Universidade de Lisboa
Instituto Superior de Economia e Gestão
Msc in Economics, Mathematical Finance,
Monetary and Financial Economics

<u>Mathematical Economics</u> – 1st Semester - 2025/2026

Resit Assessment - 8th of January 2026

Duration: $(120 + \varepsilon)$ minutes, $|\varepsilon| \leq 30$

Version A

Name: ....

Student ID #: ....

Part I

- Complete the following sentences in order to obtain true propositions. The items are independent from each other.
- There is no need to justify your answers.

(a) *(7)* For $D_f \subset \mathbb{R}^2$, the (maximal) domain of $f : D_f \to \mathbb{R}$ defined by

$$f(x, y) = \frac{\ln(x - y^2 - 1)}{\sqrt{y}}$$

is the set

$D_f = \{$..$\}$

and its **planar representation** in the cartesian plane (x, y) is:

(b) *(7)* With respect to the set

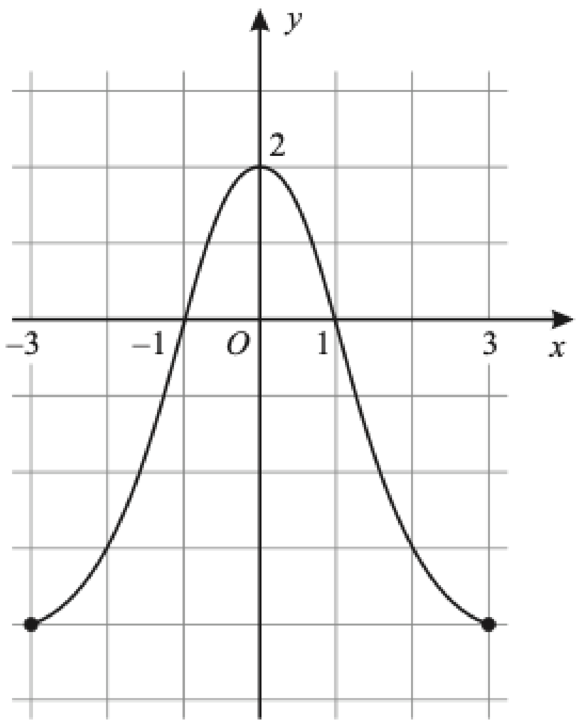
$$\Omega = \{(x, y) \in \mathbb{R}^2 : |x - 5| < 1 \ \wedge \ 4 \leq y < 7\} \cup \{(0, 0)\},$$

we may conclude that $(0, 0)$ is not a/an .... point of Ω,

$int(\Omega) = \{$...$\}$,

and, since $int(\Omega) \neq \Omega$, then Ω is not ....

(c) *(6)* Let $f : [-3, 3] \to \mathbb{R}$ the map whose graphical representation is below. Integer objects have integer images.



Let $g : [e^{-3}, e^3] \to \mathbb{R}$ be the map whose analytical expression is $g(x) = \ln x$. Then:

- $Im(f) =$ ....
- The map $f \circ g$ has **two** zeros: .... and ....
- $\lim_{n \to +\infty} f\left(-\frac{1}{n}\right) =$ ....

(d) *(4)* The **gradient** vector of $f : \mathbb{R}^2 \to \mathbb{R}$ is given by $(2xy^3, 3x^2y^2 - \cos y)$. If $f(x,y)$ **does not have constant terms**, then

$$f(x,y) = \text{...}$$

(e) *(4)* With respect to the C^2 map $f : \mathbb{R}^2 \to \mathbb{R}$, we know that $\nabla f(3,2) = (0,0)$ and

$$H_f(3,2) = \begin{pmatrix} \ldots\ldots & 0 \\ 0 & \ldots\ldots \end{pmatrix}.$$

Then, $f(3,2)$ is a local **maximum** of f.

(f) *(4)* With respect to a C^∞ map $f : \mathbb{R}^2 \to \mathbb{R}$, we know that

$$\frac{\partial^4 f}{\partial x^2 \partial y^2}(x,y) = \cos(3x).$$

Then:

$$\frac{\partial^{12} f}{\partial x^{10} \partial y^2}(x,y) = \text{............}$$

(g) *(6)* The **continuous** map $f(x,y) = \dfrac{1}{x^2+y^2}$ defined in $\mathbb{R}^2 \backslash \{(0,0)\}$ has a **global maximum and a global minimum** when restricted to the set

$$M = \{(x,y) \in \mathbb{R}^2 \backslash \{(0,0)\} : \text{..}\}$$

This is a consequence of ....' Theorem (since f is continuous and M is compact).

(h) *(4)* In $\mathbb{R}^2$, consider the sets

$$I_1 = \left[-\frac{1}{10}, \frac{1}{10}\right] \times \left[-\frac{1}{10}, \frac{1}{10}\right] \qquad \text{and} \qquad I_2 = \{(x,y) \in \mathbb{R}^2 : 4 < x^2 + y^2 \leq 9\}.$$

The **Hyperplane Separation Theorem** cannot be applied to separate I_1 and I_2 through a line because

....

(i) *(8)* For $a \in [0,2]$, consider the correspondence $H_a \colon [0,2] \rightrightarrows [0,2]$ defined by:

$$H_a(x) = \begin{cases} \left[\sqrt{x}, 2 - \dfrac{x}{2}\right] & x \leq 1 \\ \\ \{a\} & x > 1 \end{cases}$$

The **closed graph property** is valid to H_a if and only if $a \in$ ....

Combining the latter information with the fact that $graph(H_a)$ is compact, one concludes that the correspondence H_a is .... at $x \in [0,2]$.

The set of **fixed points** of H_1 is explicitly given by: ....

(j) *(5)* Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be an **affine contracting** map given by:

$$f\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 3.4 \end{pmatrix} + \begin{pmatrix} 0.5 & 0 \\ 0 & -0.7 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}.$$

Then, the equation $f\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$ has a **unique** solution as a consequence of the

.... Theorem. In particular,

$$\lim_{n \to +\infty} f^n(0,0) = \text{................}$$

(k) *(6)* In $(\mathbb{R}_0^+)^2$, consider the following **optimization** problem

$$\text{minimize} \quad U(x,y) = 6xy - 2x, \qquad \text{subject to} \quad 8x + 3y \leq 17.$$

The **Karush-Kuhn-Tucker conditions** are:

$$(*) \begin{cases} 6y - 2 - 8\mu = 0 \\ \\ \text{........................} = 0 \\ \\ \mu \leq 0 \\ \\ \mu(\text{........................}) = 0 \end{cases}$$

If (x_0, y_0) is one possible solution of (*) associated to $\mu =$ ...., then

$$(x_0, y_0) \in \text{fr}\{(x,y) \in (\mathbb{R}_0^+)^2 : 8x + 3y \leq 17\}.$$

(l) *(8)* The **differential equation** associated to the IVP (y depends on x)

$$\begin{cases} (x^2+1)y' + y^2 x = 0 \\ y(0) = 2 \end{cases}$$

is .... because it can be written as $\frac{y'}{y^2} = -\frac{x}{x^2+1}, y \neq 0$. The unique solution of the IVP is given by:

$$y(x) = ...,$$

whose maximal domain is $D =$ ....

(m) *(5)* Assuming that y depends on x, the IVP

$$\begin{cases} y' = \sqrt{y-1} \\ y(0) = 1 \end{cases}$$

admits two different solutions passing through $x = 0$. This does not contradict the **Existence and Uniqueness Theorem** for ordinary differential equations (Picard Theorem) because

....

(n) *(10)* Assuming that x and y depend on t, the equilibria of

$$(*) \begin{cases} \dot{x} = -x + xy \\ \dot{y} = y - xy \end{cases}$$

are .... and .... The linearization of (*) around $(1,1)$ is

$$(**) \begin{cases} \dot{x} = ... \\ \dot{y} = ... \end{cases}$$

The point $(1,1)$ is **non-hyperbolic** because...

(o) *(10)* A given population of size p depends on the time $t \geq 0$ and follows the **logistic law**:

$$p' = 20p - 10p^2.$$

The equilibria of the differential equation are .... and its phase portrait is:

If $p(0) = 0.25$, then:

- the solution of the previous differential equation is monotonic ....
- $\lim_{t \to +\infty} p(t) =$ ....
- the graph of p has an inflexion point when $p =$ .....

(p) *(16)* Consider the following problem of optimal control where $x : [0, 1] \to \mathbb{R}$ is the *state* and $u : [0, 1] \to \mathbb{R}$ is the *control*:

$$\max_{u(t) \in \mathbb{R}} \int_0^1 (1 - tx(t) - u(t)^2) dt, \quad x'(t) = u(t), \quad x(0) = 1 \quad \text{and} \quad x(1) \in \mathbb{R}.$$

Then the **Hamiltonian** is given by (*specify the formulas to the case under consideration*):

$$H(t, x, u, p) = \text{...}$$

The **Pontryagin maximum principle** says that the optimal control $u^\star$ should satisfy the equality .... which is equivalent to $u(t) = p(t)/2$.

The **Hamiltonian equations** are given by:

$$\begin{cases} \dot{x} = \text{.....................} \\ \dot{p} = \text{.....................} \end{cases}$$

The *transversality condition* is given by ....

Integrating the differential equation above, using the initial conditions and the transversality condition, we conclude that

$$x^\star(t) = \text{.................................} \quad \text{and} \quad u^\star(t) = \text{.......................................}$$

Since H is .... in (x, u) then the above solutions are the sought solutions.

Part II

- Give your answers in exact form.
- In order to receive credit, you must show all of your work. If you do not indicate the way in which you solve a problem, you may get little or no credit for it, even if your answer is correct.

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1. Consider the map $F : [0,1]^2 \to \mathbb{R}^2$ defined by:

$$F(x,y) = \left(\frac{(x+y)^2}{4}, x \right)$$

   (a) Show that F satisfies the conditions of the **Brouwer fixed point Theorem**.

   (b) Find the fixed point(s) of F.

2. Using the **Lagrange multipliers method**, find the global maximum of $f : \mathbb{R}^3 \to \mathbb{R}$ defined as

$$f(x,y,z) = \frac{x}{2} + \frac{y}{2} - 2z$$

   restricted to the compact set

$$M = \left\{ (x,y,z) \in \mathbb{R}^3 : \frac{x^2}{2} + \frac{y^2}{4} + z^2 = 1 \right\}.$$

3. Consider the following IVP of **second order** (y is a function of x):

$$\begin{cases} y'' + \alpha y' + 6y = 12x + 4 \\ y(0) = -1, \quad y'(0) = 1 \end{cases}$$

   for which $y_p(x) = 2x - 1, x \in \mathbb{R}$, is a **particular solution**.

   Show that $\alpha = 5$ <u>and</u> solve the problem.

4. Consider the linear system of ODEs in $\mathbb{R}^2$ given by (x and y depend on $t \in \mathbb{R}$):

$$\begin{cases} \dot{x} = x + 2y \\ \dot{y} = -2x + y \end{cases}$$

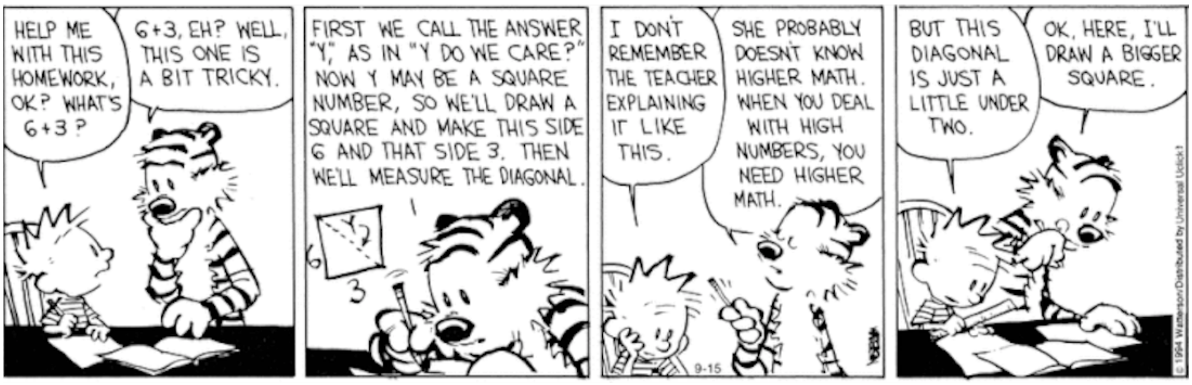
(a) Classify the equilibrium of the system according to its **Lyapunov stability**.

(b) Sketch the **phase portrait** of the system.

5. Consider the following Problem on *Calculus of Variations*, where $x : [0,1] \to \mathbb{R}$ is a smooth function on t:

$$\min_{x \in \mathcal{A}} \int_0^1 [2tx(t) + \dot{x}^2(t)]dt, \quad \text{with} \quad x(0) = 0 \quad \text{and} \quad x(1) = 7/6,$$

and $\mathcal{A}$ is the set of admissible maps. Find the solution of the problem.



Credits:

I	II.1(a)	II.1(b)	II.2
110	10	10	15

II.3	II.4(a)	II.4(b)	I.5
15	10	10	20