



Mathematical Economics – 1st Semester - 2026/2027

Exercises - Group I

1. (DeMorgan's laws) Let $\Omega \subset \mathbb{R}^n$, $n \in \mathbb{N}$ and let $A, B, C \subset \Omega$. Show that:
 - (a) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
 - (b) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
 - (c) $A \setminus (B \cup C) = (A \setminus B) \cap (A \setminus C)$
2. Sketch the following sets:
 - (a) $A_1 = \{x \in \mathbb{R} : x^2 > 1\}$ and $A_2 = \{(x, y) \in \mathbb{R}^2 : x^2 > 1\}$
 - (b) $B = \{x \in \mathbb{R} : x^3 > 1 \text{ and } x^4 < 16\}$
 - (c) $C = \{x \in \mathbb{Z} : x^2 < 2 \text{ and } x \text{ is even}\}$
 - (d) $B \cup C$ and $(A_1)^c$
 - (e) $A_1 \times B$
 - (f) $D = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 4\}$
 - (g) $E = \{(x, y) \in \mathbb{R}^2 : x^2 \leq y \text{ and } 1 \geq y + x^2\}$
3. Consider the following real values maps defined by $f(x) = \sqrt{x-1}$ and $g(x) = x^2 - 4$.
 - (a) Compute $(f - g)(1)$, $(fg)(0)$ and $f(3)/g(2)$ if they exist.
 - (b) Define the maps $f \circ g$ and $g \circ f$ (including their domains).
 - (c) Do f and g commute?
 - (d) Define the inverse of f and g (in a suitable domain).
4. Sketch the graph of the following real valued functions $f : D \rightarrow \mathbb{R}$, determine if are injective/surjective and compute its inverse (if it exists):
 - (a) $f(x) = x^3 + 1$ with $x \in \mathbb{R}$
 - (b) $f(x) = x^2 - x$ with $x \in \mathbb{R}$
 - (c) $f(x) = \sqrt{x+1}$ with $x > 0$.
 - (d) $f(x) = \frac{x-1}{x+1}$ with $x > -1$
 - (e) $f(x) = 2e^{-x}$ with $x \in \mathbb{R}$
 - (f) $f(x) = \log(x^2 + 1)$ with $x > 0$

5. (a) Find a bijective function mapping $\mathbb{N}$ to $\mathbb{Z}$.
 (b) Find a bijective function mapping $\mathbb{R}$ to $] - 10, 10[$.
6. Show that the following functions $d : \mathbb{R}^2 \times \mathbb{R}^2 \rightarrow \mathbb{R}$ satisfy the four axioms of a metric:

(a) $d(x, y) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2}$

(b) $d(x, y) = |x_1 - y_1| + |x_2 - y_2|$

(c) $d(x, y) = \max\{|x_1 - y_1|, |x_2 - y_2|\}$

For each of the metrics above sketch the set

$$\{x \in \mathbb{R}^2 : d(x, 0) = 1\}.$$

7. Consider the open intervals $A_n =] - 1/n, 1/n[$ with $n \in \mathbb{N}$. Show that

$$\bigcap_{n \in \mathbb{N}} A_n = \{0\}.$$

Conclude that the intersection of a countable set of open sets might be closed.

8. Consider the following sequence in $\mathbb{R}^3$:

$$u_n = \left(\left(\frac{n+3}{n} \right)^{2n}; \frac{n^2(-1)^{2n+1}}{5n^2+1}; \frac{4n^5+5n^3}{8n^5+7n^2} \right), \quad n \in \mathbb{N}$$

Does $(u_n)_n$ converge?

9. The (maximal) domain of $f : D_f \rightarrow \mathbb{R}$ is the set

$$D_f = \{(x, y) \in \mathbb{R}^2 : y > (x-1)^2 \wedge (x, y) \neq (1, 5)\}.$$

- (a) Define the closure and the set of accumulation points of D_f .
- (b) Is D_f an open set? Why?
- (c) Given one possible analytical expression for f .

10. Which is the set of accumulation points of $\{(\cos(n\pi), (\frac{1}{2})^n), n \in \mathbb{N}\} \subset \mathbb{R}^2$?

11. Consider the map $f : D_f \rightarrow \mathbb{R}$, $D_f \subset \mathbb{R}^2$, given by:

$$f(x, y) = \frac{\ln(4 - (x-1)^2 - y^2)}{\sqrt{y-x}}$$

- (a) Define analytically the domain of f , D_f and represent it in the plane (x, y)
- (b) Define analytically $\text{int}(D_f)$, $\text{cl}(D_f)$ and $\partial(D_f)$.
- (c) Is D_f compact?

12. Consider $A = \{(x, y) \in \mathbb{R}^2 : x^2 + (y+2)^2 < 4 \wedge xy \neq 0\}$.

- (a) Is A a compact set?
- (b) Define analytically its topological border.
13. Define the topological interior, closure and accumulation points of the following sets:
- (a) $A = \{(1, 1), (1, 0)\}$
- (b) $B = \left\{ (x, y) \in \mathbb{R}^2 : x \geq 0 \wedge y = \frac{(-1)^n}{n}, n \in \mathbb{N} \right\}$
14. Sketch the following sets and decide which are open, closed, bounded and compact.
- (a) $A = \{(x, y) \in \mathbb{R}^2 : x \geq 0\}$
- (b) $B = \{(x, y) \in \mathbb{R}^2 : 1 \leq x^2 + y^2 \leq 4\}$
- (c) $A \cap B$
- (d) $C = \{(x, y) \in \mathbb{R}^2 : x^2 + 2x > y\}$
- (e) $C \setminus A$
- (f) $D = \{(x, y, z) \in \mathbb{R}^3 : z > x^2 + y^2\}$
- (g) $E = \{(x, y, z) \in \mathbb{R}^3 : -1 \leq z \leq 1\}$
- (h) $F = \{(x, y, z) \in \mathbb{R}^3 : z^2 \geq x^2 + y^2 \text{ and } z \leq 0\}$
- (i) $D \cap E$
- (j) $(\overline{D} \cup F) \cap E$
15. Show that:
- (a) Any finite union of compact sets is compact.
- (b) $A \subset \mathbb{R}^n$ is bounded if and only if $\overline{A}$ is compact.
16. Exhibit an example of a compact set of $\mathbb{R}^2$ without accumulation points.
17. In this exercise you will study the Cantor set C . Let $A_0 = [0, 1]$ and define A_1 by cutting A_0 in three equal parts and then removing the middle part from A_0 . i.e., $A_1 = [0, 1/3] \cup [2/3, 1]$. The set A_1 is a union of two disjoint intervals. Now for each interval of A_1 we proceed as before. Cut in three equal parts and remove from each its middle part. Call this new set A_2 . Continue this process indefinitely to obtain a sequence of sets A_n , $n \geq 0$. The Cantor set is the intersection of all these sets, i.e.,
- $$C = \bigcap_{n \geq 0} A_n$$
- (a) Obtain the analytic expression for A_2 .
- (b) Show that each set A_n is a union of 2^n disjoint closed intervals.
- (c) Is A_n closed? Why?
- (d) Prove that C is compact.

18. Show that the limit of a converging sequence is unique, i.e., a converging sequence cannot more than one limit.
19. For each of the following functions $f: D \rightarrow \mathbb{R}$ determine its continuity points and, using the Weierstrass theorem, decide if the function has a maximum or minimum.
- (a) $f(x) = x^2$ where $D = \{x \in \mathbb{R} : |x| \leq 1\}$
 - (b) $f(x) = x^3 - x^2 + x - 1$ where $D = [-2, -1] \cup [1, 2]$
 - (c) $f(x) = x \cos^2(1/x)$ where $D = \left\{ \frac{(-1)^n}{2\pi n} : n \in \mathbb{N} \right\} \cup \{0\}$
 - (d) $f(x, y) = xy$ where $D = [-1, 1]^2$
 - (e) $f(x, y) = x \log(y)$ where $D =]0, 1]^2$
 - (f) $f(x, y) = e^{-x^2-y^2}$ where $D = \mathbb{R}^2$
20. Consider the function $f: [-1, 1] \rightarrow [-1, 1]$ defined by $f(x) = \sin(\lambda x)$ where $\lambda \in]0, 1[$. Sketch the graph of f and interpret the fixed point geometrically (also draw the bissectrix, i.e., $y = x$).
21. Complete the sentence:
The continuous map $f(x, y) = x^2 + y^2$ has a maximum but not a minimum when restricted to the set

$$M = \{(x, y) \in \mathbb{R}^2 : \dots\dots\dots\}$$

.....'s Theorem cannot be applied because M is not

22. Show that $f:]0, 1/4[\rightarrow]0, 1/4[$ defined by $f(x) = x^2$ is a Lipschitz contraction. Can you conclude that f has a unique fixed point?
23. Determine whether the following functions $f: D \rightarrow \mathbb{R}$ are Lipschitz contractions. For each function determine its fixed points.
- (a) $f(x) = \frac{1}{4}x(1 - x^2)$ with $D = [-1, 1]$
 - (b) $f(x) = \arctan(x/2)$ with $D = \mathbb{R}$
 - (c) $f(x) = \frac{1}{4}\sin(x^3)$ with $D = [-1, 1]$
 - (d) $f(x) = \sqrt{1+x}$ with $D = [0, +\infty[$
 - (e) $f(x, y) = \left(\frac{x}{2} + 5, \frac{y}{3} - 1\right)$ with $D = \mathbb{R}^2$
24. Use the Banach fixed point theorem to show that the sequence $x_{n+1} = \sqrt{1+x_n}$, $n \in \mathbb{N}$ and $x_0 = 0$ converges to the golden ratio

$$\frac{1+\sqrt{5}}{2} = \sqrt{1+\sqrt{1+\sqrt{1+\dots}}}$$

25. Decide which of the following sets are convex:

- (a) $A = \{(x, y) \in \mathbb{R}^2: y \geq x^2\}$
- (b) $B = \{(x, y) \in \mathbb{R}^2: y < x^2\}$
- (c) $C = \{(x, y) \in \mathbb{R}^2: \frac{x^2}{4} + y^2 \leq 1\}$
- (d) $A \cap C$
- (e) $C \setminus B$
- (f) $D = \{(x, y, z) \in \mathbb{R}^3: x + y + z = 1, x, y, z \geq 0\}$

26. Consider the function $f(x) = \frac{1}{2}(x + 1)$ defined on the interval $]0, 1[$. Show that $f(]0, 1[) \subset]0, 1[$. Does f has a fixed point in $]0, 1[$? Can you apply the Brouwer fixed point theorem to f ?

27. Consider the following matrix

$$A = \begin{pmatrix} 0 & 1/2 & 1 \\ 1 & 0 & 0 \\ 0 & 1/2 & 0 \end{pmatrix}$$

and define the function $f(v) = Av$ where $v \in \Delta^2 = \{(x, y, z) \in \mathbb{R}^3: x + y + z = 1, x, y, z \geq 0\}$.

- (a) Show that $f(\Delta^2) \subset \Delta^2$.
 - (b) Can you apply the Brouwer fixed point theorem to f ? Can you show that f has a fixed point?
 - (c) If yes, compute the fixed points of f explicitly.
28. Find a continuous function $f: [0, 1[\rightarrow [0, 1[$ with no fixed points. Can you apply the Brouwer fixed point theorem to f ?
29. Find a continuous function $f: [0, 1] \rightarrow [0, 1]$ with an infinite number of fixed points.
30. Consider the function $f: [0, 1] \rightarrow [0, 1]$ defined by

$$f(x) = \begin{cases} 2x, & x \leq 1/2 \\ 2 - 2x, & x > 1/2 \end{cases}$$

- (a) Can you apply the Brouwer fixed point to f ?
 - (b) Compute the fixed points of f and of $f \circ f$ (hint: draw the graph of the functions).
 - (c) How many fixed points has $f^n = f \circ \dots \circ f$? Here, f^n denotes the composition of f with itself n times ($f^2 = f \circ f$, $f^3 = f \circ f \circ f$, etc.)
31. Consider a pure exchange economy with 2 commodities and 2 consumers. Denote by p_1 and p_2 the relative price of commodity 1 and 2, respectively. The 1st consumer has an initial amount of 1 unit of commodity 1 and 2 units of commodity 2. The 2nd consumer has an initial amount of 2 units of commodity 1 and 1 unit of commodity 2. Consumers engage in exchanging with the following demand functions

$$x_{1,1}(p) = \frac{\alpha(p_1 + 2p_2)}{p_1}, \quad x_{1,2}(p) = \frac{(1 - \alpha)(p_1 + 2p_2)}{p_2}$$

for the 1st consumer and

$$x_{2,1}(p) = \frac{\alpha(2p_1 + p_2)}{p_1}, \quad x_{2,2}(p) = \frac{(1 - \alpha)(2p_1 + p_2)}{p_2}$$

for the 2nd consumer, where $\alpha \in [0, 1]$. Recall that $x_{i,j}(p)$ is the demand of commodity j of consumer i . Check that the Walras's law is satisfied. Determine an equilibrium price for this economy.

32. Sketch the following hyperplanes and half-spaces:

- (a) $H((1, 1), -1)$
- (b) $H^+((2, -1), 1)$
- (c) $H((1, 0, -1), 1)$
- (d) $H^-((0, 1, 0), 2)$

33. (a) Let A be the set of points in $\mathbb{R}^n$ whose first coordinate is equal to $a \in \mathbb{R}$. Find $p \in \mathbb{R}^n$ and $c \in \mathbb{R}$ such that $A = H(p, c)$.

(b) Find a hyperplane $H(p, c)$ that separates $\Delta^2 = \{(x, y, z) \in \mathbb{R}_+^3 : x + y + z = 1\}$ and $A = \{(x, y, z) \in \mathbb{R}^3 : x = -2\}$.

(c) Find the hyperplane $H(p, c)$ that contains the points $(1, 0, 0)$, $(1, 1, 0)$ and $(0, 1, 1)$.

34. Explain, using the hyperplane separation theorem, if it is possible to prove the existence of a hyperplane separating A and B :

- (a) $A = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 < 1\}$ and $B = \{(x, y) \in \mathbb{R}^2 : x + y = 2\}$
- (b) $A = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 < 3\}$ and $B = \{(x, y) \in \mathbb{R}_+^2 : x + y = 1\}$
- (c) $A = C \cap D$ with $C = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 2\}$ and $D = [-6, 6] \times [-5, 5]$, and $B = [-1, 1] \times [-1, 1]$

35. Decide if the following correspondences $F : [0, 1] \rightrightarrows [0, 1]$ are upper hemicontinuous and/or have the closed graph property:

(a)

$$F(x) = \begin{cases} [1/4, 3/4], & x < 1/2 \\ [1/2, 3/4], & x \geq 1/2 \end{cases}$$

(b)

$$F(x) = \begin{cases} [x^2, (x+1)/2], & x < 1 \\ \{0, 1\}, & x = 1 \end{cases}$$

(c)

$$F(x) = \begin{cases}]x, 1 - x[, & x < 1/2 \\ \{1/4, 3/4\}, & x \geq 1/2 \end{cases}$$

(d)

$$F(x) =]x/2, (x+1)/2[$$

36. Explain if the following correspondences satisfy the hypothesis of the Kakutani fixed point theorem. Compute the fixed points if they exist.

(a) $F: [0, 2] \rightrightarrows [0, 2]$ defined by

$$F(x) = \begin{cases} \{1\}, & 0 \leq x < 1 \\ [1, 2], & x = 1 \\ \{2\}, & 1 < x \leq 2 \end{cases}$$

(b) $F: [0, 20] \rightrightarrows [0, 20]$ defined by

$$F(x) = \begin{cases} \{10 - x\}, & 0 \leq x < 7 \\ [3, 20], & x = 7 \\ \{x - 4\}, & 7 < x \leq 20 \end{cases}$$

(c) $F: [-6, 20] \rightrightarrows [-6, 20]$ defined by

$$F(x) = \begin{cases} \{x + 1\}, & -6 \leq x < 7 \\ [-6, 10], & x = 7 \\ \{(x + 5)/2\}, & 7 < x \leq 20 \end{cases}$$

(d) $F: [-6, 20] \rightrightarrows [-6, 20]$ defined by

$$F(x) = \begin{cases} \{x + 1\}, & -6 \leq x < 7 \\ [-6, 6] \cup [8, 10], & x = 7 \\ \{(x + 5)/2\}, & 7 < x \leq 20 \end{cases}$$