X_3) = E(X_1) + 2E(X_2) - E(X_3) = 4 + 2 * 9 - 3 = 19$$

$$Var(Z) = Var(X_1 + 2X_2 - X_3) = Var(X_1) + 2^2 Var(X_2) + (-1)^2 Var(X_3) =$$

$$= 3 + 4 * 7 + 5 = 36$$

Exercise 5 b)

$\text{cov}(X_1, X_2) = 1 = \text{Cov}(X_2, X_1)$
$\text{cov}(X_2, X_3) = -2 = \text{Cov}(X_3, X_2)$
$\text{cov}(X_1, X_3) = -3 = \text{cov}(X_3, X_1)$

With $n = 3$ and not assuming independence we have:

$$\text{Var}\left(\sum_{i=1}^{3} a_i X_i\right) = \sum_{i=1}^{3} a_i^2 \text{Var}(X_i) + 2 \sum_{i=1}^{3} \sum_{j=1,\, j<i}^{3} a_i a_j \text{Cov}(X_i, X_j)$$

The expected values, $E(Y) = -7$ and $E(Z) = 19$, remain unchanged.

$$\text{Var}(Y) = \text{Var}(2X_1 - 3X_2 + 4X_3) = 2^2 \text{Var}(X_1) + (-3)^2 \text{Var}(X_2) + 4^2 \text{Var}(X_3) + 2\big((-3 \times 2) \underbrace{\text{Cov}(X_2, X_1)}_{1} + (4 \times 2) \underbrace{\text{cov}(X_3, X_1)}_{-3} + (4 \times (-3)) \underbrace{\text{Cov}(X_3, X_2)}_{-2} =$$

$$= 4 \times 3 + 9 \times 7 + 16 \times 5 + 2(-6 + 8(-3) - 12(-2)) =$$

$$= 143$$

$$\text{Var}(Z) = \text{Var}(X_1 + 2X_2 - X_3) =$$

$$= \text{Var}(X_1) + 2^2 \text{Var}(X_2) + (-1)^2 \text{Var}(X_3) + 2\big((1 \times 2) \underbrace{\text{cov}(X_2, X_1)}_{1} + (-1 \times 1) \underbrace{\text{Cov}(X_3, X_1)}_{-3} + (-1 \times 2) \underbrace{\text{Cov}(X_3, X_2)}_{-2}\big) =$$

$$= 3 + 4 \times 7 + 5 + 2(2 + 3 + 4) = 54$$

7. Let (X, Y) be a discrete random vector with joint probability function given by:

$$f(x, y) = \frac{x+y}{21}, \quad x = 1, 2; \; y = 1, 2, 3$$

(a) Compute the means and variances of X and Y

(b) Using $E(XY)$ analyze the independence of the two random variables and compute the correlation coefficient.

(c) Compute $E(X|Y=1)$

Exercise 7 a)

$f_{X,Y}(x, y)$:

$x \backslash y$	1	2	3	$f_X(x)$
1	$\frac{2}{21}$	$\frac{3}{21}$	$\frac{4}{21}$	$\frac{9}{21}$
2	$\frac{3}{21}$	$\frac{4}{21}$	$\frac{5}{21}$	$\frac{12}{21}$
$f_Y(y)$	$\frac{5}{21}$	$\frac{7}{21}$	$\frac{9}{21}$	

$$E(X) = \sum_{x \in D_X} x f_X(x) = \frac{9}{21} + 2 \times \frac{12}{21} = \frac{33}{21} = \frac{11}{7}$$

$$E(Y) = \sum_{y \in D_Y} y f_Y(y) = \frac{5}{21} + 2 \times \frac{7}{21} + 3 \times \frac{9}{21} = \frac{46}{21}$$

$$E(X^2) = \sum_{x \in D_X} x^2 f_X(x) = \frac{9}{21} + 2^2 \times \frac{12}{21} = \frac{57}{21}$$

$$E(Y^2) = \sum_{y \in D_Y} y^2 f_Y(y) = \frac{5}{21} + 2^2 \times \frac{7}{21} + 3^2 \times \frac{9}{21} = \frac{114}{21}$$

$$Var(X) = E(X^2) - E(X)^2 = \frac{57}{21} - \left(\frac{33}{21}\right)^2 = \dots = \frac{12}{49}$$

$$Var(Y) = E(Y^2) - E(Y)^2 = \frac{114}{21} - \left(\frac{46}{21}\right)^2 = \dots = \frac{278}{441}$$

Exercise 7 b)

$$E(XY) = \sum_{(x,y)\in D_{X,Y}} x\,y\, f_{X,Y}(x,y) =$$

$$= (1\times 1)\frac{2}{21} + (1\times 2)\frac{3}{21} + (1\times 3)\frac{4}{21} + (2\times 1)\frac{3}{21} + (2\times 2)\frac{4}{21} + (2\times 3)\frac{5}{21} =$$

$$= \frac{2}{21} + \frac{6}{21} + \frac{12}{21} + \frac{6}{21} + \frac{16}{21} + \frac{30}{21} = \frac{72}{21} = \frac{24}{7} \neq E(X)E(Y) = \frac{11}{7}\times\frac{46}{21} = \frac{506}{147}$$

Therefore X and Y are not independent.

$$\rho_{X,Y} = \frac{\mathrm{cov}(X,Y)}{\sigma_X\,\sigma_Y} = \frac{E(XY) - E(X)E(Y)}{\sqrt{\mathrm{Var}(X)\,\mathrm{Var}(Y)}} =$$

$$= \frac{\frac{24}{7} - \frac{506}{147}}{\sqrt{\frac{12}{49}\times\frac{278}{441}}} \simeq -0.035 \rightarrow$$

This makes sense because, as we already know, X and Y are not independent (if they were we would have $\mathrm{cov}(X,Y) = \rho_{X,Y} = 0$).

Exercise 7 c)

$$E(X \mid Y=1) = \sum_{x \in D_X} x \, f_{X|Y=1}(x) \, dx =$$

$$= \sum_{x=1}^{2} x \frac{f_{X,Y}(x,1)}{f_Y(1)} = 1 \frac{f_{X,Y}(1,1)}{f_Y(1)} + 2 \frac{f_{X,Y}(2,2)}{f_Y(1)} =$$

$$= \frac{\frac{2}{21}}{\frac{5}{21}} + 2 \frac{\frac{3}{21}}{\frac{5}{21}} = \frac{2}{5} + \frac{6}{5} = \frac{8}{5}$$

10. Let (X, Y) be a two dimensional random variable, such that its set of discontinuities is $D_{X,Y} = \{(0,1), (0,2), (1,0), (1,1), (1,2)\}$ and its probability function is

$$f_{X,Y}(x,y) = \begin{cases} \frac{x+y}{a}, & (x,y) \in D_{X,Y} \\ 0, & \text{otherwise} \end{cases}.$$

(a) Find a and represent $f_{X,Y}$ by filling a suitable table.

(b) Compute $E(X)$ and $E(Y)$.

(c) Calculate $Cov(2X, 3Y)$ and $Var(X+Y)$.

(d) Characterize the distribution of $E(Y|X)$.

Exercise 10 a)

$f_{X,Y}(x,y)$:

$X \backslash Y$	0	1	2	$f_X(x)$
0	0	$\frac{1}{a} = \frac{1}{9}$	$\frac{2}{a} = \frac{2}{9}$	$\frac{3}{a} = \frac{3}{9}$
1	$\frac{1}{a} = \frac{1}{9}$	$\frac{2}{a} = \frac{2}{9}$	$\frac{3}{a} = \frac{3}{9}$	$\frac{6}{a} = \frac{6}{9}$
$f_Y(y)$	$\frac{1}{a} = \frac{1}{9}$	$\frac{3}{a} = \frac{3}{9}$	$\frac{5}{a} = \frac{5}{9}$	

$$\sum_{(x,y) \in D_{X,Y}} f_{X,Y}(x,y) = 1 \Leftrightarrow \frac{1}{a} + \frac{2}{a} + \frac{1}{a} + \frac{2}{a} + \frac{3}{a} = \frac{9}{a} = 1 \Leftrightarrow a = 9$$

Exercise 10 b)

b)

$$f_X(x) = \begin{cases} \frac{3}{9} = \frac{1}{3} & (x = 0) \\ \frac{6}{9} = \frac{2}{3} & (x = 1) \\ 0 & (\text{elsewhere}) \end{cases} \qquad f_Y(y) = \begin{cases} \frac{1}{9} & (y = 0) \\ \frac{3}{9} = \frac{1}{3} & (y = 1) \\ \frac{5}{9} & (y = 2) \\ 0 & (\text{elsewhere}) \end{cases}$$

$$E(X) = \sum_{x \in D_X} x\, f_X(x) = 0 \times \frac{1}{3} + 1 \times \frac{2}{3} = \frac{2}{3}$$

$$E(Y) = \sum_{y \in D_Y} y\, f_Y(y) = 0 \times \frac{1}{9} + 1 \times \frac{3}{9} + 2 \times \frac{5}{9} = \frac{3}{9} + \frac{10}{9} = \frac{13}{9}$$

Exercise 10 c)

$$Cov(2X, 3Y) = 6\,Cov(X, Y) = 6 \times \left(-\frac{2}{27}\right) = -\frac{12}{27} = -\frac{4}{9}$$

$$Var(X+Y) = Var(X) + Var(Y) + 2\,Cov(X,Y) =$$
$$= \frac{2}{9} + \frac{38}{81} - 2\left(\frac{2}{27}\right) = \cdots = \frac{44}{81}$$

→ This is the correct value of $Var(X+Y)$. The one in the solutions is wrong.

Cálculos auxiliares:

$$E(XY) = \sum_{(x,y)\in D_{X,Y}} xy\, f_{X,Y}(x,y) = (1\times1)\frac{2}{9} + (1\times2)\frac{3}{9} = \frac{2}{9} + \frac{6}{9} = \frac{8}{9}$$

$$Cov(X,Y) = E(XY) - E(X)E(Y) = \frac{8}{9} - \frac{2}{3}\times\frac{13}{9} = \frac{8}{9} - \frac{26}{27} = \frac{24}{27} - \frac{26}{27} = -\frac{2}{27}$$

$$Var(X) = E(X^2) - E(X)^2 = \frac{2}{3} - \left(\frac{2}{3}\right)^2 = \frac{2}{3} - \frac{4}{9} = \frac{6}{9} - \frac{4}{9} = \frac{2}{9}$$

$$E(X^2) = \sum_{x\in D_X} x^2 f_X(x) = 0^2\times\frac{1}{3} + 1^2\times\frac{2}{3} = \frac{2}{3}$$

$$Var(Y) = E(Y^2) - E(Y)^2 = \frac{23}{9} - \left(\frac{13}{9}\right)^2 = \frac{207}{81} - \frac{169}{81} = \frac{38}{81}$$

$$E(Y^2) = \sum_{y\in D_Y} y^2 f_Y(y) = 0^2\times\frac{1}{9} + 1^2\times\frac{3}{9} + 2^2\times\frac{5}{9} = \frac{3}{9} + \frac{20}{9} = \frac{23}{9}$$

Exercise 10 d)

Let $Z = E(Y|X)$

$$E(Y|X=x) = \sum_{y \in D_Y} y f_{Y|X=x}(y) = \sum_{y=0}^{2} y \frac{f_{X,Y}(x,y)}{f_X(x)} \quad (x = 0, 1)$$

Therefore:

$$E(Y|X=0) = \frac{1 \times \frac{1}{9} + 2 \times \frac{2}{9}}{\frac{3}{9}} = \frac{5}{3}$$ so, $f_Z\left(\frac{5}{3}\right) = f_X(0) = \frac{1}{3}$

$$E(Y|X=1) = \frac{0 \times \frac{1}{9} + 1 \times \frac{2}{9} + 2 \times \frac{3}{9}}{\frac{6}{9}} = \frac{8}{6} = \frac{4}{3}$$ so, $f_Z\left(\frac{4}{3}\right) = f_X(1) = \frac{2}{3}$

$D_Z = \left\{ \frac{4}{3}, \frac{5}{3} \right\}$

$$f_Z(z) = \begin{cases} \frac{1}{3} & \left(z = \frac{5}{3}\right) \\ \frac{2}{3} & \left(z = \frac{4}{3}\right) \\ 0 & (\text{elsewhere}) \end{cases}$$

Thanks!

Questions?