Mathematical Programming I

BSc in Applied Mathematics for Economics and Management (MAEG)

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Pos-optimization and Sensitivity analysis

Motivation

- After solving a linear program (LP), what happens if the data changes slightly?
- Can we reuse the solution of the original problem to solve the new one efficiently?

Post-optimality (or re-optimization) consists of:

- Using information from the original optimal solution
- To quickly solve a modified version of the problem

Topics to be studied:

- Discrete changes in:
  - objective function coefficients
  - right-hand side values of constraints
- Introduction of:
  - a new variable
  - a new constraint
- Sensitivity analysis with respect to:
  - objective function coefficients
  - right-hand side values

Example 1

Formulation and the corresponding optimal table

$$\begin{array}{ll} \max & z = 2x_1 + x_2 - x_3 \\ \text{s. a:} & \\ & x_1 + 2x_2 + x_3 \leq 8 \\ & -x_1 + x_2 - 2x_3 \leq 4 \\ & x_1, x_2, x_3 \geq 0 \end{array}$$

x_B	x_1	x_2	x_3	x_4	x_5	$\bar{b}$
x_1	1	2	1	1	0	8
x_5	0	3	-1	1	1	12
$z_j - c_j$	0	3	3	2	0	16

Without solving the problem from the beginning, obtain a new optimal solution when the initial data of the problem are changed.

Example 2 – ToyCo Example (Taha)

Problem description

- ToyCo assembles three types of toys:
  - Trains (x_1), Trucks (x_2), Cars (x_3).

  Each day, 430, 460, and 420 minutes are available for assembly operations M1, M2, and M3, respectively.
- Daily, the available time (in minutes) for assembly operations M1, M2, and M3 are:
  - Operation M_1: 430
  - Operation M_2: 460
  - Operation M_3: 420
- The revenue per unit is \$3, \$2, and \$5 for trains, trucks, and cars, respectively.
- The processing times (minutes per unit) per toy and per operation are:

	M_1	M_2	M_3
Train	1	3	1
Truck	2	0	4
Car	1	2	0

Goal:

- Determine how many units of each toy to produce in order to **maximize total revenue**

Example 2 – ToyCo Example (Taha)

Mathematical Formulation:

- x_1 number of Trains to assemble,
- x_2 number of Trucks to assemble,
- x_3 number of Cars to assemble,

$$\begin{aligned}
\text{max} \quad & z = 3x_1 + 2x_2 + 5x_3 \\
\text{s.t.} \quad & x_1 + 2x_2 + x_3 \leq 430 \quad (M_1) \\
& 3x_1 + 0x_2 + 2x_3 \leq 460 \quad (M_2) \\
& x_1 + 4x_2 + 0x_3 \leq 420 \quad (M_3) \\
& x_1, x_2, x_3 \geq 0
\end{aligned}$$

Obtain the optimal production plan.

Example 2 – ToyCo Example (Taha)

Final Simplex table – optimal table

x_B	x_1	x_2	x_3	x_4	x_5	x_6	$\bar{b}$
x_2	-1/4	1	0	1/2	-1/4	0	100
x_3	3/2	0	1	0	1/2	0	230
x_6	2	0	0	-2	1	1	20
$z_j - c_j$	4	0	0	1	2	0	1350

Optimal Solution – Optimal production plan:

$$x_1 = 0, \quad x_2 = 100, \quad x_3 = 230,$$

Maximum revenue:

$$z^* = 3(0) + 2(100) + 5(230) = 0 + 200 + 1150 = 1350$$

Resource usage:

- M_1: $0 + 2(100) + 230 = 430 \leq 430$ $\Rightarrow$ binding
- M_2: $3(0) + 2(230) = 460$ $\Rightarrow$ binding
- M_3: $0 + 4(100) = 400 \leq 420$ (slack 20)

I – Discrete changes in the objective function coefficients

I – Discrete changes in the objective function coefficients, c

the cost of one or more variables is modified from c_k to $\bar{c}_k$

1. if x_k is a nonbasic variable

   the vector c_B of the costs of the basic variables does not change, therefore $z_j = c_B B^{-1} A_j$ remains unchanged for all variables; replace the reduced cost $z_k - c_k \leq 0$ by the reduced cost

   $$z_k - \bar{c}_k = z_k - c_k + (c_k - \bar{c}_k),$$

   - if $z_k - \bar{c}_k \leq 0$ the solution remains optimal,
   - if $z_k - \bar{c}_k > 0$ then x_k must enter the basis, proceeding as in the usual simplex algorithm

I – Discrete changes in the objective function coefficients, c

2. if x_k is a basic variable

the vector c_B of the costs of the basic variables is modified,
therefore $z_j = c_B B^{-1} A_j$ changes for all nonbasic variables;
it is necessary to update the

- reduced costs $z_j - c_j$,
- value of the objective function $c_B B^{-1} b$;

after updating the last row of the simplex table (reduced costs and objective function value), if optimality is no longer satisfied, proceed with the simplex algorithm as usual

Example 1

Formulation and the corresponding optimal table

$$\begin{array}{ll} \max & z = 2x_1 + x_2 - x_3 \\ \text{s. a:} & \\ & x_1 + 2x_2 + x_3 \leq 8 \\ & -x_1 + x_2 - 2x_3 \leq 4 \\ & x_1, x_2, x_3 \geq 0 \end{array}$$

x_B	x_1	x_2	x_3	x_4	x_5	$\bar{b}$
x_1	1	2	1	1	0	8
x_5	0	3	-1	1	1	12
$z_j - c_j$	0	3	3	2	0	16

Without solving the problem from the beginning, obtain a new optimal solution when the initial data of the problem are changed.

- 1. (a) $\bar{c}_2 = 5$ (b) $\bar{c}_2 = 2$ (c) $\bar{c}_2 = -2$
- 2. (a) $\bar{c}_1 = 1$ (b) $\bar{c}_1 = -2$

Example 2 – ToyCo Example (Taha)

Formulation

$$\max \quad z = 3x_1 + 2x_2 + 5x_3$$

$$\begin{aligned} \text{s.t.} \quad & x_1 + 2x_2 + x_3 \leq 430 \quad (M_1) \\ & 3x_1 + 0x_2 + 2x_3 \leq 460 \quad (M_2) \\ & x_1 + 4x_2 + 0x_3 \leq 420 \quad (M_3) \\ & x_1, x_2, x_3 \geq 0 \end{aligned}$$

Optimal simplex table

x_B	x_1	x_2	x_3	x_4	x_5	x_6	$\bar{b}$
x_2	$-1/4$	1	0	$1/2$	$-1/4$	0	100
x_3	$3/2$	0	1	0	$1/2$	0	230
x_6	2	0	0	-2	1	1	20
$z_j - c_j$	4	0	0	1	2	0	1350

New situation – Change in Selling Prices:

Unit selling prices change from $(3,\ 2,\ 5)$ to $(6,\ 8,\ 3)$ for trains (x_1), trucks (x_2), cars (x_3)

Question:

- What happens to the optimal production plan?
- What happens to the total revenue?

II – Discrete changes in the constraint matrix A

II – Discrete changes in the constraint matrix A

the column of a variable in the constraint matrix is modified from A_k to $\bar{A}_k$

1. if x_k is a nonbasic variable

   the new column $B^{-1}\bar{A}_k$ must be computed, as well as the new reduced cost $\bar{z}_k - c_k = c_B B^{-1}\bar{A}_k - c_k$

   if $\bar{z}_k - c_k \leq 0$ the solution remains optimal,
   if $\bar{z}_k - c_k > 0$ then x_k must enter the basis, proceeding as in the usual simplex algorithm

2. if x_k is a basic variable

   the new set of basic vectors may no longer form a valid basis;
   even if it does, modifying the column of a basic variable changes the basis and therefore B^{-1}, which leads to a modification of the entire simplex table

Example 1

Formulation and the corresponding optimal table

$$\begin{array}{ll} \max & z = 2x_1 + x_2 - x_3 \\ \text{s. a:} & \\ & x_1 + 2x_2 + x_3 \leq 8 \\ & -x_1 + x_2 - 2x_3 \leq 4 \\ & x_1, x_2, x_3 \geq 0 \end{array}$$

x_B	x_1	x_2	x_3	x_4	x_5	$\bar{b}$
x_1	1	2	1	1	0	8
x_5	0	3	-1	1	1	12
$z_j - c_j$	0	3	3	2	0	16

Without solving the problem from the beginning, obtain a new optimal solution when the initial data of the problem are changed.

- 3. (a) $\bar{A}_3^t = [1 \;\; 2]$ (b) $\bar{A}_3^t = [-2 \;\; 3]$

Example 2 – ToyCo Example (Taha)

Formulation

$$\begin{array}{ll} \text{max} & z = 3x_1 + 2x_2 + 5x_3 \\ \text{s.t.} & x_1 + 2x_2 + x_3 \leq 430 \quad (M_1) \\ & 3x_1 + 0x_2 + 2x_3 \leq 460 \quad (M_2) \\ & x_1 + 4x_2 + 0x_3 \leq 420 \quad (M_3) \\ & x_1, x_2, x_3 \geq 0 \end{array}$$

Optimal simplex table

x_B	x_1	x_2	x_3	x_4	x_5	x_6	$\bar{b}$
x_2	$-1/4$	1	0	$1/2$	$-1/4$	0	100
x_3	$3/2$	0	1	0	$1/2$	0	230
x_6	2	0	0	-2	1	1	20
$z_j - c_j$	4	0	0	1	2	0	1350

New decision: ToyCo considers replacing **trains (x_1)** with a new product: **police cars**

Characteristics of the new product:

- Unit revenue: $3
- Processing times (minutes):

	M_1	M_2	M_3
police cars	1	1	2

Question:

- Is it profitable to replace **trains** with **police cars** in production?

III – Introduction of a new variable

III – Introduction of a new variable (activity)

Consider a new variable x_{n+1} with cost c_{n+1} and column in the constraint matrix A_{n+1}

without solving the problem we can determine whether it will or will not be advantageous to produce (bring into the basis) x_{n+1}, for that compute

$$z_{n+1} - c_{n+1} = c_B B^{-1} A_{n+1} - c_{n+1}$$

- if $z_{n+1} - c_{n+1} \leq 0$ the solution remains optimal and we will have $x_{n+1} = 0$ in the optimal solution,
- if $z_{n+1} - c_{n+1} > 0$ then x_{n+1} must enter the basis, compute $B^{-1} A_{n+1}$ and proceed as usual with the simplex algorithm

Example 1

Formulation and the corresponding optimal table

$$
\begin{array}{ll}
\max & z = 2x_1 + x_2 - x_3 \\
\text{s. a:} & \\
 & x_1 + 2x_2 + x_3 \leq 8 \\
 & -x_1 + x_2 - 2x_3 \leq 4 \\
 & x_1, x_2, x_3 \geq 0
\end{array}
$$

x_B	x_1	x_2	x_3	x_4	x_5	$\bar{b}$
x_1	1	2	1	1	0	8
x_5	0	3	-1	1	1	12
$z_j - c_j$	0	3	3	2	0	16

Without solving the problem from the beginning, obtain a new optimal solution when the initial data of the problem are changed.

- 4. (a) $c_6 = 4$ and $A_6^t = [1\ 2]$ (b) $c_6 = 2$ and $A_6^t = [2\ 3]$

Example 2 – ToyCo Example (Taha)

Formulation

$$
\begin{array}{lll}
\text{max} & z = 3x_1 + 2x_2 + 5x_3 & \\
\text{s.t.} & x_1 + 2x_2 + x_3 \leq 430 & (M_1) \\
 & 3x_1 + 0x_2 + 2x_3 \leq 460 & (M_2) \\
 & x_1 + 4x_2 + 0x_3 \leq 420 & (M_3) \\
 & x_1, x_2, x_3 \geq 0 &
\end{array}
$$

Optimal simplex table

x_B	x_1	x_2	x_3	x_4	x_5	x_6	$\bar{b}$
x_2	$-1/4$	1	0	$1/2$	$-1/4$	0	100
x_3	$3/2$	0	1	0	$1/2$	0	230
x_6	2	0	0	-2	1	1	20
$z_j - c_j$	4	0	0	1	2	0	1350

New decision: ToyCo is considering producing a new product: **Fire trucks** (x_7)

Characteristics of the new product:

- Unit revenue: \$4
- Processing times (minutes):

	M_1	M_2	M_3
Fire truck	1	1	2

Question:

- Is it profitable to include fire trucks in production?

Example 2 – ToyCo Example (Taha)

Formulation

$$\begin{array}{ll} \text{max} & z = 3x_1 + 2x_2 + 5x_3 \\ \text{s.t.} & x_1 + 2x_2 + x_3 \leq 430 \quad (M_1) \\ & 3x_1 + 0x_2 + 2x_3 \leq 460 \quad (M_2) \\ & x_1 + 4x_2 + 0x_3 \leq 420 \quad (M_3) \\ & x_1, x_2, x_3 \geq 0 \end{array}$$

Optimal simplex table

x_B	x_1	x_2	x_3	x_4	x_5	x_6	$\bar{b}$
x_2	$-1/4$	1	0	$1/2$	$-1/4$	0	100
x_3	$3/2$	0	1	0	$1/2$	0	230
x_6	2	0	0	-2	1	1	20
$z_j - c_j$	4	0	0	1	2	0	1350

New decision: ToyCo is considering producing a new product: **Trailers** (x_7)

Characteristics of trailers:

- Unit revenue: \$4
- Processing times (minutes):

	M_1	M_2	M_3
Trailer	1	2	1

Questions:

- Is it worthwhile to produce trailers?
- What are the consequences for the optimal plan and total revenue?

IV – Discrete changes in the RHS vector

IV – Discrete changes in the RHS vector b

the right-hand-side vector is modified from b to $\tilde{b}$

update the last column of the simplex table:

compute $B^{-1}\tilde{b}$ as well as the new value of the objective function $c_B B^{-1}\tilde{b}$

since the previous table was optimal, after

- if primal feasibility is satisfied, $x_B = B^{-1}\tilde{b} \geq 0$, the current basis is optimal and we have a new optimal solution and new optimal objective function value,
- if primal feasibility is violated, proceed with the **dual simplex** algorithm to restore primal feasibility,

Example 1

Formulation and the corresponding optimal table

$$\begin{array}{ll} \max & z = 2x_1 + x_2 - x_3 \\ \text{s. a:} & \\ & x_1 + 2x_2 + x_3 \leq 8 \\ & -x_1 + x_2 - 2x_3 \leq 4 \\ & x_1, x_2, x_3 \geq 0 \end{array}$$

x_B	x_1	x_2	x_3	x_4	x_5	$\bar{b}$
x_1	1	2	1	1	0	8
x_5	0	3	-1	1	1	12
$z_j - c_j$	0	3	3	2	0	16

Without solving the problem from the beginning, obtain a new optimal solution when the initial data of the problem are changed.

- 5. (a) $\tilde{b}^t = [10 \quad 1]$ (b) $\tilde{b}^t = [3 \quad -12]$

Example 2 – ToyCo Example (Taha)

Formulation

$$\begin{aligned} \text{max} \quad & z = 3x_1 + 2x_2 + 5x_3 \\ \text{s.t.} \quad & x_1 + 2x_2 + x_3 \leq 430 \quad (M_1) \\ & 3x_1 + 0x_2 + 2x_3 \leq 460 \quad (M_2) \\ & x_1 + 4x_2 + 0x_3 \leq 420 \quad (M_3) \\ & x_1, x_2, x_3 \geq 0 \end{aligned}$$

Optimal simplex table

x_B	x_1	x_2	x_3	x_4	x_5	x_6	$\bar{b}$
x_2	$-1/4$	1	0	$1/2$	$-1/4$	0	100
x_3	$3/2$	0	1	0	$1/2$	0	230
x_6	2	0	0	-2	1	1	20
$z_j - c_j$	4	0	0	1	2	0	1350

New situation: Increase in Production Capacity – ToyCo increases its available production time to:

$$M_1 = 600, \quad M_2 = 640, \quad M_3 = 590$$

Questions:

- Will the company still produce only trucks and cars?
- How is the daily revenue affected?

Example 2 – ToyCo Example (Taha)

Formulation

$$\begin{array}{ll} \text{max} & z = 3x_1 + 2x_2 + 5x_3 \\ \text{s.t.} & x_1 + 2x_2 + x_3 \leq 430 \quad (M_1) \\ & 3x_1 + 0x_2 + 2x_3 \leq 460 \quad (M_2) \\ & x_1 + 4x_2 + 0x_3 \leq 420 \quad (M_3) \\ & x_1, x_2, x_3 \geq 0 \end{array}$$

Optimal simplex table

x_B	x_1	x_2	x_3	x_4	x_5	x_6	$\bar{b}$
x_2	$-1/4$	1	0	$1/2$	$-1/4$	0	100
x_3	$3/2$	0	1	0	$1/2$	0	230
x_6	2	0	0	-2	1	1	20
$z_j - c_j$	4	0	0	1	2	0	1350

New situation: ToyCo increases its available production time to:

$$M_1 = 600, \quad M_2 = 640, \quad M_3 = 590$$

$$b^T = [430 \ 460 \ 420] \qquad \rightarrow \qquad \tilde{b}^T = [600 \ 640 \ 590]$$

Questions:

- Will the company still produce only trucks and cars?
- How is the daily revenue affected?

Example 2 – ToyCo Example (Taha)

Formulation

$$\begin{array}{ll} \text{max} & z = 3x_1 + 2x_2 + 5x_3 \\ \text{s.t.} & x_1 + 2x_2 + x_3 \leq 430 \quad (M_1) \\ & 3x_1 + 0x_2 + 2x_3 \leq 460 \quad (M_2) \\ & x_1 + 4x_2 + 0x_3 \leq 420 \quad (M_3) \\ & x_1, x_2, x_3 \geq 0 \end{array}$$

Optimal simplex table

x_B	x_1	x_2	x_3	x_4	x_5	x_6	$\bar{b}$
x_2	$-1/4$	1	0	$1/2$	$-1/4$	0	100
x_3	$3/2$	0	1	0	$1/2$	0	230
x_6	2	0	0	-2	1	1	20
$z_j - c_j$	4	0	0	1	2	0	1350

New situation: Due to a major restructuring, the available time becomes:

$$M_1 = 100, \quad M_2 = 600, \quad M_3 = 100$$

$$b^T = [430\ 460\ 420] \qquad \rightarrow \qquad \tilde{b}^T = [100\ 600\ 100]$$

Questions:

- Will the company still produce only trucks and cars?
- How is the daily revenue affected?

Example 2 – ToyCo Example (Taha)

Formulation

$$\max \quad z = 3x_1 + 2x_2 + 5x_3$$

$$\begin{aligned} \text{s.t.} \quad & x_1 + 2x_2 + x_3 \leq 430 \quad (M_1) \\ & 3x_1 + 0x_2 + 2x_3 \leq 460 \quad (M_2) \\ & x_1 + 4x_2 + 0x_3 \leq 420 \quad (M_3) \\ & x_1, x_2, x_3 \geq 0 \end{aligned}$$

Optimal simplex table

x_B	x_1	x_2	x_3	x_4	x_5	x_6	$\bar{b}$
x_2	$-1/4$	1	0	$1/2$	$-1/4$	0	100
x_3	$3/2$	0	1	0	$1/2$	0	230
x_6	2	0	0	-2	1	1	20
$z_j - c_j$	4	0	0	1	2	0	1350

New situation:

- Suppose the available time for operation M_2 changes:

$$460 \rightarrow 420$$

Question:

- Do we need to solve the problem from scratch?

Dual Simplex Method

Why Dual Simplex?

When to use:

- Initial solution is **dual feasible but primal infeasible**
- After changes in the RHS
- After adding new constraints (re-optimization)

This is the setting for:

$$\boxed{\text{Dual Simplex Method}}$$

- Avoids restarting from scratch
- Uses previous optimal table (when available)

Dual Simplex Method

- Reduced costs satisfy dual feasibility

$$z_j - c_j \geq 0 \quad \Rightarrow \text{dual feasible(maximization primal)}$$

$$z_j - c_j \leq 0 \quad \Rightarrow \text{dual feasible(minimization primal)}$$

- But:

$$x_B = B^{-1}b \ngeq 0 \quad \Rightarrow \text{primal infeasible}$$

Key idea:

- In the **primal simplex** (maximization)
  - Maintain **primal feasibility** ($x_B \geq 0$)
  - Obtain primal optimality (dual feasibility) ($z_j - c_j \geq 0$)
- In the **dual simplex**, we do the opposite:
  - Maintain **dual feasibility** ($z_j - c_j \geq 0$)
  - Obtain (or restore) **primal feasibility** ($x_B \geq 0$)

Dual Feasibility and Optimality

Dual feasibility condition (primal max):

$$z_j - c_j = c_B B^{-1} A_j - c_j \geq 0 \quad \forall j$$

Primal feasibility condition:

$$x_B = B^{-1} b \geq 0$$

Optimality condition:

$$\text{Primal feasible} + \text{Dual feasible} \;\Rightarrow\; \text{Optimal solution}$$

Dual simplex strategy:

- Keep $z_j - c_j \geq 0$
- Eliminate negative components of x_B

Geometric Interpretation

- Primal simplex:
  - Moves along feasible region edges
- Dual simplex:
  - Starts outside feasible region
  - Moves toward feasibility
- Each pivot:
  - Improves feasibility
  - Keeps dual optimal structure

Outline

Step 1: Leaving variable

- Choose the most infeasible basic variable:

$$r = \arg\min_i \{\bar{b}_i\}, \quad \bar{b}_r < 0$$

- This corresponds to the **most violated constraint**

Step 2: Entering variable

- Only consider columns with:

$$\bar{a}_{rj} < 0$$

- Ratio test:

$$\frac{|z_j - c_j|}{|\bar{a}_{rj}|} \quad \text{(choose minimum)}$$

Goal:

- Fix infeasibility without breaking dual feasibility

Why the Ratio Test Works

- After pivoting, reduced costs must remain:

$$z_j - c_j \geq 0$$

- The ratio test ensures:
    - Dual feasibility is preserved
    - No reduced cost becomes negative
- If no $\overline{a}_{rj} < 0$ exists:
    - No way to restore feasibility
    - $\Rightarrow$ Dual unbounded
    - $\Rightarrow$ Primal infeasible

Algorithm Summary (primal: maximization)

Repeat:

1. Compute $\bar{b} = B^{-1}b$
2. If $\bar{b} \geq 0$:
$$\text{STOP (optimal solution)}$$
3. Choose leaving variable:
$$r = \arg\min \bar{b}_i$$
4. If $\bar{a}_{rj} \geq 0$ for all j:
$$\text{STOP (primal infeasible)}$$
5. Choose entering variable:
$$k = \arg\min \left\{ \frac{|z_j - c_j|}{|\bar{a}_{rj}|} \ : \ \bar{a}_{rj} < 0 \right\}$$
6. Pivot and update basis

Dual Simplex Method for a Minimization Primal Problem

0. Choose an initial dual feasible and basic solution (i.e. such that $z_j - c_j = c_B B^{-1} A_j - c_j \leq 0$ for all j) and let B be the associated basis.
1. Solve the system $B\bar{b} = b$ and let $\bar{b}$ be its unique solution.
   If $\bar{b} \geq 0$, $STOP$, the current basic solution is an optimal solution.
   Otherwise, select the pivot row r such that $\bar{b}_r = \min_{i \in I}\{\bar{b}_i\} < 0$.
   The variable x_r leaves the basis.
2. If $\bar{a}_{rj} \geq 0$, for all j, $STOP$, the dual solution is unbounded and the primal is infeasible.
   Otherwise, the index k of the pivot column is determined by the following ratio
   $$\frac{z_k - c_k}{\bar{a}_{rk}} = \min_{j \in J}\left\{\frac{z_j - c_j}{\bar{a}_{rj}} \ : \bar{a}_{rj} < 0\right\}.$$
   The variable x_k enters the basis.
3. Update the basis B where column A_k replaces column A_{B_r}.
   Update the index sets I_B and I_N.
   Return to step 1.

Example 1 for dual simplex method

$$\begin{array}{ll} \min & z = 4x_1 + 12x_2 + 18x_3 \\ \text{s. t.:} & \\ & x_1 \quad + 3x_3 \geq 3 \\ & \quad 2x_2 + 2x_3 \geq 5 \\ & x_1, x_2, x_3 \geq 0 \end{array}$$

Example 2 for dual simplex method

$$\begin{array}{ll} \max & z = -10x_1 - 5x_2 \\ \text{s. t.:} & \\ & -20x_1 - 50x_2 \leq -200 \\ & -50x_1 - 10x_2 \leq -150 \\ & -30x_1 - 30x_2 \leq -210 \\ & x_1, x_2 \geq 0 \end{array}$$

initial basic solution

$$x = (0, 0, -200, -150, -210) \qquad \text{and} \qquad y = (0, 0, 0, 10, 5)$$

V – Adding a New Constraint

V – Adding a New Constraint

1. If the current optimal solution satisfies the new constraint,

   the solution remains optimal for the updated problem.

2. If the current optimal solution does not satisfy the new constraint,

   the solution is no longer optimal for the updated problem
   Let B be the optimal basis before adding the new constraint. The new basis is $\bar{B} = \left[\begin{array}{c|c} B & 0 \\ \hline A_B^{m+1} & \pm 1 \end{array}\right]$ and the corresponding inverse is
   $\bar{B}^{-1} = \left[\begin{array}{c|c} B^{-1} & 0 \\ \hline \mp A_B^{m+1} B^{-1} & \pm 1 \end{array}\right]$.

2.1 Let $A^{m+1}x \leq b_{m+1}$ be the new constraint and x_{n+1} the corresponding *slack* variable. We can rewrite the constraint as

$$A_B^{m+1}x_B + A_N^{m+1}x_N + x_{n+1} = b_{m+1}.$$

Since the current solution is $x_B = B^{-1}b - B^{-1}Nx_N$, we obtain

$$(A_N^{m+1} - A_B^{m+1}B^{-1}N)x_N + x_{n+1} = b_{m+1} - A_B^{m+1}B^{-1}b.$$

Adding this row to the simplex table with basic variable x_{n+1} yields a basic solution to the new problem.

If $b_{m+1} - A_B^{m+1}B^{-1}b \geq 0$, the solution is optimal.

If $b_{m+1} - A_B^{m+1}B^{-1}b < 0$, the dual simplex algorithm is used to restore primal feasibility.

2.2 Let $A^{m+1}x \geq b_{m+1}$ be the new constraint and x_{n+1} the corresponding *slack* variable. We can rewrite the constraint as

$$A_B^{m+1}x_B + A_N^{m+1}x_N - x_{n+1} = b_{m+1}.$$

Since the tableau solution is $x_B = B^{-1}b - B^{-1}Nx_N$, we obtain

$$(A_B^{m+1}B^{-1}N - A_N^{m+1})x_N + x_{n+1} = A_B^{m+1}B^{-1}b - b_{m+1}.$$

Adding this row to the simplex tableau with basic variable x_{n+1} yields a basic solution to the new problem.

If $A_B^{m+1}B^{-1}b - b_{m+1} \geq 0$, the solution is optimal.

If $A_B^{m+1}B^{-1}b - b_{m+1} < 0$, the dual simplex algorithm is used to restore primal feasibility.

2.3 Let $A^{m+1}x = b_{m+1}$ be the new constraint and x^a_{n+1} the corresponding artificial variable. We can rewrite the constraint as

$$A_B^{m+1}x_B + A_N^{m+1}x_N \pm x^a_{n+1} = b_{m+1},$$

where the artificial variable enters with a $+$ or $-$ sign depending on whether the constraint is violated by excess or by deficiency, respectively.

Example 1

Formulation and the corresponding optimal table

$$\begin{array}{ll} \max & z = 2x_1 + x_2 - x_3 \\ \text{s. a:} & \\ & x_1 + 2x_2 + x_3 \leq 8 \\ & -x_1 + x_2 - 2x_3 \leq 4 \\ & x_1, x_2, x_3 \geq 0 \end{array}$$

x_B	x_1	x_2	x_3	x_4	x_5	$\bar{b}$
x_1	1	2	1	1	0	8
x_5	0	3	-1	1	1	12
$z_j - c_j$	0	3	3	2	0	16

Without solving the problem from the beginning, obtain a new optimal solution when the initial data of the problem are changed.

- 6.
  - (a) $x_1 + x_2 + x_3 \leq 10$
  - (b) $x_2 + x_3 = 10$
  - (c) $x_2 + x_3 \geq 2$
  - (d) $2x_1 - x_2 = 1$

Example 1

introduction of a new constraint: $2x_1 - x_2 = 1$

the optimal solution $x^* = (8, 0, 0, 0, 12)$ does not satisfy this new constraint, it is necessary to reoptimize

as the new constraints is an equality, it will be necessary to use the two-phase method

consider the artificial variable x_a, the new constraint becomes

$$2x_1 - x_2 + x_a = 1$$

x_1 is a basic variable for which we have in the current basis

$$x_1 = 8 - 2x_2 - x_3 - x_4$$

x_2 is a non-basic variable

the new constraint becomes $-5x_2 - 2x_3 - 2x_4 + x_a = -15$

Example 1

introduction of a new constraint: $2x_1 - x_2 = 1$

the new table, for solving with the two-phase method, is

c'_j	0	0	0	0	0	1	
x_B	x_1	x_2	x_3	x_4	x_5	x_a	$\bar{b}$
x_1	1	2	1	1	0	0	8
x_5	0	3	-1	1	1	0	12
x_a	0	$\mathbf{-5}$	-2	-2	0	1	-15
$z'_j - c'_j$	0	-5	-2	-2	0	0	-15

we now use the dual simplex method in the two-phase method

Example 1

introduction of a new constraint: $2x_1 - x_2 = 1$

we obtain the following final table of phase 1, which is also optimal for the problem

x_B	x_1	x_2	x_3	x_4	x_5	x_a	$\bar{b}$
x_1	1	0	$1/5$	$1/5$	0	$2/5$	2
x_5	0	0	$-11/5$	$-1/5$	1	$3/5$	3
x_2	0	1	$2/5$	$2/5$	0	$-1/5$	3
$z'_j - c'_j$	0	0	0	0	0	-1	0
$z_j - c_j$	0	0	$9/5$	$4/5$	0	$3/5$	7

optimal solution $x^* = (2, 3, 0, 0, 3)$ with $z^* = 7$

Example 2 – ToyCo Example (Taha)

Formulation

$$\begin{aligned} \text{max} \quad & z = 3x_1 + 2x_2 + 5x_3 \\ \text{s.t.} \quad & x_1 + 2x_2 + x_3 \leq 430 \quad (M_1) \\ & 3x_1 + 0x_2 + 2x_3 \leq 460 \quad (M_2) \\ & x_1 + 4x_2 + 0x_3 \leq 420 \quad (M_3) \\ & x_1, x_2, x_3 \geq 0 \end{aligned}$$

Optimal simplex table

x_B	x_1	x_2	x_3	x_4	x_5	x_6	$\bar{b}$
x_2	$-1/4$	1	0	$1/2$	$-1/4$	0	100
x_3	$3/2$	0	1	0	$1/2$	0	230
x_6	2	0	0	-2	1	1	20
$z_j - c_j$	4	0	0	1	2	0	1350

New decision: to make the toys shinier, it was decided that they must also go through the varnishing section.

- Each train requires 2 minutes in this section, each truck requires 2 minutes, and each car requires 1 minute.
- There are 4000 minutes available in the varnishing section.

Question:

- What is the optimal plan and the corresponding revenue?

Example 2 – ToyCo Example (Taha)

Formulation

$$\begin{array}{ll} \text{max} & z = 3x_1 + 2x_2 + 5x_3 \\ \text{s.t.} & x_1 + 2x_2 + x_3 \le 430 \quad (M_1) \\ & 3x_1 + 0x_2 + 2x_3 \le 460 \quad (M_2) \\ & x_1 + 4x_2 + 0x_3 \le 420 \quad (M_3) \\ & x_1, x_2, x_3 \ge 0 \end{array}$$

Optimal simplex table

x_B	x_1	x_2	x_3	x_4	x_5	x_6	$\bar{b}$
x_2	$-1/4$	1	0	$1/2$	$-1/4$	0	100
x_3	$3/2$	0	1	0	$1/2$	0	230
x_6	2	0	0	-2	1	1	20
$z_j - c_j$	4	0	0	1	2	0	1350

New situation: An error led to writing 4000 instead of 400 minutes in the varnishing section constraint.

Question:

- What is the optimal plan and the corresponding revenue?

Sensitivity analysis

Sensitivity analysis

Determine the allowable range of variation for each coefficient of the problem, considered individually, such that the optimal basis remains unchanged.

- Changes to the RHS of constraints
- Changes to the objective function coefficients

Changes to the RHS

As long as the binding constraints remain the same; equivalently, as long as the basis remains unchanged.

How much can the RHS b_i of constraint i change?

$$\tilde{b}_i = b_i + \Delta b_i$$

What is the range of Δb_i?

$$B^{-1}\tilde{b} \;=\; B^{-1}(b + \Delta b) \;=\; B^{-1}b + B^{-1}\Delta b \geq 0$$

Example: the RM model

Recall the Reddy–Mikks example: the formulation and the optimal table.

$$
\begin{aligned}
\max \quad & z = 4x_1 + 5x_2, \\
\text{s.t.} \quad & 4x_1 + 6x_2 \leq 24, \\
& 2x_1 + x_2 \leq 6, \\
& x_1 - x_2 \leq 1, \\
& x_1 \leq 2, \\
& x_1, x_2 \geq 0
\end{aligned}
$$

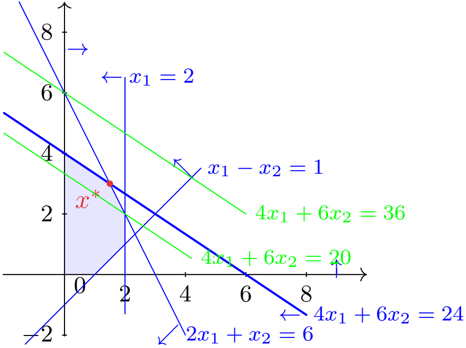
x_B	x_1	x_2	x_3	x_4	x_5	x_6	RHS
x_2	0	1	$\frac{1}{4}$	$-\frac{1}{2}$	0	0	3
x_1	1	0	$-\frac{1}{8}$	$\frac{3}{4}$	0	0	$3/2$
x_5	0	0	$\frac{3}{8}$	$-\frac{5}{4}$	1	0	$5/2$
x_6	0	0	$\frac{1}{8}$	$-\frac{3}{4}$	0	1	$1/2$
$z_j - c_j$	0	0	$\frac{3}{4}$	$\frac{1}{2}$	0	0	21

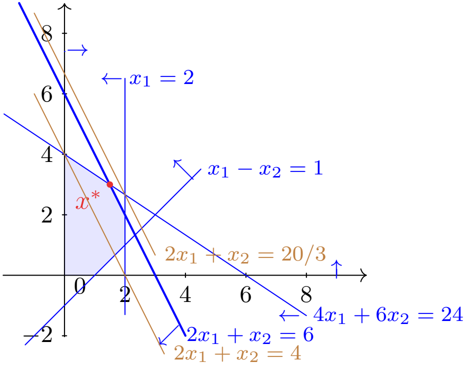
$$x^* = \left(\tfrac{3}{2}, 3, 0, 0, \tfrac{5}{2}, \tfrac{1}{2}\right), \qquad z^* = 21$$

Algebraically determine the allowable range of variation for the RHS of the constraints.

Changes to the RHS – graphically for the RM example

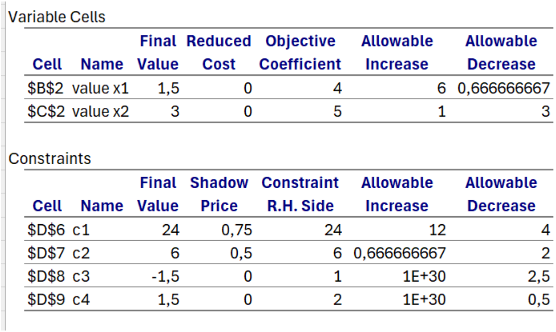
The following interpretation is only valid as long as the binding constraints are the same (first and second in this example), i.e. the basis remains the same.





Changes to the RHS

The Sensitivity Report: see allowable increase and allowable decrease columns for the constraints



Variable Cells

Cell	Name	Final Value	Reduced Cost	Objective Coefficient	Allowable Increase	Allowable Decrease
B2	value x1	1,5	0	4	6	0,666666667
C2	value x2	3	0	5	1	3

Constraints

Cell	Name	Final Value	Shadow Price	Constraint R.H. Side	Allowable Increase	Allowable Decrease
D6	c1	24	0,75	24	12	4
D7	c2	6	0,5	6	0,666666667	2
D8	c3	-1,5	0	1	1E+30	2,5
D9	c4	1,5	0	2	1E+30	0,5

$b_1 = 24$ and can decrease 4 and increase 12, i.e. $-4 \leq \Delta b_1 \leq 12$

$b_2 = 6$ and can decrease 2 and increase $2/3$, i.e. $-2 \leq \Delta b_2 \leq 2/3$

Changes to the RHS

$b_1 = 24$ and can decrease 4 and increase 12, $-4 \leq \Delta b_1 \leq 12$ thus

$$20 = 24 - 4 \leq b_1 \leq 24 + 12 = 36$$

shadow prices maintain their value
minor change in the solution value $\hat{z}^* = z^* + \Delta b_1 y_1^*$

$b_2 = 6$ and can decrease 2 and increase $2/3$, $-2 \leq \Delta b_2 \leq 2/3$ thus

$$4 = 6 - 2 \leq b_2 \leq 6 + 2/3 = 20/3$$

shadow prices maintain their value
minor change in the solution value $\hat{z}^* = z^* + \Delta b_2 y_2^*$

Changes to the objective function coefficients

As long as the binding constraints are the same; equivalently, as long as the basis is the same.

How much can the coefficient c_j of variable x_j can change?

$$\tilde{c}_j = c_j + \Delta c_j$$

Which is the range of Δc_j?

$$\tilde{c}_B B^{-1} A - \tilde{c} \geq 0$$

Example: the RM model

The Reddy–Mikks example: the formulation and the optimal table.

$$\begin{aligned} \max \quad & z = 4x_1 + 5x_2, \\ \text{s.t.} \quad & 4x_1 + 6x_2 \leq 24, \\ & 2x_1 + x_2 \leq 6, \\ & x_1 - x_2 \leq 1, \\ & x_1 \leq 2, \\ & x_1, x_2 \geq 0 \end{aligned}$$

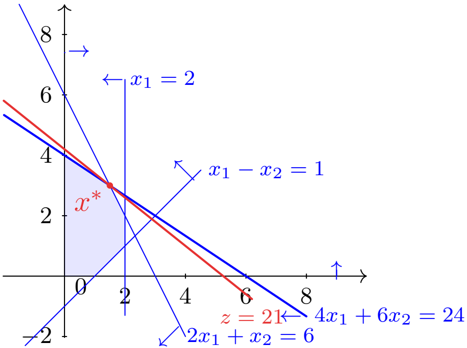
x_B	x_1	x_2	x_3	x_4	x_5	x_6	RHS
x_2	0	1	$\frac{1}{4}$	$-\frac{1}{2}$	0	0	3
x_1	1	0	$-\frac{1}{8}$	$\frac{3}{4}$	0	0	$3/2$
x_5	0	0	$\frac{3}{8}$	$-\frac{5}{4}$	1	0	$5/2$
x_6	0	0	$\frac{1}{8}$	$-\frac{3}{4}$	0	1	$1/2$
$z_j - c_j$	0	0	$\frac{3}{4}$	$\frac{1}{2}$	0	0	21

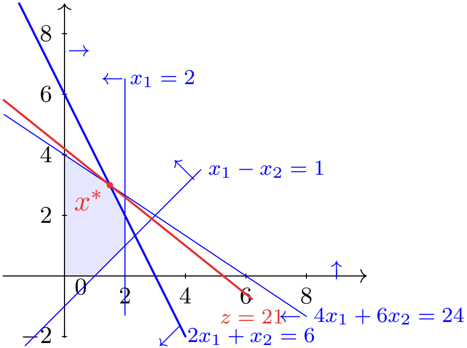
$$x^* = (\tfrac{3}{2}, 3, 0, 0, \tfrac{5}{2}, \tfrac{1}{2}), \qquad z^* = 21$$

Algebraically determine the allowable range of variation for the objective function coefficients.

Changes to the objective function coefficients – graphically for the RM example

$z = c_1x_1 + c_2x_2 \iff x_2 = \frac{1}{c_2}z - \frac{c_1}{c_2}x_1 \implies \nabla z = (c_1, c_2)$ and slope $= -\frac{c_1}{c_2}$





Changes to the objective function coefficients

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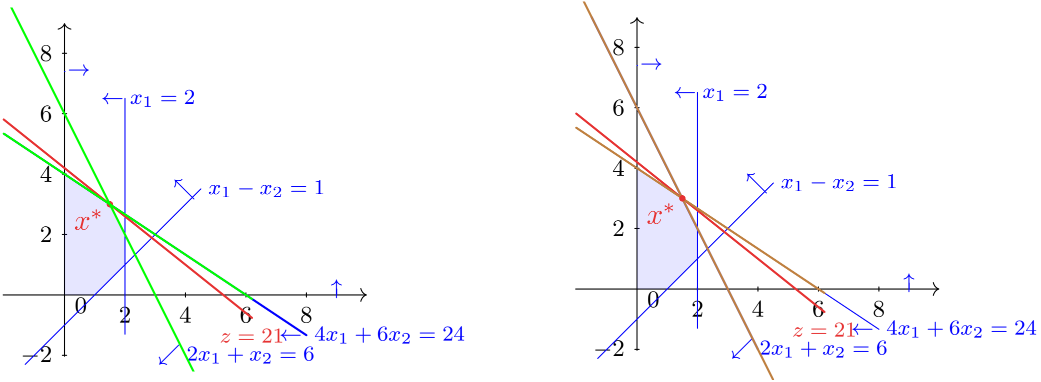
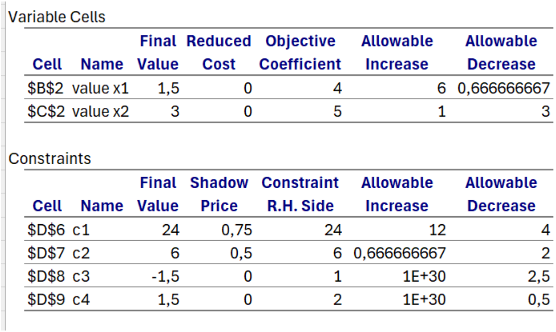


Changes to the objective function coefficients

The Sensitivity Report: see allowable increase and allowable decrease columns for the variables



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D8	c3	-1,5	0	1	1E+30	2,5
D9	c4	1,5	0	2	1E+30	0,5

$c_1 = 4$ and can decrease $2/3$ and increase 6, i.e. $-2/3 \leq \Delta c_1 \leq 6$

$c_2 = 5$ and can decrease 3 and increase 1, i.e. $-3 \leq \Delta b_2 \leq 1$

Changes to the objective function coefficients

$c_1 = 4$ and can decrease $2/3$ and increase 6, $-2/3 \leq \Delta c_1 \leq 6$ thus

$$10/3 = 4 - 2/3 \leq c_1 \leq 4 + 6 = 10$$

primal solution maintain their value
minor change in the solution value $\hat{z}^* = z^* + \Delta c_1 x_1^*$

$c_2 = 5$ and can decrease 3 and increase 1, $-3 \leq \Delta c_2 \leq 1$ thus

$$2 = 5 - 3 \leq c_2 \leq 5 + 1 = 6$$

primal solution maintain their value
minor change in the solution value $\hat{z}^* = z^* + \Delta c_2 x_2^*$

Example 1

Formulation and the corresponding optimal table

$$\begin{array}{ll} \max & z = 2x_1 + x_2 - x_3 \\ \text{s. a:} & \\ & x_1 + 2x_2 + x_3 \leq 8 \\ & -x_1 + x_2 - 2x_3 \leq 4 \\ & x_1, x_2, x_3 \geq 0 \end{array}$$

x_B	x_1	x_2	x_3	x_4	x_5	$\bar{b}$
x_1	1	2	1	1	0	8
x_5	0	3	-1	1	1	12
$z_j - c_j$	0	3	3	2	0	16

Obtain the allowable range of variation for the RHS of the constraints and for the objective function coefficients.

Example 2 – ToyCo Example (Taha)

Formulation

$$\begin{array}{ll} \text{max} & z = 3x_1 + 2x_2 + 5x_3 \\ \text{s.t.} & x_1 + 2x_2 + x_3 \leq 430 \quad (M_1) \\ & 3x_1 + 0x_2 + 2x_3 \leq 460 \quad (M_2) \\ & x_1 + 4x_2 + 0x_3 \leq 420 \quad (M_3) \\ & x_1, x_2, x_3 \geq 0 \end{array}$$

Optimal simplex table

x_B	x_1	x_2	x_3	x_4	x_5	x_6	$\bar{b}$
x_2	$-1/4$	1	0	$1/2$	$-1/4$	0	100
x_3	$3/2$	0	1	0	$1/2$	0	230
x_6	2	0	0	-2	1	1	20
$z_j - c_j$	4	0	0	1	2	0	1350

Obtain the allowable range of variation for the RHS of the constraints and for the objective function coefficients.