Universidade de Lisboa
Instituto Superior de Economia e Gestão
Msc in Economics, Mathematical Finance,
Monetary and Financial Economics

Mathematical Economics – 1st Semester - 2025/2026

Regular Assessment - 16th of December 2025

Duration: $(120 + \varepsilon)$ minutes, $|\varepsilon| \leq 30$

Version A

Name: ....

Student ID #: ....

Part I

- Complete the following sentences in order to obtain true propositions. The items are independent from each other.
- There is no need to justify your answers.

(a) *(4)* For $D_f \subset \mathbb{R}^2$, the (maximal) domain of $f : D_f \to \mathbb{R}$ defined by

$$f(x, y) = \frac{\sqrt{y}}{\ln x^2}$$

is the set

$D_f =$ ....

(simplify the expression)

(b) *(6)* Consider the set

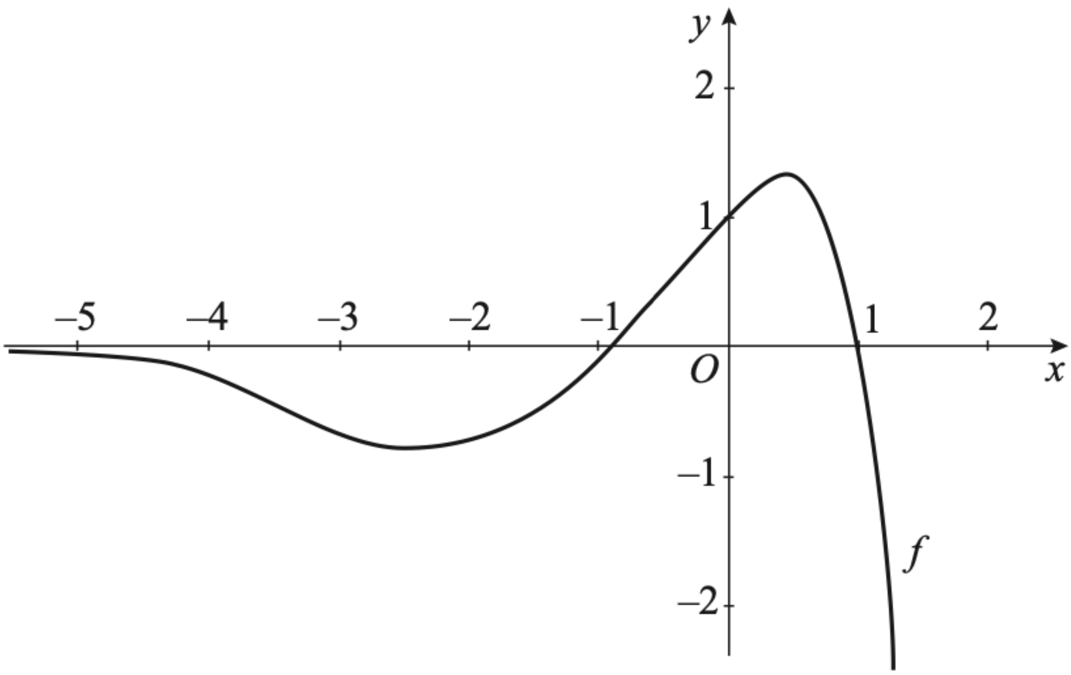
$$A = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 < 4 \quad \wedge \quad y \geq x\}.$$

The **topological closure** of A is analytically defined by

$\text{cl}(A) =$ ....

Since cl(A) $\neq A$, then A is not .... and, consequently, A is not compact.

(c) *(6)* The graph of the differentiable map $f : \mathbb{R} \to \mathbb{R}$ is below.



Then:

$$\lim_{n \to +\infty} f\left(-\frac{2}{n}\right) = \text{.........} \quad \text{and} \quad \lim_{x \to -\infty} \frac{1}{f(x)} = \text{.........}$$

(d) *(5)* With respect to the map $f : \mathbb{R}^2 \to \mathbb{R}$, one knows that

$$\nabla f(x, y) = (-2x \sin(x^2 + y); 7 - \sin(x^2 + y)).$$

If $f(x, y)$ does not have constant terms, then

$$f(x, y) = \text{...}$$

(e) *(5)* Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by $f(x, y) = e^{3x-5y}$. Then:

$$\frac{\partial^{200} f}{\partial x^{200}}(0, 0) = \text{............}$$

(f) *(4)* In $\mathbb{R}^2$, the sets

$$I_1 = \left[-\frac{1}{10}, \frac{1}{10}\right] \times \left[-\frac{1}{10}, \frac{1}{10}\right] \qquad \text{and} \qquad I_2 = \{(x, y) \in \mathbb{R}^2 : (x-5)^2 + y^2 \leq 9\}$$

are non-empty, disjoint and .... Hence, the **Hyperplane Separation Theorem** may be applied. One possible hyperplane that separates I_1 and I_2 is

....

(g) *(10)* The graph of the correspondence $H : [0, 4] \rightrightarrows [0, 4]$

$$H(x) = \begin{cases} \left[\frac{3}{2}x, x+2\right] & x < 2 \\ [0,1] \cup [3,4], & x = 2 \\ \left[x-2, \frac{3}{2}x - 2\right], & x > 2 \end{cases}$$

is

Since the .... property is valid, then H is **upper hemicontinuous**. The following hypothesis of the **Kakutani fixed point Theorem** is failing (*apply to the correspondence under consideration*):

....

The set of **fixed points** of H is explicitly given by ....

(h) *(4)* The map $f : \mathbb{R} \to \mathbb{R}$ is continuous, $f(-2) = -4$ and $f(1) = 0$. The **Intermediate Value Theorem** ensures that the equation $f(x) =$ .... has at least one solution in $]-2, 1[$. Moreover, if f is .... then the solution is unique.

(i) *(6)* With respect to the map $f : \mathbb{R}^2 \to \mathbb{R}$, one knows that $\nabla f(1,2) = (0,0)$ and

$$\forall (x,y) \in \mathbb{R}^2, \quad H_f(x,y) = \begin{pmatrix} -1 & -2 \\ 2 & 1 \end{pmatrix},$$

then f is strictly .... in $\mathbb{R}^2$ and thus $f(1,2)$ is an .... of f.

(j) *(5)* In $\mathbb{R}^2$, consider the following **optimization** problem

$$\text{minimize} \quad U(x,y) = 6xy - 2x, \qquad \text{subject to} \quad 8x + 3y \leq 17.$$

The **Karush-Kuhn-Tucker conditions** are:

$$(*) \begin{cases} 6y - 2 - 8\mu = 0 \\ \text{........................} = 0 \\ \mu \geq 0 \\ \mu(\text{........................}) = 0 \end{cases}$$

If (x_0, y_0) is one possible solution of (*) associated to $\mu =$ ...., then

$$(x_0, y_0) \in \text{fr}\{(x,y) \in \mathbb{R}^2 : 8x + 3y \leq 17\}.$$

(k) *(5)* Let $a \in \mathbb{R}$. The differential equation (y depends on t):

$$t + ye^{2ty} + a\,te^{2ty}\frac{dy}{dt} = 0.$$

is **exact** if and only if

$$a = \text{..........}$$

(l) *(8)* The map $y(x) = e^{5x}(\cos(x) + \sin(x))$, $x \in \mathbb{R}$, is a solution of the IVP

$$\begin{cases} y'' + \ldots\ldots y' + \ldots\ldots y = 0 \\ y(0) = \ldots\ldots \\ y'(0) = \ldots\ldots \end{cases} .$$

(m) *(8)* Assuming that x and y depend on t, the **linearisation** of

$$(*) \begin{cases} \dot{x} = -x + xy \\ \dot{y} = y - xy \end{cases}$$

around the equilibrium $(0, 0)$ is

$$(**) \begin{cases} \dot{x} = \ldots\ldots\ldots\ldots \\ \dot{y} = \ldots\ldots\ldots\ldots \end{cases}$$

With respect to the Lyapunov's **stability**, we may conclude that (0,0) is ....
Furthermore, since $(0, 0)$ is hyperbolic, ...-.........................Theorem says that there exists a small neighbourhood of $(0, 0)$ where the dynamics of (*) and (**) are "qualitatively" the same (topologically conjugate).

(n) *(10)* A given population of size p depends on the time $t \geq 0$ and follows the **logistic law**:

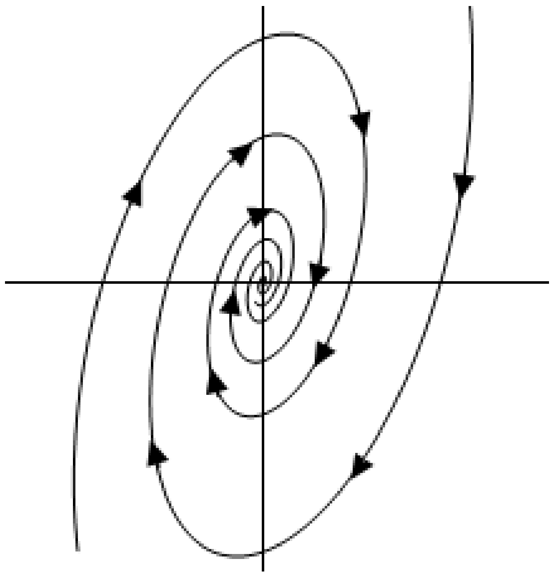
$$p' = 10p - 5p^2.$$

The equilibria of the differential equation are .... and its phase portrait is:

If $p(0) = \frac{1}{10}$, then:

- the solution of the previous differential equation is monotonic ....
- $\lim_{t \to -\infty} p(t) = \ldots\ldots$
- the graph of p has an inflexion point when $p = \ldots\ldots$

(o) *(4)* The phase portrait of $\dot{X} = AX$, where $A \in M_{2\times 2}(\mathbb{R})$ and $X \in \mathbb{R}^2$, is given by:



Then the following **inequalities** hold: det(A)............ and Tr(A).............

(det(A) and Tr(A) denote the usual determinant and trace operators)

(p) *(10)* Consider the following Problem on *Calculus of Variations*, where $x : [0,1] \to \mathbb{R}$ is a smooth function on t and $\mathcal{A}$ is the set of admissible maps:

$$\max_{x\in\mathcal{A}} \int_0^1 [1 - x(t)^2 - \dot{x}(t)^2]\, dt, \quad \text{with} \quad x(0) = 1 \quad \text{and} \quad x(1) = e.$$

The **Euler-Lagrange equation** applied to the case under consideration is:

....

(simplify the equation)

The solution of the problem is:

$x^\star(t) =$ ...., $t \in [0,1]$.

This map is, in fact, the solution of the problem because

$$F(t, x, \dot{x}) = 1 - x^2 - \dot{x}^2$$

is .... in $(x, \dot{x})$.

Part II

- Give your answers in exact form.
- In order to receive credit, you must show all of your work. If you do not indicate the way in which you solve a problem, you may get little or no credit for it, even if your answer is correct.

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1. Consider the map $F : [3/2, +\infty[\to [3/2, +\infty[$ defined by:

$$F(x) = 1 + \frac{1}{x}$$

   (a) Show that F satisfies the conditions of the **Banach fixed point Theorem**.

   (b) Find the fixed point of F.

   (c) Consider the **convergent** sequence $(x_n)_{n\in\mathbb{N}}$ such that

$$x_1 = 3/2 \qquad \text{and} \qquad x_{n+1} = F(x_n).$$

   Compute $\lim\limits_{n\to+\infty} x_n$ and justify your reasoning.

2. Consider the map $f : \mathbb{R}^2 \to \mathbb{R}$ defined as

$$f(x, y) = xy\, e^{x+y}$$

   (a) Identify and classify the **critical points** of f.

   (b) Find possible extrema of f restricted to $M = \{(x, y) \in \mathbb{R}^2 : y = 1 - x\}$.

3. Consider the following **linear of first order** IVP (y is a function of x):

$$\begin{cases} x^2y' + xy = x^3 \\ y(1) = \frac{4}{3} \end{cases}$$

   Solve the differential equation and write its maximal domain.

4. Consider the linear system of ODEs in $\mathbb{R}^2$ given by (x and y depend on $t \in \mathbb{R}$):

$$\begin{cases} \dot{x} = x + y \\ \dot{y} = 3x + 3y \end{cases}$$

   (a) Write the **general form** of the solution.

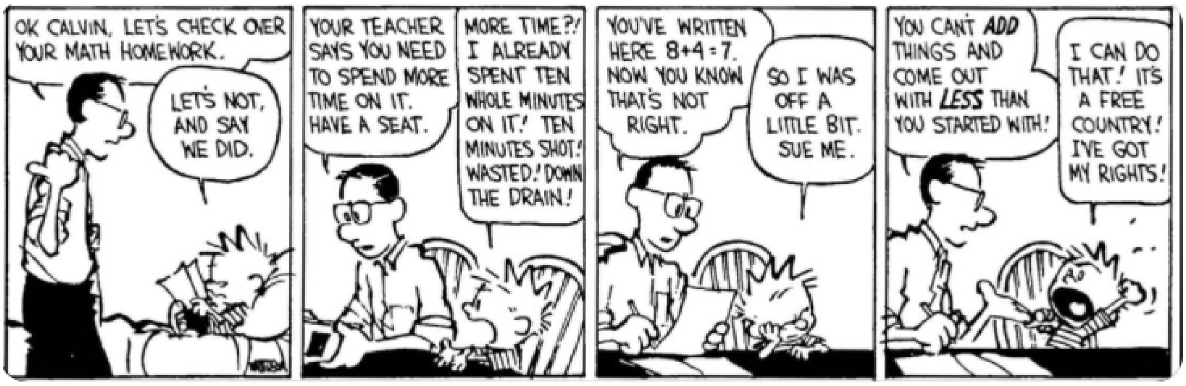
   (b) Sketch the **phase portrait** of the system.

   (c) In the phase portrait of (b), locate the **unique** solution such that $x(0) = 0$ and $y(0) = 3$.

5. Consider the following Problem on *Optimal Control*, where $x : [0,1] \to \mathbb{R}$ is a smooth function on t and $\mathcal{A}$ is the set of admissible maps:

$$\min_{u \in \mathcal{A}} \int_0^1 \left[t x(t) + u(t)^2 - 1 \right] dt, \quad \text{with} \quad \dot{x}(t) = u(t) \quad \text{and} \quad x(0) = 1,\ x(1) \in \mathbb{R}.$$

   Using the **Pontryagin maximum principle**, compute the **control**, the **state** and the **co-state** maps.



Credits:

I	II.1(a)	II.1(b)	II.1(c)	II.2(a)	II.2(b)
100	10	5	10	10	10

II.3	II.4(a)	II.4(b)	II.4(c)	II.5
15	10	10	5	15