



Mathematical Economics – 1st Semester - 2026/2027

Exercises - Group II

1. Complete the following sentences:

- (a) If $f(x, y) = \frac{3}{(x-1)^2+y^2}$ and $g : \mathbb{R} \setminus \{3\} \rightarrow \mathbb{R}$ is the map defined by $g(x) = 2 + \frac{5}{x-3}$, then

$$\lim_{(x,y) \rightarrow (0,0)} [g \circ f(x, y)] = \dots\dots\dots$$

- (b) The gradient vector of $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ is given by $(2x \cos y, 1 - x^2 \sin y)$. If $f(x, y)$ does not have constant terms in both components, then $f(1, \pi) = \dots\dots$

- (c) With respect to the map $f : \mathbb{R}^2 \rightarrow \mathbb{R}$, we know that $\nabla f(3, 2) = (0, 0)$ and

$$H_f(3, 2) = \begin{pmatrix} \dots\dots & 0 \\ 0 & \dots\dots \end{pmatrix}.$$

Then, $f(3, 2)$ is a local maximum of f .

- (d) The point $x = 5$ is a saddle-point of the map $f(x) = \dots\dots\dots$, $x \in \mathbb{R}$.

2. Classify the critical points of the following functions,

- (a) $f(x, y, z) = x^2 + 2y^2 + 3z^2 + 2xy + 2xz$ on $\mathbb{R}^3$
(b) $f(x, y, z, w) = 20y + 48z + 6w + 8xy - 4x^2 - 12z^2 - w^2 - 4y^3$ on $\mathbb{R}^4$
(c) $f(x, y, z) = z \log(x^2 + y^2 + z^2)$ on $\mathbb{R}^3 \setminus \{(0, 0, 0)\}$

3. Consider the map f defined in $\mathbb{R}^2$ as

$$f(x, y) = x^3 + y^3 + x^3y^3.$$

- (a) Compute $f(x, 0)$, $x \in \mathbb{R}$.
(b) Identify and classify the critical points of f .

4. Find and classify the critical points of $f(x, y) = e^{x^2 - ay^2}$ as function of the parameter $a \in \mathbb{R} \setminus \{0\}$.
5. Consider the map $f(x, y) = (y - \alpha)xe^x$, where $\alpha \in \mathbb{R} \setminus \{0\}$.
 - (a) If $(0, 1)$ is a critical point of f , find α and classify the critical point.
 - (b) Show that f is not limited.
6. Consider the map $f(x, y) = x^2 e^{y^3 - 3y}$.
 - (a) Find the maximal domain of f .
 - (b) Find and classify all critical points of f .
 - (c) Show that f attains its global minimum at points of the form $(0, b), b \in \mathbb{R}$.
 - (d) Show that f is not limited.
 - (e) Show that f has a maximum and a minimum on

$$B = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 5\}.$$

7. Consider the following function defined on $\mathbb{R}^2$:

$$f(x, y) = (1 + y)^3 x^2 + y^2$$

Show that f has a unique critical point (x^*, y^*) which is a local minimizer. Is (x^*, y^*) a global minimizer ?

8. Determine the maximum and minimum distance to the origin of the points in the ellipse defined by $5x^2 + 6xy + 5y^2 = 8$.
9. Determine the point in the ellipse $x^2 + 2xy + 2y^2 = 2$ with smallest x coordinate.
10. Find and classify all critical points of

$$f(x, y) = 8y^2 - 4x^2(y - 1).$$

Among the critical points, is there some global one?

11. Find the extrema of $f(x, y) = x$ when restricted to the set $y^2 + x^4 - x^3 = 0$.
12. Find the extrema of $f(x, y) = x - 2y + 2z$ when restricted to

$$M = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\}.$$

13. Determine the global extrema of f over the set M , where:

- (a) $f(x, y, z) = x - 2y + 2z$ and $M = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\}$
- (b) $f(x, y, z) = x^2 + 2xy + y^2$ and $M = \{(x, y, z) \in \mathbb{R}^3 : (x - 3)^2 + y^2 = 2\}$

14. Determine the global extrema of f over the set M , where:

(a) $f(x, y, z) = x - 2y + 2z$ and $M = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 \leq 1\}$

(b) $f(x, y, z) = x^2 + 2xy + y^2$ and $M = \{(x, y, z) \in \mathbb{R}^3 : (x - 3)^2 + y^2 \leq 2\}$

15. Determine if the following functions are (strictly) convex/concave:

(a) $f(x, y) = 2x - y - x^2 + 2xy - y^2$ on $\mathbb{R}^2$.

(b) $f(x, y) = x^a y^b$ on $\mathbb{R}_+^2$ and $a + b \leq 1$ with $a, b \geq 0$.

16. Find the largest domain $D \subset \mathbb{R}^2$ on which the following function is concave,

$$f(x, y) = x^2 - y^2 - xy - x^3.$$

17. Use Lagrange convexity's Theorem to solve the following optimization problems:

(a)

$$\begin{aligned} &\text{maximize } 2x + y \\ &\text{subject to } x^2 + y^2 = 1 \end{aligned}$$

(b)

$$\begin{aligned} &\text{minimize } x^2 y^2 \\ &\text{subject to } (1/x)^2 + (1/y)^2 = 1 \end{aligned}$$

(c)

$$\begin{aligned} &\text{maximize } x + 4y + z \\ &\text{subject to } x + 2y + 3z = 0 \\ &\quad x^2 + y^2 + z^2 = 42 \end{aligned}$$

(d)

$$\begin{aligned} &\text{minimize } x + 4z \\ &\text{subject to } x - y + z = 2 \\ &\quad x^2 + y^2 = 1 \end{aligned}$$

18. Find the local optimal points of f on D where:

(a)

$$\begin{aligned} f(x, y) &= \log(xy) \\ D &= \{(x, y) \in \mathbb{R}^2 : (1/x)^2 + (1/y)^2 = 1\} \end{aligned}$$

(b)

$$\begin{aligned} f(x, y) &= x + y \\ D &= \{(x, y) \in \mathbb{R}^2 : xy = 16\} \end{aligned}$$

(c)

$$f(x, y, z) = x^2 - z^2$$
$$D = \{(x, y, z) \in \mathbb{R}^3 : 2x + z = a, x - y = b\}$$

19. Use Kuhn-Tucker Theorem to solve the following optimization problems:

(a)

$$\begin{aligned} &\text{maximize } x^2 + 2y \\ &\text{subject to } x^2 + y^2 \leq 5 \\ &\quad y \geq 0 \end{aligned}$$

(b)

$$\begin{aligned} &\text{maximize } \frac{1}{2}x - y \\ &\text{subject to } x + e^{-x} + z^2 \leq y \\ &\quad x \geq 0 \end{aligned}$$

(c)

$$\begin{aligned} &\text{minimize } 2x^2 + 3y^2 \\ &\text{subject to } x + 2y \leq 11 \\ &\quad x \geq 0 \\ &\quad y \geq 0 \end{aligned}$$

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