

Mathematical Programming I

BSc in Applied Mathematics for Economics and Management (MAEG)



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Duality

Primal and Dual Problems in Linear Programming

- ▶ Every linear programming (LP) problem has an associated **dual** problem.
- ▶ The original problem is called the **primal**, and the associated problem is the **dual**.

Primal (Maximization)

$$\begin{array}{ll} \max & z = c^T x \\ \text{s.t.} & Ax \leq b \\ & x \geq 0 \end{array}$$

Dual (Minimization)

$$\begin{array}{ll} \min & w = b^T y \\ \text{s.t.} & y^T A \geq c \\ & y \geq 0 \end{array}$$

Construction of the Dual Problem

Construction of the Dual Problem

max		min	
constraint i	\leq	≥ 0	variable i
	\geq	≤ 0	
	$=$	$\in \mathbb{R}$	
variable j	≥ 0	\geq	constraint j
	≤ 0	\leq	
	$\in \mathbb{R}$	$=$	
matrix A		matrix A^t	
objective coefficients		rhs terms	
rhs terms		objective coefficients	

Example

PRIMAL

$$\max \quad z = 4x_1 + 5x_2,$$

$$\text{s. to: } 4x_1 + 6x_2 \leq 24,$$

$$2x_1 + x_2 \leq 6,$$

$$x_1 - x_2 \leq 1,$$

$$x_1 \leq 2,$$

$$x_1, x_2 \geq 0,$$

Example

PRIMAL

$$\max \quad z = 4x_1 + 5x_2,$$

$$\text{s. to: } 4x_1 + 6x_2 \leq 24,$$

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$$x_1 - x_2 \leq 1,$$

$$x_1 \leq 2,$$

$$x_1, x_2 \geq 0,$$

DUAL

$$\min \quad w = 24y_1 + 6y_2 + y_3 + 2y_4,$$

$$\text{s. to: } 4y_1 + 2y_2 + y_3 + y_4 \geq 4,$$

$$6y_1 + y_2 - y_3 \geq 5,$$

$$y_1, y_2, y_3, y_4 \geq 0,$$

Fundamental Properties of the Dual

Primal and Dual

Primal (Maximization)

$$\begin{aligned} \max \quad & z = c^T x \\ \text{s.t.} \quad & Ax \leq b \\ & x \geq 0 \end{aligned}$$

Dual (Minimization)

$$\begin{aligned} \min \quad & w = b^T y \\ \text{s.t.} \quad & y^T A \geq c \\ & y \geq 0 \end{aligned}$$

Consider the corresponding standard forms (and simplify notation). We have

$$Ax = b \Leftrightarrow x_B = B^{-1}b - B^{-1}Nx_N \Leftrightarrow x_B = B^{-1}b - \sum_{j \in J} B^{-1}A_j x_j$$

$$y^T A = c \Leftrightarrow y = c_B B^{-1}, \quad y^T N = c_N \Leftrightarrow y = c_B B^{-1}, \quad c_B B^{-1} N = c_N$$

$$z = cx = c_B B^{-1}b - \sum_{j \in J} (c_B B^{-1}A_j - c_j)x_j = z_0 - \sum_{j \in J} (c_B B^{-1}A_j - c_j)x_j$$

Let x^* and y^* be optimal solutions

$$z^* = c^T x^* = c_B B^{-1}b = y^* b = w^*$$

Fundamental Properties of the Dual

- ▶ The dual of the dual is the primal.
- ▶ [Weak Duality] The objective function value of any feasible solution of a maximization problem is never greater than that of a feasible solution of the corresponding minimization problem.
- ▶ [Fundamental Theorem of Duality (1)] If x and y are feasible solutions for a pair of dual problems and their objective function values coincide, then x and y are optimal for their respective problems.

Fundamental Properties of the Dual

[Fundamental Theorem of Duality (2)]

- ▶ For any pair of dual problems, if one has a finite optimal solution, the other also has a finite optimal solution, and their optimal objective function values are equal.
- ▶ A linear programming program has a finite optimal solution iff both primal and dual have feasible solutions.
- ▶ If one problem is unbounded, the other has no feasible solution.
- ▶ If one problem has no feasible solution, the other either has no feasible solution or is unbounded.

		Primal	
		$X_p \neq \emptyset$	$X_p = \emptyset$
Dual	$X_d \neq \emptyset$	finite optimal solution	dual has no finite optimum
	$X_d = \emptyset$	primal has no finite optimum	impossible

$X_p \neq \emptyset$: primal problem has feasible solutions

$X_d \neq \emptyset$: dual problem has feasible solutions

$X_p = \emptyset$: primal problem is infeasible

$X_d = \emptyset$: dual problem is infeasible



Fundamental Properties of the Dual

▶ [Complementary Slackness]

- ▶ If a variable is nonzero in the optimal solution, the corresponding constraint in the other problem is tight.
- ▶ If a constraint is not tight in the optimal solution, the associated variable in the other problem is zero.
- ▶ Feasible solutions satisfying complementary slackness are optimal.

If x and y are feasible solutions for the primal and dual problems, respectively, and satisfy complementary slackness, then x and y are optimal solutions for the primal and dual.

Complementary Slackness: Primal-Dual Form

$$\begin{array}{ll} \max & z = \sum_{j \in J} c_j x_j \\ \text{s.t.} & \\ & \sum_{j \in J} a_{ij} x_j \leq b_i, \quad i \in I \\ & x_j \geq 0, \quad j \in J \end{array}$$

$$\begin{array}{ll} \min & w = \sum_{i \in I} b_i y_i \\ \text{s.t.} & \\ & \sum_{i \in I} a_{ij} y_i \geq c_j, \quad j \in J \\ & y_i \geq 0, \quad i \in I \end{array}$$

$$\begin{array}{l} x_j^* > 0 \Rightarrow \sum_{i \in I} a_{ij} y_i^* = c_j \\ y_i^* > 0 \Rightarrow \sum_{j \in J} a_{ij} x_j^* = b_i \end{array}$$

$$\begin{array}{l} \sum_{i \in I} a_{ij} y_i^* > c_j \Rightarrow x_j^* = 0 \\ \sum_{j \in J} a_{ij} x_j^* < b_i \Rightarrow y_i^* = 0 \end{array}$$

$$x_j^* \left(\sum_{i \in I} a_{ij} y_i^* - c_j \right) = 0, \quad j \in J$$

$$y_i^* \left(\sum_{j \in J} a_{ij} x_j^* - b_i \right) = 0, \quad i \in I$$

Exercise

Consider the following LP problem

$$\begin{aligned} \min z &= 2x_1 + 3x_2 + 5x_3 + 2x_4 + 3x_5 \\ \text{s.t.} \quad &x_1 + x_2 + 2x_3 + x_4 + 3x_5 \geq 4 \\ &2x_1 - 2x_2 + 3x_3 + x_4 + x_5 \geq 3 \\ &x_1, x_2, x_3, x_4, x_5 \geq 0 \end{aligned}$$

Solve the problem without using the Simplex algorithm.

Economic Interpretation

Shadow prices

Definition: shadow price

The **shadow price** or **marginal value** of a constraint is the rate of change in the optimal value of the objective function when the right-hand side of that constraint changes by a small amount, given that the current non basic variables are held at zero.

Let x^* and y^* be optimal solutions

$$z^* = c^T x^* = c_B B^{-1} b = y^* b = w^* = y_1^* b_1 + \cdots + y_m^* b_m$$

therefore

$$\frac{\partial z^*}{\partial b_i} = c_B B_i^{-1} = y_i^*$$

Shadow prices

suppose

$$\tilde{b}_i = b_i + \delta, \quad \delta \in \mathbb{R}$$

thus

$$\tilde{z}^* = y^* \tilde{b} = y_1^* b_1 + \cdots + y_i^* (b_i + \delta) + \cdots + y_m^* b_m$$

$$\tilde{z}^* = y_1^* b_1 + \cdots + y_i^* b_i + \cdots + y_m^* b_m + y_i^* \delta = z^* + y_i^* \delta$$

$$y_i^* = \frac{\tilde{z}^* - z^*}{\delta}$$

- ▶ The i -th **shadow price** represents the proportion of the change in the optimal value of the primal due to a change in the i -th right-hand-side, assuming all other data remain constant and the change stays within the allowable range.

Shadow prices

Theorem

Let (P) be a LP with optimal solution (non degenerate). The **shadow price** or **marginal value** of a constraint i is the value of the corresponding dual variable y_i^* .

Let x^* and y^* be optimal solutions

$$\begin{aligned} z^* &= c^T x^* = c_B B^{-1} b + c_N x_N = c_B B^{-1} b - (c_B B^{-1} N - c_N) x_N \\ &= c_B B^{-1} b - (y^* N - c_N) x_N = c_B B^{-1} b - \sum_j (y^* A_j - c_j) x_j \\ &= c_B B^{-1} b - \sum_j (z_j - c_j) x_j \end{aligned}$$

$$\text{if } x_j^* > 0 \Rightarrow y^* A_j - c_j = 0;$$

$$\text{if } x_j^* = 0, \quad y^* A_j - c_j \geq 0$$

$(z_j - c_j) = y^* A_j - c_j$ are the reduced costs

Shadow prices

$$\text{if } x_j^* > 0 \Rightarrow y^* A_j - c_j = 0; \quad \text{if } x_j^* = 0, \quad y^* A_j - c_j \geq 0$$

when x_j corresponds to a slack variable, $A_j = e_j$ and $c_j = 0$

thus the corresponding reduced costs

$$(z_j - c_j) = y^* A_j - c_j = y^* e_j = y_j^*$$

The Simplex Optimal Table of the RM example

\bar{z} and basic var	x_1	x_2	x_3	x_4	x_5	x_6	RHS
x_2	0	1	$\frac{1}{12}$	$-\frac{1}{2}$	0	0	3
x_1	1	0	$-\frac{1}{8}$	$\frac{3}{4}$	0	0	$\frac{3}{2}$
x_5	0	0	$\frac{3}{8}$	$-\frac{5}{4}$	1	0	$\frac{5}{2}$
x_6	0	0	$\frac{1}{8}$	$-\frac{3}{4}$	0	1	$\frac{1}{2}$
\bar{z}	0	0	$\frac{3}{4}$	$\frac{1}{2}$	0	0	21

The optimal values of the dual solution can be found in the row of the Z:

The value of the dual decision variable y_i is the value in the Z row of the i -th constraint slack variable

The value of the dual slack variable corresponding to constraint j is the value of the j -th primal decision variable x_j

Economic Interpretation

The shadow price represents the marginal value of a resource. If a constraint represents a limited resource, and $\delta = 1$, the shadow price tells you how much the objective would improve if you had one more unit (δ extra units) of that resource.

the dual model provides economic information about limited resources

the **shadow price** or **marginal value** of constraint $i \in I$ is the value of the dual variable y_i associated with that constraint and represents the variation in the optimal value z^* of the problem when one unit is added to the right-hand side of that constraint, keeping everything else constant;

it also corresponds to the **fair price** we would be willing to pay to increase the right-hand side of that constraint by one unit

Example: the RM model

The optimal solution of the (PRIMAL) model

$$\max \quad z = 4x_1 + 5x_2,$$

$$\text{s. to: } 4x_1 + 6x_2 \leq 24,$$

$$2x_1 + x_2 \leq 6,$$

$$x_1 - x_2 \leq 1,$$

$$x_1 \leq 2,$$

$$x_1, x_2 \geq 0$$

$$\text{is } x^* = \left(\frac{3}{2}, 3\right) = (1.5, 3)$$

$$\text{with value } z^* = 21$$

the augmented optimal solution is

$$\begin{aligned} x^* &= \left(\frac{3}{2}, 3, 0, 0, \frac{5}{2}, \frac{1}{2}\right) \\ &= (1.5, 3, 0, 0, 2.5, 0.5) \end{aligned}$$

the binding constraints are

$$\begin{cases} 4x_1 + 6x_2 = 24, \\ 2x_1 + x_2 = 6, \end{cases}$$

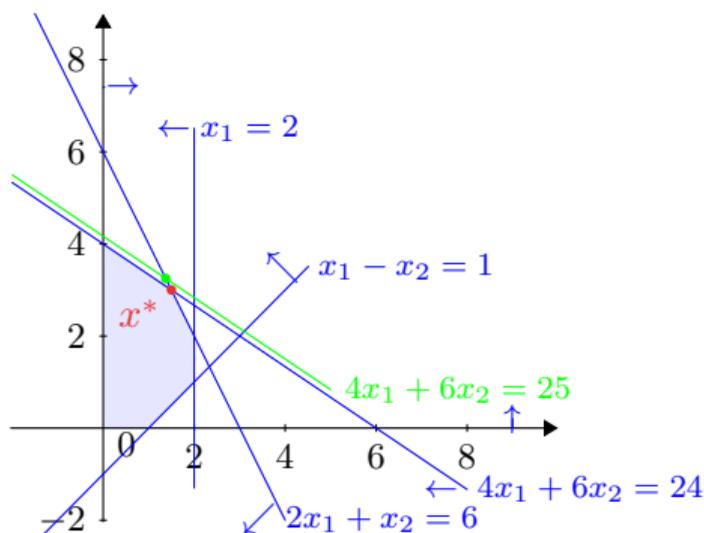
we want to obtain the augmented optimal dual solution which is

$$y^* = \left(\frac{3}{4}, \frac{1}{2}, 0, 0, 0, 0\right) = (0.75, 0.5, 0, 0, 0, 0)$$

Graphically: value of y_1

The optimal value of dual variable y_1 represents the proportion of the change in the optimal value of the primal z^* due to an increase in the right-hand-side of the constraint

$$4x_1 + 6x_2 \leq 24$$



$$\Delta b_1 = \hat{b}_1 - b_1 = 1 \implies \hat{b}_1 = 25$$

$$\begin{cases} 4x_1 + 6x_2 = 25, \\ 2x_1 + x_2 = 6, \end{cases}$$

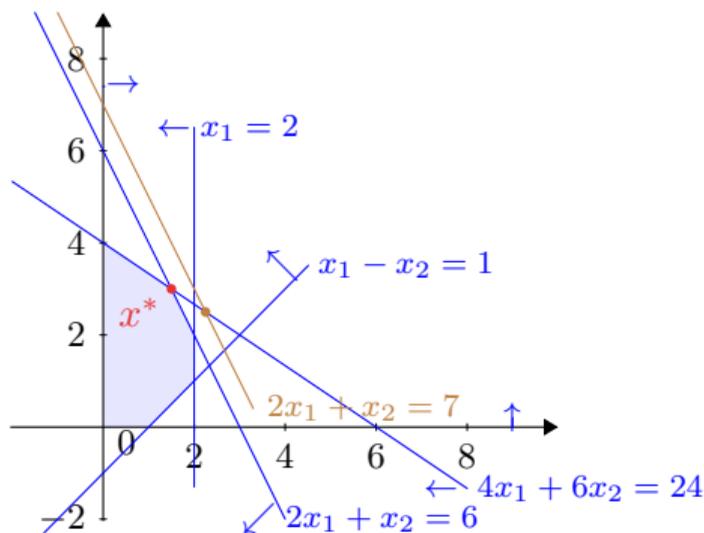
solving we obtain

$$\begin{cases} x_1 = \frac{11}{8}, \\ x_2 = \frac{13}{4}, \end{cases} \implies \hat{z}_1 = \frac{87}{4}$$

$$\text{hence } y_1^* = \Delta z = \hat{z}_1 - z^* = \frac{87}{4} - 21 = \frac{3}{4}$$

Graphically: value of y_2

The optimal value of dual variable y_2 represents the proportion of the change in the optimal value of the primal z^* due to an increase in the right-hand-side of the constraint $2x_1 + x_2 \leq 6$



$$\Delta b_2 = \hat{b}_2 - b_2 = 1 \implies \hat{b}_2 = 7$$

$$\begin{cases} 4x_1 + 6x_2 = 24, \\ 2x_1 + x_2 = 7, \end{cases}$$

solving we obtain

$$\begin{cases} x_1 = \frac{9}{4}, \\ x_2 = \frac{5}{2}, \end{cases} \implies \hat{z}_2 = \frac{86}{4}$$

$$\text{hence } y_2^* = \Delta z = \hat{z}_2 - z^* = \frac{86}{4} - 21 = \frac{1}{2}$$

Example 1

A company produces two products P1 and P2 that generate, respectively, a profit of 3 and 2 monetary units per unit produced.

A single machine produces both products. The maximum production capacity of the machine is 9 ($\times 1000$) units (of either P1 or P2).

The production requires wood and boxes (specific to each product).

The company has available 18 ($\times 1000$) m^3 of wood. Producing one unit of P1 requires 3 ($\times 1000$) m^3 of wood, and producing one unit of P2 requires 1 ($\times 1000$) m^3 of wood.

The company has available 7 ($\times 1000$) boxes for P1 and 6 ($\times 1000$) boxes for P2.

Assume that all production can be sold.

How should the company allocate its resources in order to maximize profit?

Example 1: models and solutions

Decision variables:

x_1 : units of P_1 , x_2 : units of P_2

$$\max \quad z = 3x_1 + 2x_2$$

$$\text{s. t.} \quad x_1 + x_2 \leq 9 \quad (\text{machine capacity})$$

$$3x_1 + x_2 \leq 18 \quad (\text{wood})$$

$$x_1 \leq 7 \quad (\text{boxes for } P_1)$$

$$x_2 \leq 6 \quad (\text{boxes for } P_2)$$

$$x_1, x_2 \geq 0$$

Dual variables:

y_1 : machine capacity, y_2 : wood

y_3 : boxes for P_1 , y_4 : boxes for P_2

$$\min \quad w = 9y_1 + 18y_2 + 7y_3 + 6y_4$$

s. t.:

$$y_1 + 3y_2 + y_3 \geq 3$$

$$y_1 + y_2 + y_4 \geq 2$$

$$y_1, y_2, y_3, y_4 \geq 0$$

$$z^* = 22.5$$

$$x^* = (4.5, 4.5, 0, 0, 2.5, 1.5)$$

$$w^* = 22.5$$

$$y^* = (1.5, 0.5, 0, 0, 0, 0)$$

Example 1: economic interpretation

$z^* = 22.5$	total profit is 22.5 monetary units
$x_1^* = 4.5$	production of 4.5 units of product P1
$x_2^* = 4.5$	production of 4.5 units of product P2
$x_3^* = 0$	machine capacity is fully used
$x_4^* = 0$	the wood resource is fully used
$x_5^* = 2.5$	2.5 units of boxes for P1 remain unused
$x_6^* = 1.5$	1.5 units of boxes for P2 remain unused
$w^* = 22.5$	internal valuation of production and resources
$y_1^* = 1.5$	shadow price (internal value) of machine operating capacity
$y_2^* = 0.5$	shadow price of the wood resource
$y_3^* = 0$	shadow price of boxes for P1
$y_4^* = 0$	shadow price of boxes for P2
$y_5^* = 0$	opportunity loss of producing P1
$y_6^* = 0$	opportunity loss of producing P2

Example 1: economic interpretation of the dual

Resource (Primal constraint)	Availability	Used amount	Shadow price
Machine capacity	9	9	$y_1^* = 1.5$
Wood	18	18	$y_2^* = 0.5$
Boxes for P_1	7	4.5	$y_3^* = 0$
Boxes for P_2	6	4.5	$y_4^* = 0$

- ▶ Shadow prices measure the **marginal value of resources**.
- ▶ Example: increasing machine capacity by one unit would increase the optimal profit by **1.5 monetary units**.
- ▶ Resources with zero shadow price are **not binding**.

Example 1: Link Between Primal and Dual Solutions

Constraint / Resource	Slack	Shadow price	Interpretation
Machine capacity	0	$y_1^* = 1.5$	binding resource
Wood	0	$y_2^* = 0.5$	binding resource
Boxes for P_1	2.5	$y_3^* = 0$	unused resource
Boxes for P_2	1.5	$y_4^* = 0$	unused resource

- ▶ Binding constraints \Rightarrow positive shadow prices.
- ▶ Non-binding constraints \Rightarrow zero shadow prices.
- ▶ This relationship follows from the **complementary slackness conditions**.

$$x_j^*(A_j^T y^* - c_j) = 0 \quad y_i^*(b_i - A_i x^*) = 0$$

Example 2

A company produces three products (P1, P2, P3). For their production, it considers one constraint related to the minimum production level (MP) and another related to the available raw material (RM).

With the objective of maximizing total profit (in monetary units), the company determined the optimal production plan by solving the following linear programming problem:

$$\begin{array}{ll} \max & z = x_1 + 5x_2 + 4x_3 \\ \text{s. a:} & 5x_1 + 10x_2 + 2x_3 \geq 10 \quad (MP) \\ & 4x_1 + 4x_2 + 2x_3 \leq 16 \quad (RM) \\ & x_1, x_2, x_3 \geq 0 \end{array}$$

Example 2: optimal table

whose optimal table is

	x_1	x_2	x_3	x_4	x_5	x_6	
	2	2	1	0	$1/2$	0	8
	-1	-6	0	1	1	-1	6
$z_j - c_j$	7	3	0	0	2	0	32

where x_4 and x_5 are the slack variables associated with the 1st and 2nd constraints, respectively.

The economic interpretation, valid only while the basis corresponding to the current production plan remains unchanged, is the following:

Example 2: economic interpretation

- ▶ $x_1 = 0$ and $y_3 = 7$ ($\Leftrightarrow 5y_1 + 4y_2 \geq 1$): product P1 is not produced because its opportunity cost is nonzero ($= 7$). This means that the internal valuation of the minimum production level and the raw material required to produce one unit of P1 is greater than its unit profit; therefore producing one unit of P1 would decrease the profit by 7 monetary units.
- ▶ $x_2 = 0$ and $y_4 = 3$ ($\Leftrightarrow 10y_1 + 4y_2 \geq 5$): product P2 is not produced because its opportunity cost is nonzero ($= 3$). This means that the internal valuation of the minimum production level and the raw material required to produce one unit of P2 is greater than its unit profit; therefore producing one unit of P2 would decrease the profit by 3 monetary units.
- ▶ $x_3 = 8$ and $y_5 = 0$ ($\Leftrightarrow 2y_1 + 2y_2 = 4$): 8 units of product P3 are produced, therefore its opportunity cost is zero.

Example 2: economic interpretation

- ▶ $x_4 = 6$ ($\Leftrightarrow 5x_1 + 10x_2 + 2x_3 \geq 10 \leftarrow \text{MP}$) and $y_1 = 0$: the minimum production level is exceeded by 6 units, therefore its internal valuation is zero.
- ▶ $x_5 = 0$ ($\Leftrightarrow 4x_1 + 4x_2 + 2x_3 = 16 \leftarrow \text{RM}$) and $y_2 = 2$: the available raw material is fully used; it is therefore a scarce resource because its internal valuation is nonzero. This resource is internally valued at 2 units, which means that for each additional unit of raw material the profit would increase by 2 monetary units.

Economic Interpretation: Primal & Dual

Variable / Resource	Optimal value	Slack / Surplus	Reduced cost / Opportunity cost	Dual value / Shadow price
x_1 (P1)	0	6	7	$y_3 = 7$
x_2 (P2)	0	0	3	$y_4 = 3$
x_3 (P3)	8	0	0	$y_5 = 0$
x_4 (Slack MP)	6	–	–	$y_1 = 0$
x_5 (Slack RM)	0	–	–	$y_2 = 2$

- ▶ Positive reduced cost for non-produced products \rightarrow producing one unit would decrease total profit.
- ▶ Dual values (shadow prices) for binding resources \rightarrow indicate marginal value of adding one unit of that resource.
- ▶ Slack $> 0 \rightarrow$ resource is non-binding \rightarrow shadow price = 0.

Primal and Dual Problems in Linear Programming

- ▶ Every linear programming (LP) problem has an associated dual problem.
- ▶ The original problem is called the **primal**, and the associated problem is the **dual**.
- ▶ Solving either the primal or the dual provides important information about the other:
 - ▶ Optimal values of the dual variables correspond to **shadow prices** of resources.
 - ▶ Reduced costs in the primal indicate the **opportunity cost** of non-produced activities.
 - ▶ Complementary slackness links primal and dual solutions.
- ▶ Understanding the primal-dual relationship is essential for sensitivity analysis and economic interpretation.