Mathematical Programming I

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Transportation Problem and Assignment Problem

Transportation Problems (TP) and Assignment Problems (AP)

- TP and AP are special classes of **Linear Programming (LP)** problems, notable for their:
  - mathematical properties
  - wide range of applications
- Typical applications include:
  - transportation planning
  - human resource management
  - task scheduling
  - ...
- They often arise as **subproblems** within more complex optimization problems.
- **Solution algorithms:**
  - TP: Dantzig's method (adapted simplex)
  - AP: Hungarian Algorithm

Transportation Problem

Transportation Problem (TP) | Example 1 | Problem Description

- A transportation company ships grain (9 in trucks) from **3 silos** to **4 mills**.

The capacities (in number of trucks) of the silos S_1, S_2, and S_3 are 15, 25, and 10, respectively: $S_1 = 15 \quad S_2 = 25 \quad S_3 = 10$
The demands (in number of trucks) at the mills are 5, 15, 15, and 15, respectively for M_1, M_2, M_3, and M_4: $M_1 = 5 \quad M_2 = 15 \quad M_3 = 15 \quad M_4 = 15$
The transportation costs per truck (in hundreds of dollars) for the different routes are given as follows:

	M_1	M_2	M_3	M_4
S_1	10	2	20	11
S_2	12	7	9	20
S_3	4	14	16	18

- Determine the shipping plan that **minimizes total transportation cost**.
- Formulate the problem as a **Linear Programming (LP)** model.
- Obtain a feasible solution for the problem.

Transportation Problem (TP)

Determine the quantities of a commodity to be shipped from a set of distribution centers - `the origins (or sources)` - to a set of receiving centers - `the destinations` - such that the total cost is minimized.

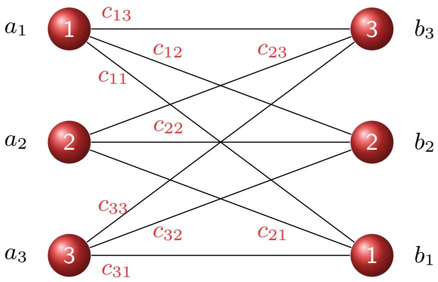
Applications:

- Transportation of products
- Production planning
- Scheduling human resources

Transportation Problem

Given the Data

- m origin points, each with a_i $(i = 1, \ldots, m)$ units of a certain product;
- n destination points, each requiring b_j $(j = 1, \ldots, n)$ units of the same product;
- c_{ij} unit transportation cost between each origin i and destination j.



Determine the transportation plan between origins and destinations with minimal cost.

Balanced Transportation Problem

Assume that, with a_i and b_j are non-negative, the TP is **balanced** when

$$\sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j$$

that is, the total supply and the total demand are equal,

if $\sum_{i=1}^{m} a_i > \sum_{j=1}^{n} b_j$ a destination is created fictitious;

if $\sum_{i=1}^{m} a_i < \sum_{j=1}^{n} b_j$ a source is created fictitious;

Variants of the Transportation Problem

- total supply $>$ total demand: new (dummy) destination
  with demand $\sum_{i=1}^{m} a_i - \sum_{j=1}^{n} b_j$
  Opt. Sol. : part of the supply is not transported.
- total supply $<$ total demand: new (dummy) source
  with supply $\sum_{j=1}^{n} b_j - \sum_{i=1}^{m} a_i$
  Opt. Sol. : part of the demand is not satisfied.
- in both cases the associated transportation costs (c_{ij}) will be zero
- Infeasible link: corresponding variable is set to zero or assign a huge cost (in a minimization problem).

Transportation model: Linear Programming model

Define the decision variables

x_{ij} as the number of units transported from source i to destination j,

the LP formulation of the transportation problem (TP) is:

$$
\begin{aligned}
\min \quad & \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij} x_{ij} \\
\text{s.t.:} \quad & \\
& \sum_{j=1}^{n} x_{ij} = a_i, \quad i = 1, \ldots, m \\
& \sum_{i=1}^{m} x_{ij} = b_j, \quad j = 1, \ldots, n \\
& x_{ij} \geq 0, \quad i = 1, \ldots, m, \; j = 1, \ldots, n
\end{aligned}
$$

Transportation Problem: Matrix Formulation

Linear programming formulation in matricial form:

$$\begin{aligned} \min \quad & c^T x \\ \text{s.t.} \quad & Ax = b \\ & x \geq 0 \end{aligned}$$

Vectors:

- $x = (x_{11}, \ldots, x_{1n}, x_{21}, \ldots, x_{mn})$ (flows = decision variables)
- $c = (c_{11}, \ldots, c_{1n}, c_{21}, \ldots, c_{mn})$ (costs = o.f. coefficients)
- $B = (a_1, \ldots, a_m, b_1, \ldots, b_n)$ (supply & demand = RHS)

Matrix A:

- $A \in \mathbb{R}^{(m+n)\times(mn)}$, real matrix with dimension $(m+n) \times (mn)$
- each column (i, j) is $A_{ij} = e_i + e_{m+j}$

Structure of the Constraint Matrix A

General structure:

$$A = \begin{bmatrix} \mathbf{1} & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \mathbf{1} & \cdots & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{1} \\ \\ I & I & \cdots & I \end{bmatrix}$$

Where:

- $\mathbf{1}$ is a row vector of ones (dimension n)
- $\mathbf{0}$ is a row vector of zeros (dimension n)
- I is the identity matrix ($n \times n$)

Any basic submatrix of A is triangular

Example 1

The T.P. has 3 origins and 4 destinations, with

$$
\begin{array}{llllllllll}
\min & z = & 10\,x_{11} & + & 2\,x_{12} & + & 20\,x_{13} & + & 11\,x_{14} & + \\
 & & 12\,x_{21} & + & 7\,x_{22} & + & 9\,x_{23} & + & 20\,x_{24} & + \\
 & & 4\,x_{31} & + & 14\,x_{32} & + & 16\,x_{34} & + & 18\,x_{34} &
\end{array}
$$

$$
\begin{array}{lllllllll}
\text{s. a:} & x_{11} & + & x_{12} & + & x_{13} & + & x_{14} & = 15 \\
 & x_{21} & + & x_{22} & + & x_{23} & + & x_{24} & = 25 \\
 & x_{31} & + & x_{32} & + & x_{33} & + & x_{34} & = 10 \\
 & & & & & & & & \\
 & x_{11} & + & x_{21} & + & x_{31} & = 5 & & \\
 & x_{12} & + & x_{22} & + & x_{32} & = 15 & & \\
 & x_{13} & + & x_{23} & + & x_{33} & = 15 & & \\
 & x_{14} & + & x_{24} & + & x_{34} & = 15 & & \\
 & & & & & & & & \\
 & \multicolumn{8}{l}{x_{ij} \geq 0, \quad i = 1, \ldots, m, \; j = 1, \ldots, n}
\end{array}
$$

Example 1

$$a^T = [a_i]^T = \begin{bmatrix} 15 & 25 & 10 \end{bmatrix}$$
$$b^T = [b_j]^T = \begin{bmatrix} 5 & 15 & 15 & 15 \end{bmatrix}$$

$$C = [c_{ij}] = \begin{bmatrix} 10 & 2 & 20 & 11 \\ 12 & 7 & 9 & 20 \\ 4 & 14 & 16 & 18 \end{bmatrix}$$

omitting the zeros, the matrix A has the form

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 & & & & & & & & \\ & & & & 1 & 1 & 1 & 1 & & & & \\ & & & & & & & & 1 & 1 & 1 & 1 \\[1em] 1 & & & & 1 & & & & 1 & & & \\ & 1 & & & & 1 & & & & 1 & & \\ & & 1 & & & & 1 & & & & 1 & \\ & & & 1 & & & & 1 & & & & 1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 15 \\ 25 \\ 10 \\ 5 \\ 15 \\ 15 \\ 15 \end{bmatrix}$$

Example 2

MG Auto has three plants in Los Angeles, Detroit, and New Orleans and two major distribution centers in Denver and Miami. The quarterly capacities of the three plants are 1000, 1500, and 1200 cars, and the demands at the two distribution centers for the same period are 2300 and 1400 cars. Mileage between plants and distribution centers is shown in the table at left.

	Distribution Centers		
Plants	Denver	Miami	Capacity
Los Angeles	1027	2342	1000
Detroit	1158	1086	1500
New Orleans	1303	661	1200
Demand	2300	1400	3700

The trucking company in charge of transporting the cars charges 8 cents per mile per car. Formulate as a linear programming problem to find the transportation plan that minimizes the total cost.

Example 2

Transportation cost per car (rounded to the nearest $):

	Distribution Centers	
Plants	Denver	Miami
Los Angeles	82	187
Detroit	92	86
New Orleans	104	52

Decision variables:

x_{ij} as the number of cars transported between plant $i = 1, 2, 3$ and distribution center $j = 1, 2$,

with $i = \begin{cases} 1 \rightarrow & \text{Los Angeles} \\ 2 \rightarrow & \text{Detroit} \\ 3 \rightarrow & \text{New Orleans} \end{cases}, \quad j = \begin{cases} 1 \rightarrow & \text{Denver} \\ 2 \rightarrow & \text{Miami} \end{cases}.$

Example 2

The data of this example can be summarized in the following table

	Distribution Centers		
Plants	Denver	Miami	Capacity
Los Angeles	82	187	1000
Detroit	92	86	1500
New Orleans	104	52	1200
Demand	2300	1400	3700

that shows transportation cost per car (rounded to the nearest $), the capacity and the demands.

the model is

$$
\begin{aligned}
\min \quad & 82x_{11} + 187x_{12} + 92x_{21} + \\
& + 86x_{22} + 104x_{31} + 52x_{32} \\
\text{s. t.:} \quad & \\
& x_{11} + x_{12} = 1000 \\
& x_{21} + x_{22} = 1500 \\
& x_{31} + x_{32} = 1200 \\
& \\
& x_{11} + x_{21} + x_{31} = 2300 \\
& x_{12} + x_{22} + x_{32} = 1400 \\
& \\
& x_{ij} \geq 0, \quad i = 1, 2, 3, \;\; j = 1, 2
\end{aligned}
$$

Properties of the Transportation Problem

Properties of the Transportation Problem

Assume the TP is balanced.

- The transportation problem has at least one feasible solution which is

$$x_{ij} = \frac{a_i b_j}{\sum a_i} = \frac{a_i b_j}{\sum b_j}, \quad \forall i, j$$

- The values of the variables satisfy

$$0 \leq x_{ij} \leq \min\{a_i, b_j\}, \quad \forall i, j$$

- From the two previous items it follows that

The TP always has an optimal solution.

- When supplies a_i $(\forall i)$ and demands b_j $(\forall j)$ are integer values, then any feasible basic solution has integer values, so does the optimal solution

Properties of the Constraint Matrix A

- Of the $m+n$ constraints, exactly one is redundant.

- The matrix A has rank equal to $m+n-1$

- The matrix A is totally unimodular
(the determinant of any square submatrix of A has value 0, 1 or -1)

Transportation Problem Representation

The data of a transportation problem can be easily represented by the following table, with m rows and n columns, in addition to the column of a_i (supplies) and the row of b_j (demands):

$$
\begin{array}{|cccccc|c|}
\hline
c_{11} & c_{12} & \cdots & c_{1j} & \cdots & c_{1n} & a_1 \\
c_{21} & c_{22} & \cdots & c_{2j} & \cdots & c_{2n} & a_2 \\
\vdots & & & \vdots & & \vdots & \vdots \\
c_{i1} & c_{i2} & \cdots & c_{ij} & \cdots & c_{in} & a_i \\
\vdots & \vdots & & \vdots & & \vdots & \vdots \\
c_{m1} & c_{m2} & \cdots & c_{mj} & \cdots & c_{mn} & a_m \\
\hline
b_1 & b_2 & \cdots & b_j & \cdots & b_n & \multicolumn{1}{c}{} \\
\cline{1-6}
\end{array}
$$

Representation and Characterization of a Basis

Let B be a basic submatrix of A. Since A is totally unimodular, we have:

$$\det(B) = \pm 1, \quad \det(B^{-1}) = \pm 1$$

and all elements of B^{-1} are in $\{0, 1, -1\}$.

Thus, any updated column of the simplex table,

$$B^{-1} A_j,$$

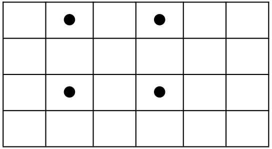
contains only elements $0, 1$, or -1. Therefore, each such column can be obtained by simple addition and subtraction of basic columns.

This property can be visualized in the transportation table through a cycle.

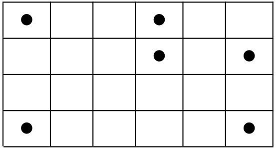
Cycles in the Transportation Table

In the transportation table, a cycle is a sequence of variables such that:

- each pair of adjacent variables lies in the same row or the same column;
- no set of three or more consecutive variables lies in the same row or column;
- the first and the last variables in the sequence lie in the same row or column.



(1,2),(1,4),(3,4),(3,2)



(1,1),(1,4),(2,4),(2,6),(4,6),(4,1)

Cycles and Bases

- Any cycle contains an even number of variables.
- Let T be a subset of columns of matrix A. These columns are linearly dependent if and only if the corresponding variables (or a subset of them) form a cycle.
- A basis can be represented by a spanning tree (a connected graph without cycles) with at least one cell in each row and each column.

B	B	B	B	B	B
					B
					B
					B

basis 1

B	B				
	B	B	B		
			B		
		B		B	B

basis 2

The Dual Problem

Dual formulation:

$$
\begin{aligned}
\max \quad & \sum_{i=1}^{m} a_i u_i + \sum_{j=1}^{n} b_j v_j \\
\text{s.t.} \quad & u_i + v_j \leq c_{ij}, \quad i = 1, \ldots, m, \; j = 1, \ldots, n \\
& u_i, v_j \text{ free}, \quad i = 1, \ldots, m, \; j = 1, \ldots, n
\end{aligned}
$$

Complementary slackness conditions:

$$(c_{ij} - u_i - v_j)\, x_{ij} = 0, \quad i = 1, \ldots, m, \; j = 1, \ldots, n$$

Optimality condition:
If we find a pair of feasible solutions for the primal and the dual that satisfy the complementary slackness conditions, then they are optimal.

The Simplex Method for the Transportation Problem

1. Determine an initial **basic feasible solution**.
2. Test the **optimality** of the current solution.

   Compute the reduced costs $z_{ij} - c_{ij}$ for each nonbasic variable. If all $z_{ij} - c_{ij} \leq 0$, then **STOP**: the solution is optimal. Otherwise, proceed to Step 3.
3. Select the **entering** and **leaving** variables.
4. Obtain a new **basic feasible solution** and return to Step 2.

Step 1: Obtain an Initial Basic Feasible Solution

How to assign values to exactly $m + n - 1$ basic variables?

1. Northwest Corner Method (NW)

- Start by making x_{11} basic and assign it the largest possible value:

$$x_{11} = \min\{a_1, b_1\}$$

  This exhausts either supply at source 1 or demand at destination 1.

- If source 1 is exhausted, move down (row 1 is removed). Set

$$x_{21} = \min\{a_2,\, b_1 - x_{11}\}$$

- If destination 1 is exhausted, move right (column 1 is removed). Set

$$x_{12} = \min\{a_1 - x_{11},\, b_2\}$$

- Repeat this process until all supplies and demands are satisfied.

Step 1: Obtain an Initial Basic Feasible Solution

Northwest Corner Method (Remarks)

- At each step of the method, either a **source** or a **destination** is exhausted. If a source is exhausted, move **down** (same column, next row). If a destination is exhausted, move **right** (same row, next column).
- At most, we obtain $m + n - 1$ positive variables, which do not form a cycle.
- If a source and a destination are exhausted simultaneously, assign a zero to the next variable in the same row or column, and proceed as before.

Drawback: the method does not take the cost matrix into account.

Exercise

5	3	2	100
4	2	1	50
80	30	40	

2. Least Cost Method

At each step of the method, select the variable corresponding to the **smallest cost** in the cost matrix and make it basic.
Apply this procedure to the previous example.

3. Vogel's Approximation Method

At each step, select the row or column with the **largest penalty**, defined as the difference between the two smallest costs in that row or column.
Then, choose the variable with the **minimum cost** in that row or column and make it basic.

Exercise

8	3	5	9	200
1	7	4	6	700
3	8	2	4	100
250	350	200	200	

Step 2: Optimality Test of the Current Basic Feasible Solution

- Compute the **reduced costs** $z_{ij} - c_{ij}$ for each nonbasic variable.
- If $z_{ij} - c_{ij} \leq 0$ for all variables, then **STOP**: the current solution is optimal. Otherwise, proceed to Step 3.

There are two ways to compute reduced costs.

(Use the example from Vogel's method, but determine the initial basic feasible solution using the Northwest Corner method.)

Methods for Computing Reduced Costs

1. Stepping-Stone Method

This method is based on the simplex method

- Construct the **cycle** associated with each nonbasic variable.
- Compute the reduced cost $z_{ij} - c_{ij}$ for each nonbasic variable, where

$$z_{ij} - c_{ij} = \sum_{x_{k_1k_2} \text{ basic variables in cycle}} -(\oplus / \ominus\, c_{k_1k_2}) - c_{ij}$$

Methods for Computing Reduced Costs

2. Dantzig's Method

This method is based on duality results.

- Construct a vector (u, v) that satisfies the **complementary slackness conditions**. To do this:
  - Build a system of $m + n - 1$ equations: if x_{ij} is a basic variable, then
$$u_i + v_j = c_{ij}$$
  - Since one of the $m + n$ constraints of the primal problem is redundant, the system is underdetermined. Therefore, we fix one dual variable (e.g., $u_1 = 0$) to obtain a unique solution.
  - Solve the resulting system.
- Compute the reduced costs for each nonbasic variable:
$$z_{ij} - c_{ij} = u_i + v_j - c_{ij}$$

Step 3: Selection of Entering and Leaving Variables

- **Select the entering variable:** choose x_{kl} such that

$$z_{kl} - c_{kl} = \max\{z_{ij} - c_{ij} \mid z_{ij} - c_{ij} > 0\}$$

- Construct the **cycle** associated with the entering variable x_{kl}.
- **Select the leaving variable:** compute

$$\theta = \min\{x_{ij} \mid x_{ij} \text{ has a } \ominus \text{ sign in the cycle of } x_{kl}\}$$

Step 4: Obtain the New Basic Feasible Solution

- Update only the variables in the cycle associated with the entering variable x_{kl}:

$$\overline{x}_{ij} = \begin{cases} x_{ij} + \theta, & \text{if } x_{ij} \text{ has a } \oplus \text{ sign in the cycle} \\ x_{ij} - \theta, & \text{if } x_{ij} \text{ has a } \ominus \text{ sign in the cycle} \end{cases}$$

- The new objective function value is:

$$\overline{z} = z - \theta(z_{kl} - c_{kl})$$

Degeneracy – Perturbation Method

Degeneracy in the transportation problem is common and occurs whenever ties arise during:

- the construction of an initial basic feasible solution, or
- the selection of the variable to leave the basis.

In case of a tie, the choice can be made arbitrarily. However, there exists a technique called the **perturbation method**, which helps identify the variables to be treated as basic in order to avoid possible cycling.

Perturbation Method

The perturbation method consists of reformulating the transportation problem into a new non-degenerate problem by slightly modifying the values of a_i and b_j as follows:

$$\begin{cases} \overline{a}_i = a_i + \epsilon, & i = 1, \ldots, m \\ \overline{b}_j = b_j, & j = 1, \ldots, n-1 \\ \overline{b}_n = b_n + m\epsilon \end{cases}$$

where $\epsilon > 0$ is arbitrarily small, so that the obtained solution is very close to the original one.

Note: An alternative formulation is:

$$\begin{cases} \overline{a}_i = a_i, & i = 1, \ldots, m-1 \\ \overline{a}_m = a_m + n\epsilon, & \\ \overline{b}_j = b_j + \epsilon, & j = 1, \ldots, n \end{cases}$$

Example

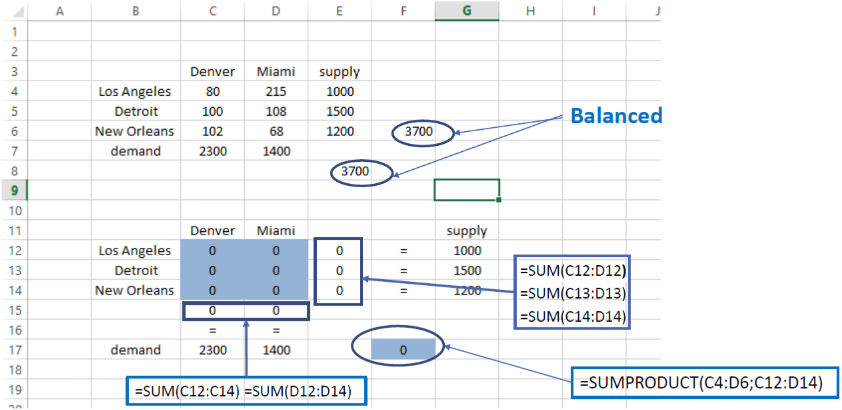
Consider the following transportation problem:

4	3	1	6	1	700
1	3	2	1	5	400
2	3	1	2	4	300
500	200	300	300	100	

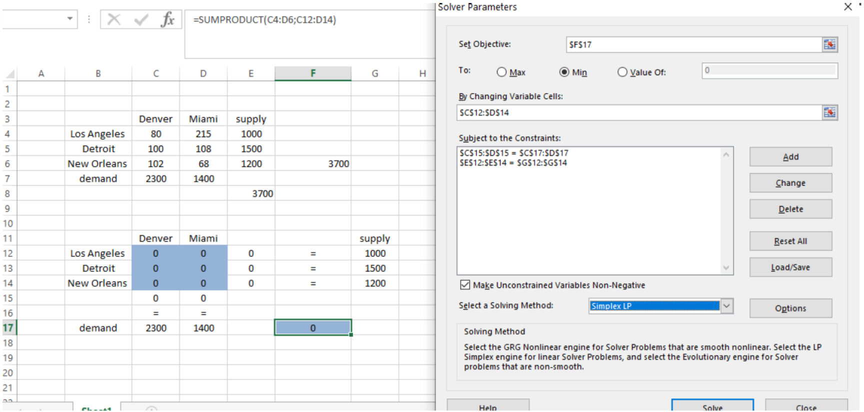
Apply the **Northwest Corner Method** to obtain an initial basic feasible solution.

Solving the problem with Excel Solver

Solving using the Solver of the Excel



Solving using the Solver of the Excel



Assignment problem

Assignment Problem

Given:

- n individuals,
- n tasks,
- and c_{ij} the cost of assigning individual i to task j.

Goal:
Assign each individual to exactly one task so that the **total cost** of performing all tasks is minimized.

Applications

- Assign people to tasks;
- Production planning (operations to machines; products to plants)

Assignment Problem

Let x_{ij} be binary variables indicating whether individual i is assigned to task j:

$$x_{ij} = \begin{cases} 1, & \text{if individual } i \text{ is assigned to task } j, \\ 0, & \text{otherwise.} \end{cases}$$

Then the Linear Programming formulation of the assignment problem is:

$$\begin{aligned} \min \quad & \sum_{i=1}^{n} \sum_{j=1}^{n} c_{ij} x_{ij} \\ \text{s.t.} \quad & \sum_{j=1}^{n} x_{ij} = 1, \quad i = 1, \ldots, n \\ & \sum_{i=1}^{n} x_{ij} = 1, \quad j = 1, \ldots, n \\ & x_{ij} \in \{0, 1\}, \quad i, j = 1, \ldots, n \end{aligned}$$

Matrix Form

In matrix notation, the assignment problem can be written as:

$$\begin{aligned} \min \quad & c^T x \\ \text{s.t.} \quad & Ax = \mathbf{1} \\ & x_{ij} \in \{0, 1\} \end{aligned}$$

where:

$x = (x_{11}, \ldots, x_{1n}, x_{21}, \ldots, x_{nn})^T$

$c = (c_{11}, \ldots, c_{1n}, c_{21}, \ldots, c_{nn})$

$\mathbf{1}$ is a vector of ones with $2n$ elements

A is a $(2n) \times (n^2)$ matrix where each column (i, j) is $a_{ij} = e_i + e_{n+j}$

Example

Consider a factory with 3 departments:
(Assembly (M), Painting (P), and Packaging (E)),
and 3 candidates (C1, C2, C3).

The assignment costs are given by:

	M	P	E
$C1$	4	5	3
$C2$	1	4	2
$C3$	3	1	5

Linear Programming Formulation

$$
\begin{aligned}
\min \quad & 4x_{11} + 5x_{12} + 3x_{13} + x_{21} + 4x_{22} + 2x_{23} + 3x_{31} + x_{32} + 5x_{33} \\
\text{s.t.} \quad & x_{11} + x_{12} + x_{13} = 1 \\
& x_{21} + x_{22} + x_{23} = 1 \\
& x_{31} + x_{32} + x_{33} = 1 \\
& \\
& x_{11} + x_{21} + x_{31} = 1 \\
& x_{12} + x_{22} + x_{32} = 1 \\
& x_{13} + x_{23} + x_{33} = 1 \\
& \\
& x_{ij} \in \{0, 1\}, \quad i, j = 1, \ldots, 3
\end{aligned}
$$

Properties of the Assignment Problem

- The polytope of the assignment problem has $n!$ extreme points. Each integer feasible solution corresponds to an assignment, and each assignment corresponds to an extreme point.
- This is a Linear Programming problem with binary $(0–1)$ variables.
- For each assignment, we select the corresponding variables $x_{ij} = 1$ as basic variables, obtaining n basic variables equal to 1. To complete the basis, we need to include $n - 1$ additional (degenerate) variables.
- It is a particular case of the transportation problem with $m = n$ and $a_i = b_j = 1$. Therefore, every basic feasible solution is integral.
- Due to its special structure, the integrality constraints can be replaced by simple non-negativity constraints, i.e. constraints $x_{ij} \in \{0, 1\}$ can be replaced by constraints $x_{ij} \geq 0, \ \forall i, j = 1, \dots, n$.

$$\begin{aligned}
\max \quad & \sum_{i=1}^{n} u_i + \sum_{j=1}^{n} v_j \\
\text{s.t.} \quad & u_i + v_j \leq c_{ij}, \quad i = 1, \ldots, n,\ j = 1, \ldots, n \\
& u_i, v_j \text{ free}, \quad i = 1, \ldots, n,\ j = 1, \ldots, n
\end{aligned}$$

Complementary slackness conditions:

$$(c_{ij} - u_i - v_j)\, x_{ij} = 0, \quad i = 1, \ldots, n,\ j = 1, \ldots, n$$

Optimality via Primal–Dual Feasible Solutions

If we find a pair of feasible solutions for the primal and dual problems that satisfy the complementary slackness conditions, then both solutions are optimal.

A feasible solution for the dual can be constructed as:

$$\begin{cases} \bar{u}_i = \min_j c_{ij}, & i = 1, \ldots, n \\ \bar{v}_j = \min_i \{c_{ij} - \bar{u}_i\}, & j = 1, \ldots, n \end{cases}$$

Define $\bar{x}$ as:

$$\bar{x}_{ij} = \begin{cases} 1, & \text{if } \bar{u}_i + \bar{v}_j - c_{ij} = 0 \\ 0, & \text{otherwise} \end{cases}$$

If $\bar{x}$ is feasible for the primal, then it is an optimal solution, and $(\bar{u}, \bar{v})$ is optimal for the dual.

Properties

- The assignment problem is completely determined by its cost matrix.
- In an assignment problem with all nonnegative costs, if there exists a feasible assignment with zero total cost, then it is an optimal assignment.

Theorem

If we add a constant to every row and/or every column of the cost matrix of an assignment problem, the resulting problem has the same optimal solution as the original problem.

How We Use This Result

We use this result in two main ways:

1. If the cost matrix contains negative elements, we add a sufficiently large constant to ensure all costs become nonnegative.
2. By appropriately choosing the values added to rows and/or columns, we can introduce zeros into the cost matrix and search for a zero-cost assignment, which will be optimal.

Key Concepts

- The values subtracted from rows and/or columns can be interpreted as a **dual solution**.
- An element (i, j) of the assignment matrix is said to be **assigned** if the corresponding variable $x_{ij} = 1$.
- A feasible solution to the assignment problem is called a **complete assignment**.
- A complete assignment consists of exactly one assigned zero in each row and each column.
- An assignment is called **maximal** if no additional zero assignments can be made in the matrix.

Key Theorems

Theorem

If k is the maximum number of assignable zeros, then there exists a set of k lines (rows and/or columns) that covers all zeros in the matrix, such that no assigned zero lies at the intersection of two lines.
If $k = n$, then we have a solution to the problem.
If $k < n$, then we introduce additional zeros into the matrix.

Theorem

Assume that the zeros in the cost matrix are covered by $k < n$ lines. Let θ be the minimum value among the uncovered elements.
If we subtract θ from all uncovered rows and add θ to all covered columns, we obtain a new cost matrix whose total sum of entries is reduced by $\theta\, n(n-k)$ compared to the previous matrix.

The finite convergence of the Hungarian algorithm (Kuhn, 1955/56), which we will now describe, follows from the previous theorem.

Hungarian Method, *Kuhn (1955)*

Step 1. Cost matrix reduction: transform the cost matrix so that all entries are nonnegative and each row and each column contains at least one zero.

1. Subtract from each row of the cost matrix its minimum value.
2. In the resulting matrix, subtract from each column its minimum value.

Hungarian Method, *Kuhn (1955)*

Step 2. Zero-cost assignment: construct a maximal assignment using only zeros in the cost matrix.

1. Draw the minimum number of lines (rows and/or columns) that cover all zeros in the matrix. Let k be the number of covering lines (or equivalently, the number of selected assigned zeros).
2. If $k = n$, **STOP**: an optimal assignment has been found, consisting of n independent zeros (one per row and one per column).
3. If $k < n$, perform the matrix adjustment step:
   - 3.1 Let θ be the smallest uncovered element.
   - 3.2 Subtract θ from all uncovered elements.
   - 3.3 Add θ to all elements covered by two lines.
   - 3.4 Reconsider all free zeros and return to Step 2.

Algorithm Interpretation

At the end of Step 1, an initial feasible solution for the dual problem is obtained as follows:

$$\begin{cases} \bar{u}_i = \min_j c_{ij}, & i = 1, \ldots, n \\ \bar{v}_j = \min_i \{c_{ij} - \bar{u}_i\}, & j = 1, \ldots, n \end{cases}$$

The corresponding primal solution can also be determined.

Steps 2.1 and 2.2 correspond to the verification of optimality of the solution.

Step 2.3 corresponds to the update of the solution. The dual solution can be updated as follows:

$$\begin{cases} \bar{u}_i \leftarrow \bar{u}_i + \theta, & \forall i \text{ (uncovered rows)} \\ \bar{v}_j \leftarrow \bar{v}_j - \theta, & \forall j \text{ (covered columns)} \end{cases}$$

Example

Example:

5	4	2	6
3	5	4	1
7	8	6	2
2	1	1	3

Step 1

1. Row reduction and dual variables

3	2	0	4
2	4	3	0
5	6	4	0
1	0	0	2

$$u_1 = 2$$
$$u_2 = 1$$
$$u_3 = 2$$
$$u_4 = 1$$

2. Column reduction and dual variables

2	2	0	4
1	4	3	0
4	6	4	0
0	0	0	2

$$v_1 = 1$$
$$v_2 = 0$$
$$v_3 = 0$$
$$v_4 = 0$$

Example

Step 2

Obtain a primal solution:
The primal solution can be obtained by trying to "assign"one zero in each row and in each column.
For example, we can assign zeros at positions $(1,3),(2,4),(4,2)$

It is not yet possible to assign $n = 4$ zeros, and therefore the solution is not yet optimal.
Alternatively, apply 1 and 2 of Step 2 (see next slide).

Example

Step 2

1. 1 Draw lines over row 4, column 4, and column 3.

2	2	0	4
1	4	3	0
4	6	4	0
0	0	0	2

2. 2 Only 3 lines were used (note that only 3 zeros can be assigned).

   Therefore, a complete assignment has not yet been obtained.

3. 3 Determine θ, the smallest value among the uncovered elements, i.e.

$$\theta = \min\{2, 2, 1, 4, 4, 6\} = 1.$$

4. 4 Update the matrix, the dual solution becomes:

$$\begin{aligned} u_1 &= 2 + 1 & \qquad v_1 &= 1 \\ u_2 &= 1 + 1 & v_2 &= 0 \\ u_3 &= 2 + 1 & v_3 &= 0 - 1 \\ u_4 &= 1 & v_4 &= 0 - 1 \end{aligned}$$

The matrix is updated to:

1	1	0	4
0	3	3	0
3	5	4	0
0	0	1	3

We can assign zeros at positions $(1,3),(3,4),(4,2),(2,1)$.

A complete assignment has been obtained, so the solution is optimal.

Alternatively (at Step 2.1), we could draw lines over row 4, column 4, column 3, and row 2.

Since 4 lines were drawn, the solution is optimal.