Simulation and Optimization

Chapter 2: Integer Programming

Academic year 2026/2027

1. The company *Imobul* intends to invest 750 thousand euros in property acquisitions. The following table gives the current cost (thousand euros) and the estimated value (thousand euros) in 10 years for the 15 apartments under consideration:

Property	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
Current	70	73	77	80	82	87	90	94	98	106	110	113	115	118	120
10 years	135	139	149	150	156	163	173	184	192	201	210	214	221	229	240

   (a) Formulate the problem as an integer linear programming model. Suggest a feasible solution that you consider "good".

   (b) Show how the following constraints could be formulated:
      i. Not all investments can be selected.
      ii. At least one investment must be selected.
      iii. Investment 1 cannot be selected if investment 3 has been selected.
      iv. Investment 4 can only be selected if investment 2 is also selected.
      v. Either investment 1 or investment 4 must be selected, or neither of these investments.
      vi. At least one of investments 1 and 3 must be selected, or at least two of investments 2, 3, and 4.

   (c) Solve the problem formulated in part (a) using appropriate software. How do you assess the solution you suggested previously?

2. A young couple, Eve and Steven, want to divide their main household chores (shopping, cooking, washing the dishes, and doing the laundry) between them so that each is assigned two chores, while minimizing the total time spent on household chores. Their efficiencies at these chores differ, and the time each would need to perform each chore is given in the following table:

	Shopping	Cooking	Washing the dishes	Doing the laundry
Eve	4.5 hours	7.8 hours	3.6 hours	2.9 hours
Steven	4.9 hours	7.2 hours	4.3 hours	3.1 hours

   Formulate this problem as an integer linear programming model and solve it using appropriate software.

3. The Research and Development Division of *Progressive* has been developing four possible new product lines. Management must now decide which of these four products will actually be produced and at what levels. An operations research study has therefore been requested to find the most profitable product mix.

   There is a substantial cost associated with starting production of any product, as shown in the first row of the following table. Management's objective is to find the product mix that maximizes total profit (total net revenue minus production start-up costs).

	Product			
	1	2	3	4
Cost	50 000	40 000	70 000	60 000
Revenue	70	60	90	80

Let the continuous decision variables x_i, for $i \in \{1, 2, 3, 4\}$, denote the production levels of product i. Management has imposed the following policy constraints:

- No more than two of the products may be produced.
- Product 3 or product 4 may only be produced if product 1 or product 2 is produced.
- Either
$$5x_1 + 3x_2 + 6x_3 + 4x_4 \leq 6\,000$$
or
$$4x_1 + 6x_2 + 3x_3 + 5x_4 \leq 6\,000.$$

Formulate the problem as a mixed-integer linear programming model and solve it using appropriate software.

4. The company *Record-a-Song* has hired a rising star to record eight songs. The sizes, in MB, of the different songs are 8, 10, 8, 7, 9, 6, 7, and 12, respectively. *Record-a-Song* uses two CDs for the recording. Each CD has a capacity of 40 MB. The company would like to distribute the songs between the two CDs so that the space used on each CD is approximately the same. Formulate the problem as an integer linear programming model and find the optimal solution.

5. The airline *Northeastern Airlines* is considering purchasing new long-, medium-, and short-range passenger jets. The purchase price would be 67 million dollars for each long-range aircraft, 50 million dollars for each medium-range aircraft, and 35 million dollars for each short-range aircraft. The board of directors has authorized a maximum commitment of 1.5 billion dollars for these purchases. Regardless of which aircraft are purchased, air travel demand at all distances is expected to be sufficiently high for these aircraft to be used essentially at full capacity. The estimated annual net profit (after deducting capital recovery costs) is 4.2 million dollars per long-range aircraft, 3 million dollars per medium-range aircraft, and 2.3 million dollars per short-range aircraft.

   It is expected that enough trained pilots will be available within the company to crew 30 new aircraft. If only short-range aircraft were purchased, the maintenance facilities could handle 40 new aircraft. However, in terms of maintenance-facility usage, each medium-range aircraft is equivalent to $\frac{4}{3}$ short-range aircraft and each long-range aircraft is equivalent to $\frac{5}{3}$ short-range aircraft.

   Management wants to know how many aircraft of each type should be purchased in order to maximize profit.

   Formulate the problem as an integer linear programming model and solve it using appropriate software.

6. The nurses at a hospital work 3 days per week. The daily nurse requirements are shown in the following table, together with the possible shift types and their respective weekly costs in euros.

	Shift type					
Day	1	2	3	4	5	No. of nurses
Monday	X		X			20
Tuesday	X				X	25
Wednesday			X	X		26
Thursday		X			X	26
Friday		X		X	X	30
Saturday		X		X		30
Sunday	X		X			35
cost	525	470	550	500	425	

   Formulate the problem as an integer linear programming model and solve it using appropriate software.

7. The company *Jobco* wants to plan the production of a batch of at least 2000 units, using up to three available machines.

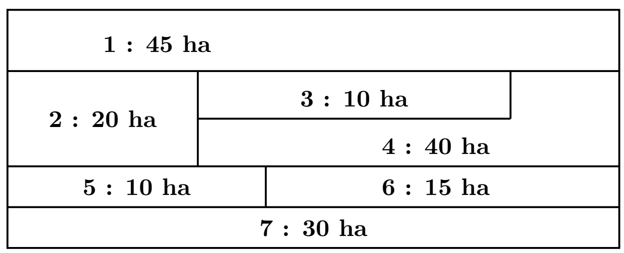
For operational reasons, if a given machine is used, a minimum production quantity of 600 units must be produced on that machine. The set-up costs, unit production costs, and maximum capacities of each machine are shown in the following table:

Machine	Set-up Cost	Unit Cost	Maximum Capacity
1	300	2	650
2	100	10	850
3	200	5	1250

Formulate the problem as an integer linear programming model to determine how many units should be produced on each machine so as to minimize total cost.

8. Suppose that the product zw appears in a constraint, where z and w are binary variables. How can this term be linearized?

9. The following map represents a forest divided into 7 stands i, with $i = 1, \ldots, 7$, with the indicated areas. The amount of timber is proportional to the planted area. Due to ecological and landscape constraints, adjacent stands cannot be harvested. The harvesting method used is "clear-cutting", i.e., all trees in the selected stands are cut. The aim is to determine which stands should be harvested so as to maximize the amount of timber harvested. Write an integer linear programming model for this problem.



10. Let $I = \{1, 2, \ldots, m\}$ be a set of agents, $J = \{1, 2, \ldots, n\}$ a set of tasks to be performed, c_{ij} the cost of task $j \in J$ being performed by agent $i \in I$, r_{ij} the resources required for agent $i \in I$ to perform task $j \in J$, and b_i the amount of resources available to agent $i \in I$. The aim is to assign the tasks to the agents at minimum cost while respecting the available resource quantities. Formulate the problem and solve it using appropriate software, considering the following data:

$$C = \begin{bmatrix} 2 & 1 & 3 & 2 & 4 \\ 3 & 2 & 2 & 1 & 1 \\ 4 & 1 & 1 & 2 & 4 \end{bmatrix} \quad R = \begin{bmatrix} 2 & 3 & 2 & 3 & 1 \\ 2 & 2 & 3 & 1 & 3 \\ 2 & 3 & 4 & 2 & 1 \end{bmatrix} \quad b = \begin{bmatrix} 3 \\ 3 \\ 7 \end{bmatrix}$$

11. Consider the following ILP problems:

(a)

$$\begin{aligned} &\max\ 5x_1 - 2x_2 \\ &\text{s.t.: } 2x_1 + x_2 \leq 6 \\ &\qquad x_1 - x_2 \geq 0 \\ &\qquad x_1, x_2 \geq 0 \text{ and integer} \end{aligned}$$

(b)

$$\begin{aligned} &\min\ 4x_1 + 3x_2 \\ &\text{s.t.: } x_1 + 3x_2 \geq 3 \\ &\qquad 2x_1 + x_2 \geq 2 \\ &\qquad x_1, x_2 \geq 0 \text{ and integer} \end{aligned}$$

Solve their linear relaxations and state what you can conclude about their optimal values.

12. Write the *branch-and-bound* algorithm assuming that P is a maximization problem.

13. Solve the following ILPs using the *branch-and-bound* algorithm, considering the following branching rules: (i) branch on the variable with the smallest index, and (ii) branch on the "most fractional" variable.

(a)

$$\begin{aligned} \max z = {} & 5x_1 + 4x_2 \\ \text{s.t.:}\ & -x_1 + 2x_2 \le 15 \\ & x_1 - x_2 \le 16 \\ & 4x_1 + 3x_2 \le 30 \\ & x_1, x_2 \ge 0 \text{ and integer} \end{aligned}$$

(b)

$$\begin{aligned} \min z = {} & 5x_1 + x_2 \\ \text{s.t.:}\ & 5x_1 - x_2 \ge 7 \\ & x_1 + x_2 \ge 4 \\ & x_1, x_2 \ge 0 \text{ and integer} \end{aligned}$$

14. Consider the following ILP problem:

$$\begin{aligned} \min z = {} & 21x_1 + 15x_2 + 12x_3 + 10x_4 \\ \text{s.t.:}\ & 1.4x_1 - 2.3x_2 + 2.4x_3 + 2.2x_4 \ge 33.3 \\ & -0.5x_1 + 3x_2 - 2x_3 + x_4 \ge 5 \\ & -x_1 - 0.8x_2 + 2.5x_3 - 3.5x_4 \ge 15 \\ & x_1, x_2, x_3, x_4 \ge 0 \text{ and integer} \end{aligned}$$

Solve using the branch-and-bound algorithm, using "solve no more than ten subproblems" as the stopping criterion, and always branching on the variable with the smallest index. State the best feasible solution found and give the optimal value or, if it has not been found, bound it using the best possible lower and upper bounds. Note: Use Excel Solver to solve the linear relaxations of the subproblems.

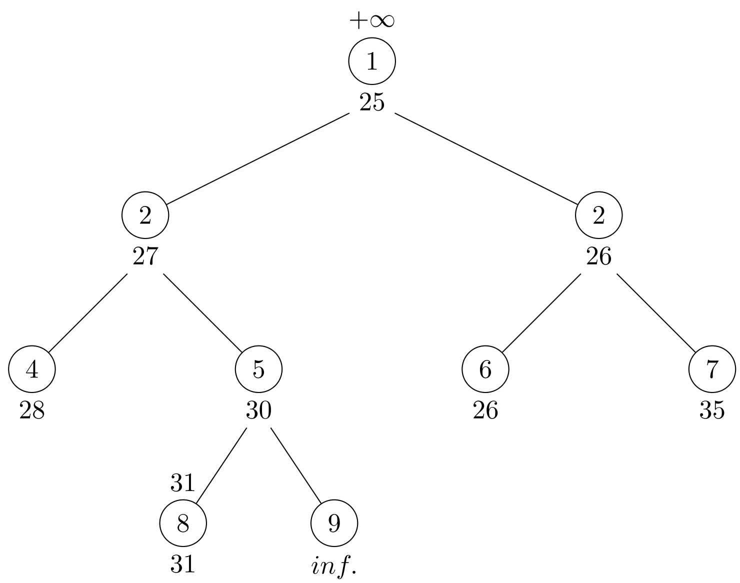
15. Consider the following ILP problem

$$\begin{aligned} (P) \equiv \max z(u) = {} & x_1 - x_2 \\ \text{s.t.:}\ & 2x_1 - x_2 \le 3 \\ & 6x_1 + 10x_2 \ge 15 \\ & x_1, x_2 \ge 0 \text{ and integer} \end{aligned}$$

(a) Solve the linear relaxation of (P) graphically. What can you conclude about the optimal value of (P)?

(b) Solve (P) using the branch-and-bound algorithm.

16. Consider the following enumeration tree for a minimization problem, where the lower bounds for the subproblems associated with each node are shown, together with upper bounds for some of the subproblems.



(a) State the best lower and upper bounds for the optimal value of the problem, as well as an upper bound on the deviation of the best solution from the optimal value.

(b) Which nodes of the tree could be fathomed, and which should be branched further?

17. Solve the following ILP problems using the branch-and-bound algorithm. Select for branching the problem with the best upper/lower bound (in case of a tie, select the most recently created problem), and the fractional variable closest to 0.5 (in case of a tie, select the variable with the largest coefficient in the objective function).

(a)

$$\begin{aligned} \max z = {} & 3x_1 + 2x_2 \\ \text{s.t.: } & 2x_1 + 5x_2 \leq 25 \\ & x_1 + x_2 \leq 7 \\ & x_1, x_2 \geq 0 \text{ and integer} \end{aligned}$$

(b)

$$\begin{aligned} \min z = {} & -3x_1 + x_2 \\ \text{s.t.: } & 16x_1 - 4x_2 \leq 13 \\ & -3x_1 - 2x_2 \leq -8 \\ & x_1, x_2 \geq 0 \text{ and integer} \end{aligned}$$

18. Consider the following integer linear programming problem:

$$\begin{aligned} \max\ & x_1 + 10x_2 \\ \text{s.t.: } & 2x_1 + x_2 \leq 4 \\ & x_1 + 2x_2 \leq 3 \\ & x_1, x_2 \geq 0 \text{ and integer} \end{aligned}$$

State which of the following inequalities are valid for the given problem and, among the valid inequalities, which are cutting planes for the optimal solution of the linear relaxation.

i. $3x_1 + 4x_2 \leq 6$

ii. $-2x_1 + 2x_2 \leq 1$

iii. $-x_1 + 2x_2 \leq 3$

iv. $x_1 + 3x_2 \leq 4$

19. Solve the following ILP problems using Gomory's cutting-plane algorithm. Graphically represent the Gomory cuts obtained for each problem.

(a)

$$\begin{aligned} \max\ & x_1 + 10x_2 \\ \text{s.t.: } & 2x_1 + x_2 \leq 4 \\ & x_1 + 2x_2 \leq 3 \\ & x_1, x_2 \geq 0 \text{ and integer} \end{aligned}$$

(b)

$$\begin{aligned} \min\ & x_1 + x_2 \\ \text{s.t.: } & 2x_1 + x_2 \geq 1 \\ & x_1 + 2x_2 \geq 1 \\ & x_1, x_2 \in \{0, 1\} \end{aligned}$$

20. Consider the following ILP in which x_2 and x_3 are the basic variables in the optimal solution of its linear relaxation.

$$\begin{aligned} \min z &= 8x_1 + 2x_2 + 4x_3 \\ \text{s.t.: } & 9x_1 + 4x_2 + 5x_3 \geq 27 \\ & 2x_1 + x_2 + 5x_3 \geq 16 \\ & x_1, x_2, x_3 \geq 0 \text{ and integer} \end{aligned}$$

Obtain the optimal value of the linear relaxation after introducing a Gomory cut. To generate the Gomory cut, use the constraint associated with the basic variable with the smallest index. Compare the value obtained with the value you would obtain by solving the given ILP. Without solving it, explain whether another cut would need to be introduced.

21. Consider the following integer linear programming problem:

$$\begin{aligned} \max\ & x_1 + 10x_2 \\ \text{s.t.: } & 2x_1 + x_2 \leq 4 \\ & x_1 + 2x_2 \leq 3 \\ & x_1, x_2 \geq 0 \text{ and integer} \end{aligned}$$

Using the Chvátal-Gomory procedure, obtain the valid inequalities $x_2 \leq 1$ and $x_1 + x_2 \leq 2$.

22. Consider the feasible region $P \subset \mathbb{R}^2_+$ defined by the following linear constraints on integer variables $x_1, x_2 \in \mathbb{Z}_+$:

$$\begin{aligned} (1) \quad & 3x_1 + 2x_2 \leq 11 \\ (2) \quad & -x_1 + 3x_2 \leq 6 \end{aligned}$$

(a) (2 points) Show that the fractional point $(x_1, x_2) = \left(\frac{21}{11}, \frac{29}{11}\right)$ is the optimal vertex of the linear relaxation of P.

(b) (3 points) Use the Chvátal rounding operator, $\lfloor \cdot \rfloor$, with multipliers $\lambda_1 = \frac{1}{11}$ and $\lambda_2 = \frac{3}{11}$, to derive a valid cutting plane that eliminates the fractional solution above.

(c) (2 points) Verify analytically that the resulting cut separates the fractional solution without eliminating any feasible integer point of P.

23. **[Cutting Planes from the Simplex Tableau]**

After solving the linear relaxation of a Pure Integer Linear Programming (ILP) problem using the Simplex algorithm, the following partial optimal tableau was obtained, with decision variables $x_1, x_2, x_3 \in \mathbb{Z}_+$ and slack variables $x_4, x_5 \geq 0$:

Basic Var.	x_1	x_2	x_3	x_4	x_5	RHS
x_1	1	0	0	$7/4$	$-1/2$	$9/4$
x_2	0	1	0	$-1/4$	$3/4$	$7/4$
x_3	0	0	1	$1/2$	$1/4$	$5/2$

(a) (3 points) Derive the equation of the Gomory constraint associated with the row of basic variable x_1.

(b) (3 points) Rewrite the cut obtained in the previous part using only the original decision variables (x_1, x_2, x_3), knowing that the slack variables were introduced from the original constraints:

$$\begin{aligned} x_1 + 2x_2 + x_3 &\leq 8 \quad (x_4) \\ 2x_1 - x_2 + 2x_3 &\leq 10 \quad (x_5) \end{aligned}$$

24. **[Successive Cut Generation and Convex Hull]**

Consider the nonconvex set of integer points $S = \{x \in \mathbb{Z}^2_+ : 4x_1 + 5x_2 \leq 18,\ x_1, x_2 \geq 0\}$.

(a) (3 points) Determine the first Chvátal-Gomory cut by applying the multiplier $\lambda = 1/4$ to the original constraint.

(b) (4 points) Show that, by iterating the process of adding Gomory cuts, the convex hull (*convex hull*) $P_I = \text{conv}(S)$ is reached in at most two cutting steps.

25. Find two cover cuts for the following constraint of a knapsack problem

$$7x_1 + 6x_2 + 3x_3 + 5x_4 \leq 12.$$

26. Consider the set

$$X = \{x_1, x_2, x_3, x_4 \in \mathbb{R}^+, y_1, y_2, y_3, y_4 \in \{0, 1\} : x_1+x_2+x_3+x_4 \geq 36, x_1 \leq 20y_1, x_2 \leq 10y_2, x_3 \leq 10y_3, x_4 \leq 8y_4\}.$$

Derive a valid constraint for X that is a knapsack constraint.

27. Determine valid inequalities for the set of points $(x_1, \dots, x_m, y)$ defined by

$$\sum_{i=1}^{m} x_i \leq my,\ 0 \leq x_i \leq 1,\ y \in \{0, 1\}.$$