Mathematics 2

Lisbon School of Economics and Management
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Lecture 5a: Limits and continuity (Functions in several variables)

Before studying limits in several variables, first we must have an understanding of functions whose input is a **point in** $\mathbb{R}^d$ and whose output may be either a **real number** or a **vector**.

Namely, we must be able to answer the following:

- Where is a function defined?
- How can we represent and interpret its values?
- Is there a geometric meaning behind it?

Definition 1

A ***scalar-valued function of several variables*** is a map

$$f : D_f \subseteq \mathbb{R}^d \longrightarrow \mathbb{R},$$
$$x \longmapsto f(x),$$

where D_f is the ***domain*** of f.

Intuitively: The domain consists of a collection of points (inputs) for which the expression defining the function is meaningful.

Remark 2

The domain of a function f can also be found in literature as $D(f)$ or even $\mathrm{dom}\, f$.

In $d = 2$ or $d = 3$, we will write

$$f : D_f \subseteq \mathbb{R}^2 \longrightarrow \mathbb{R}, \quad (x, y) \longmapsto f(x, y), \qquad \text{or} \qquad f : D_f \subseteq \mathbb{R}^3 \longrightarrow \mathbb{R}, \quad (x, y, z) \longmapsto f(x, y, z),$$

respectively.

Example 3

Let

$$f(x, y) = x^2 + y^2.$$

Since taking the square of any number always produces another number (namely, a non-negative one), it follows that

$$D_f = \mathbb{R}^2.$$

Example 4

Let

$$f(x,y) = \frac{\sqrt{1-x^2-y^2}}{x-y}.$$

Observe that the square root requires $1 - x^2 - y^2 \geq 0$, while the denominator must be nonzero. Hence

$$\begin{aligned} D_f =& \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1,\ x \neq y\} \\ =& B_1((0,0)) \cap \{(x,y) : x - y \neq 0\}. \end{aligned}$$

Geometrically, this corresponds to a filled disk (including its boundary) centered at the origin, with the diagonal of $x = y$ removed.

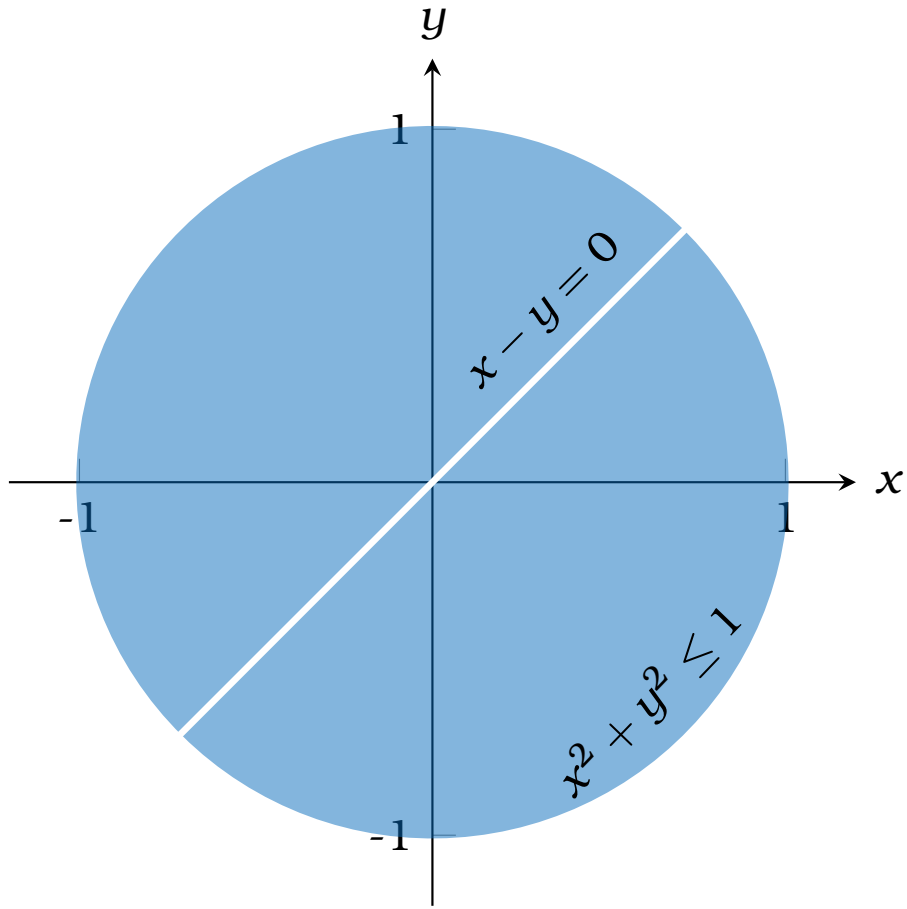


Figure: Plot of D_f, Example 4.

Example 5

Determine the domain of

$$f(x, y) = \ln(4 - x^2 - y^2).$$

Since the logarithm requires a strictly positive argument,

$$4 - x^2 - y^2 > 0 \qquad \Longrightarrow \qquad x^2 + y^2 < 4.$$

Therefore

$$D_f = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 < 4\}.$$

The domain is the open disc of radius 2 centered at the origin.

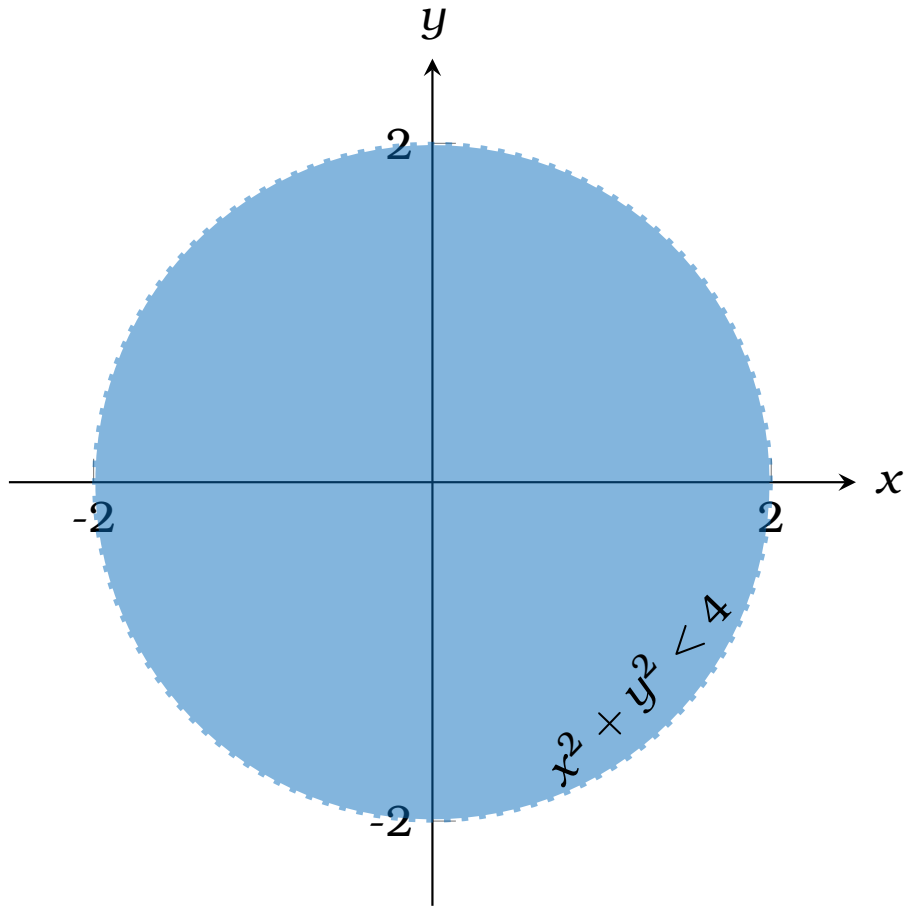


Figure: Plot of D_f, Example 5.

Example 6

Let

$$f(x, y) = \sqrt{4x^2 + y^2 - 36}\,:$$

Observe that the radical (term within the square root) must be non-negative; thus

$$D_f = \{(x, y) \in \mathbb{R}^2 : 4x^2 + y^2 \geq 36\}$$

I.e., the plane with an ellipse-shaped hole at the origin.

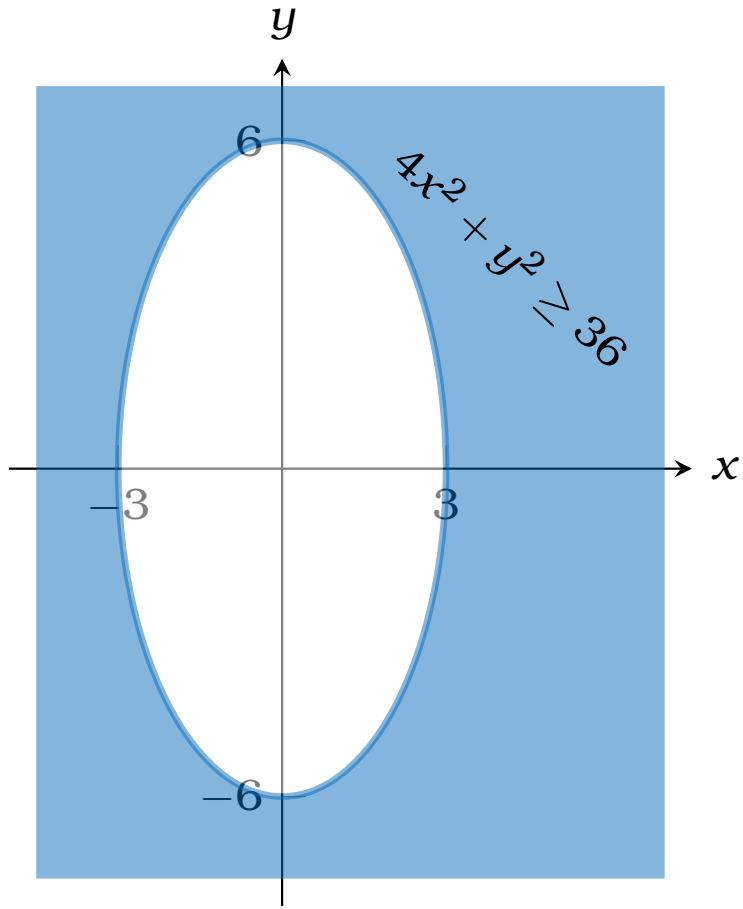


Figure: Plot of D_f, Example 6.

Example 7

Let

$$f(x, y) = \frac{1}{x^2 - y^2 - 4}.$$

Since the denominator cannot vanish, we have that

$$D_f = \{(x, y) \in \mathbb{R}^2 : x^2 - y^2 \neq 4\}.$$

This corresponds to the plane with an hyperbola removed; namely, an horizontal hyperbola with vertices at $(-2, 0)$ and $(2, 0)$.

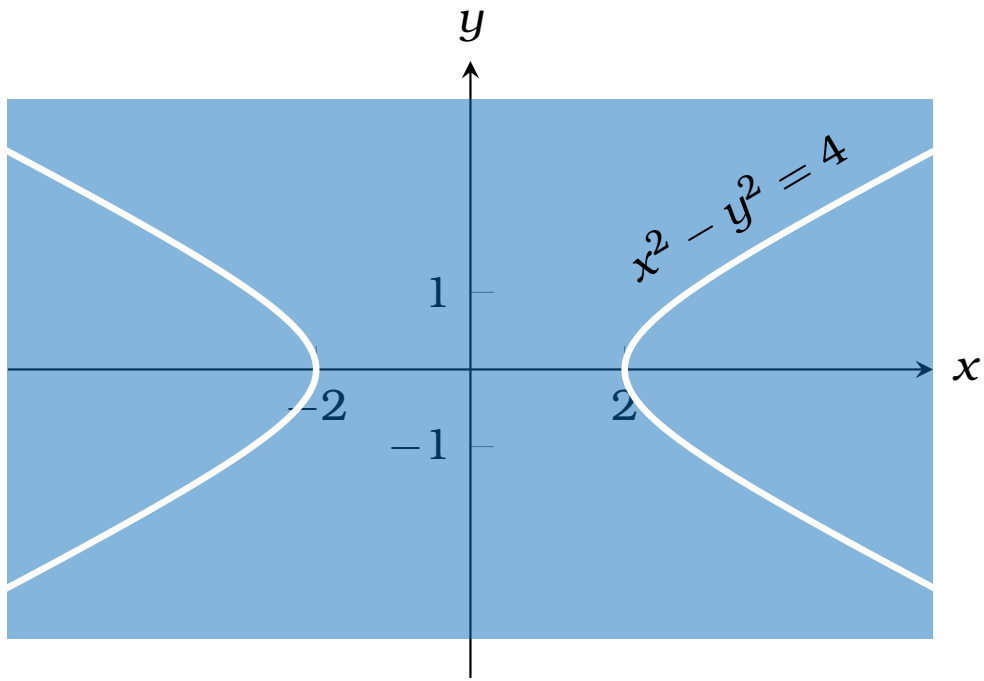


Figure: Plot of D_f, Example 7.

The “recipe”

In order to determine the domain of a function, one must first find every condition required by each component of the formula, then take their intersection.

The goal is to avoid indeterminate forms such as

$$\frac{0}{0}, \qquad \pm\infty, \qquad \sqrt{\text{negative number}}, \qquad \text{etc.}$$

Observe: A domain need not be connected and need not be open or closed.

Definition 8

For a given function $f : D_f \subseteq \mathbb{R}^d \to \mathbb{R}$, the ***graph of*** f is defined as the set

$$\text{Graph}(f) = \{(x, f(x)) \in \mathbb{R}^{d+1} : x \in D_f\}.$$

Definition 9

Let $f : D_f \subseteq \mathbb{R}^d \to \mathbb{R}$ and let $a \in \mathbb{R}$. The ***level set of*** f ***at level*** a is the subset of the domain characterized by the equality $f(x) = a$:

$$C_a = \{x \in D_f : f(x) = a\}.$$

Following our geometric intuition, the graph of a one variable function is simply its **plot on the $\mathbb{R}^2$ plane**; in the case of functions of two variables, their graphs correspond to **surfaces in $\mathbb{R}^3$**.

Consequently, the level sets of a function can be thought of as **horizontal slices** (akin to a tomography) **of the graph of f**.

Example 3 (cont.)

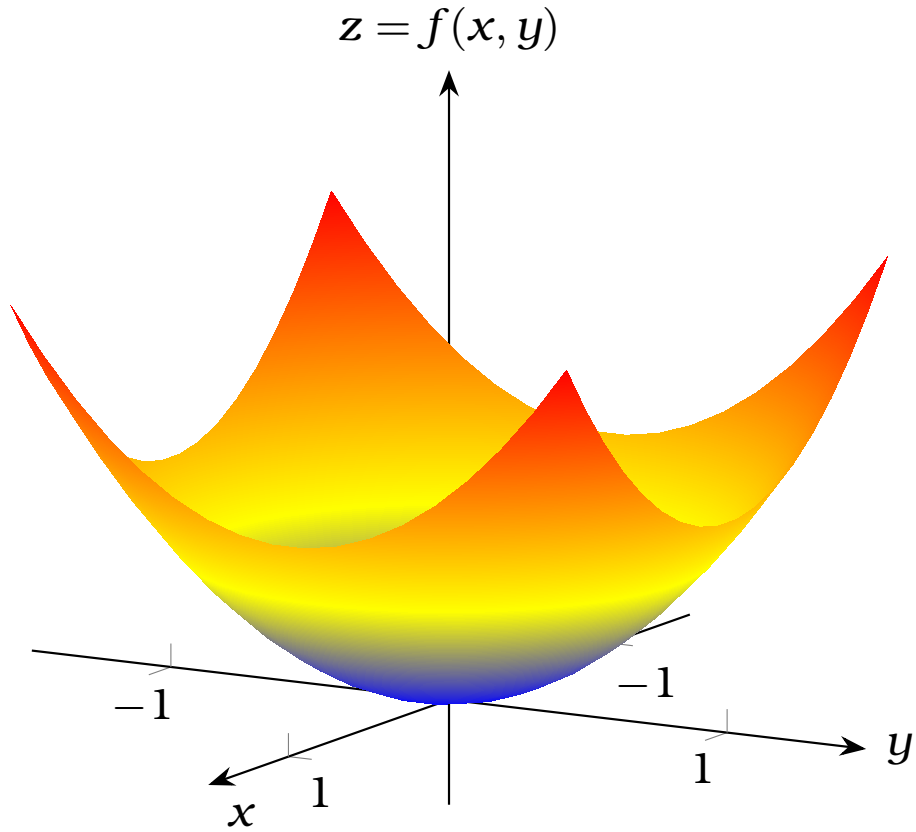


Figure: Graph of $f(x, y) = x^2 + y^2$

Example 3 (cont.)

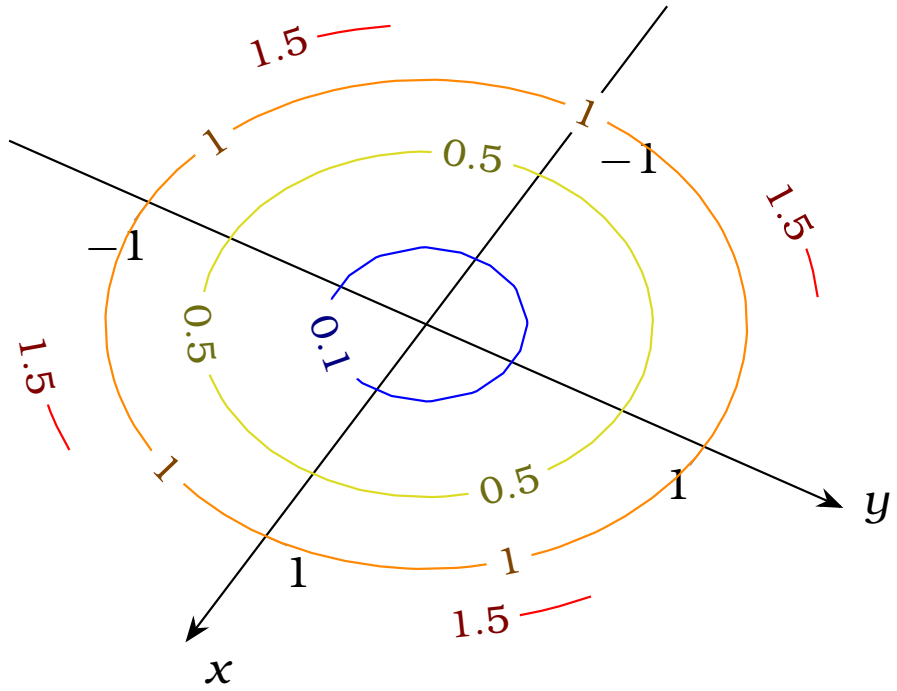


Figure: Different level curves of $f(x, y) = x^2 + y^2$

Example 4 (cont.)

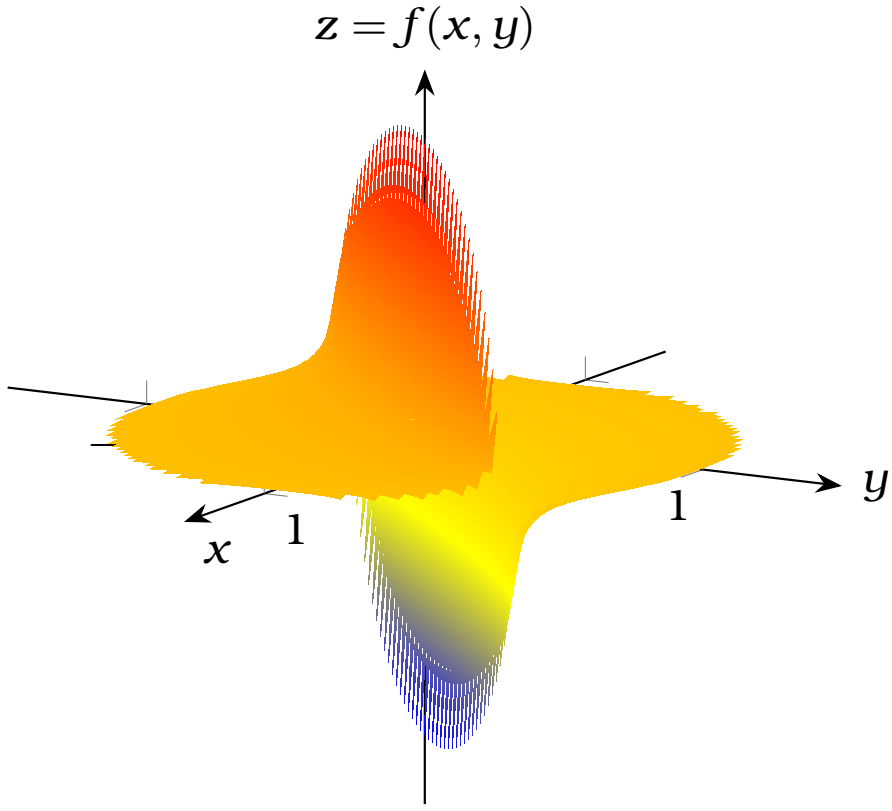


Figure: Graph of $f(x, y) = \frac{\sqrt{1-x^2-y^2}}{x-y}$

Example 4 (cont.)

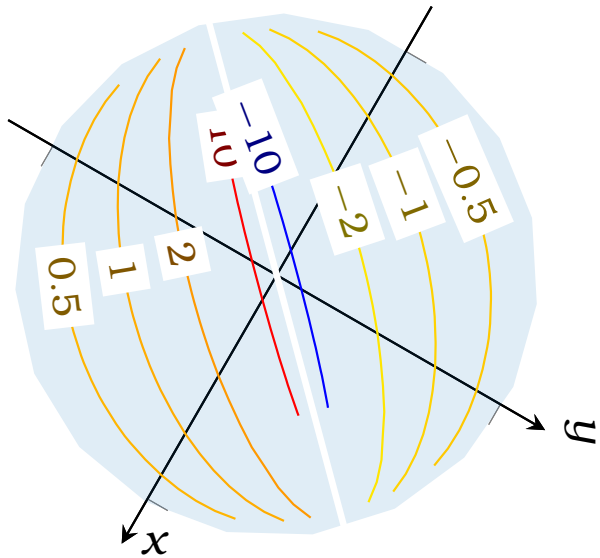


Figure: Different level curves of $f(x, y) = \frac{\sqrt{1-x^2-y^2}}{x-y}$.

Example 5 (cont.)

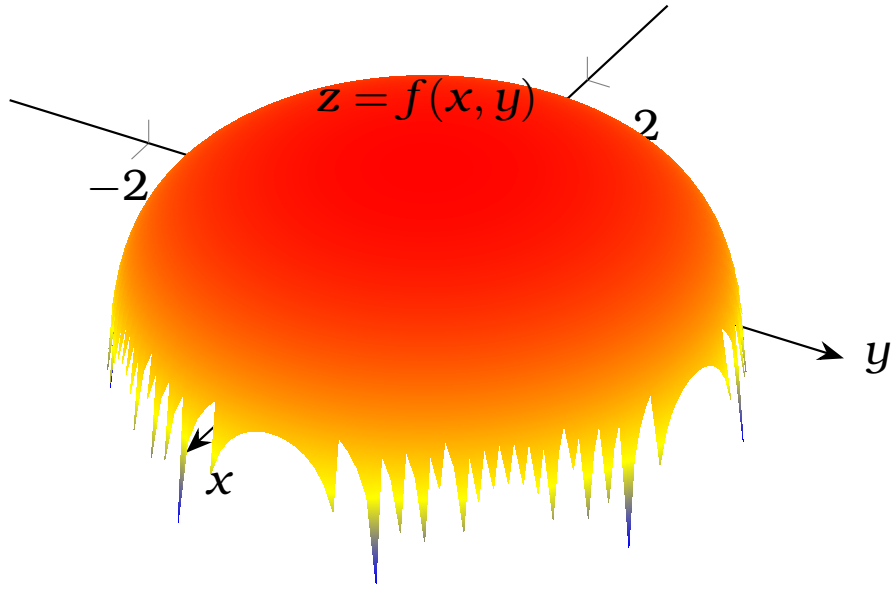


Figure: Graph of $f(x, y) = \ln(4 - x^2 - y^2)$

Example 5 (cont.)

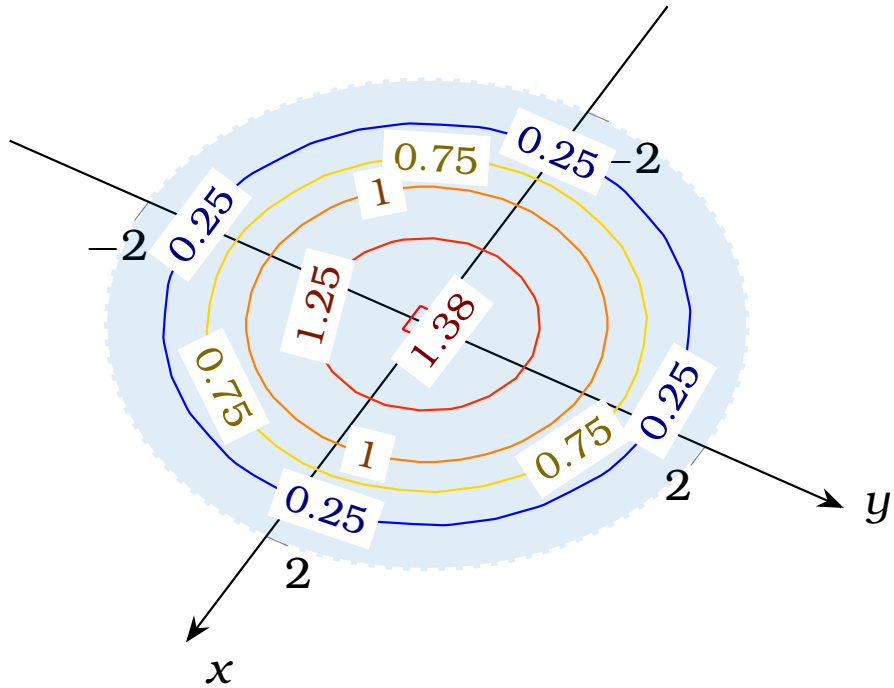


Figure: Graph of $f(x, y) = \ln(4 - x^2 - y^2)$

Example 6 (cont.)

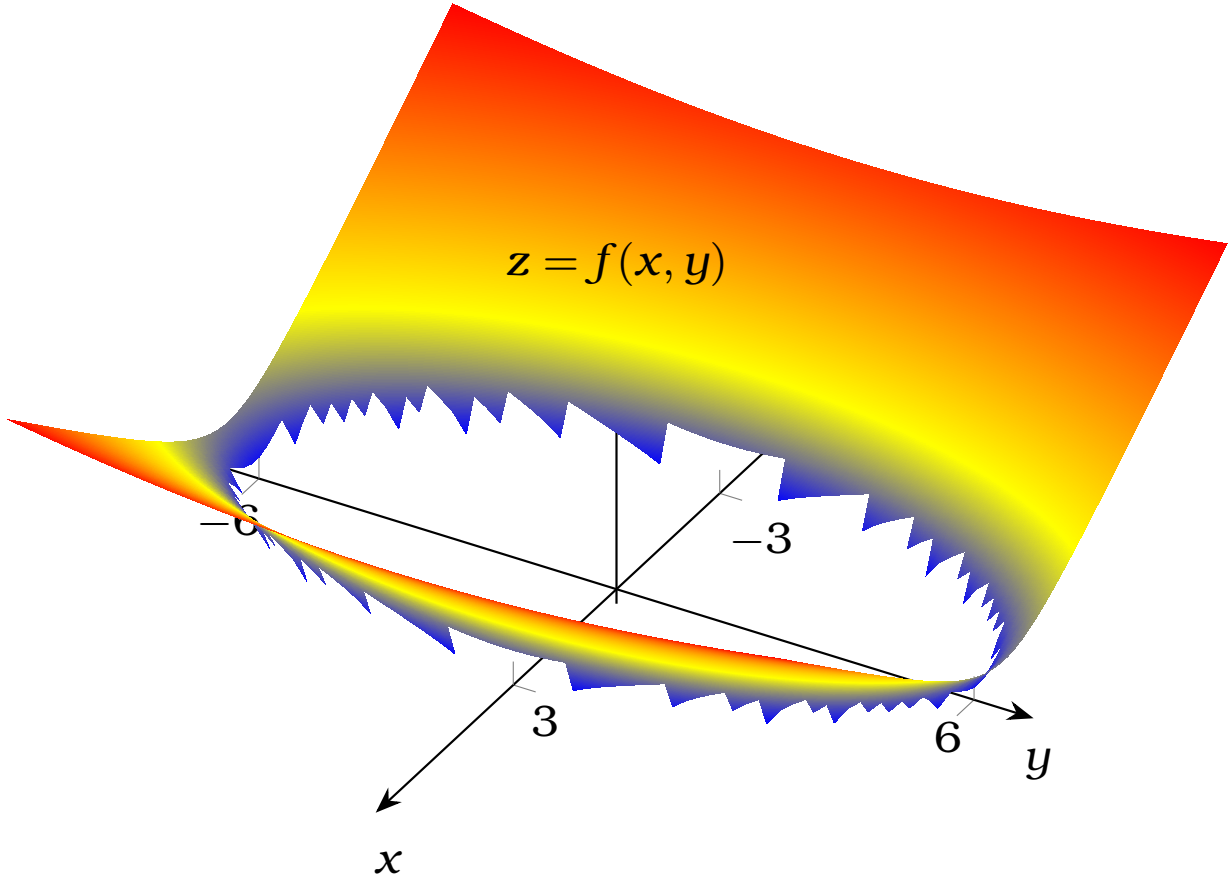


Figure: Graph of $f(x, y) = \sqrt{4x^2 + y^2 - 36}$

Example 6 (cont.)

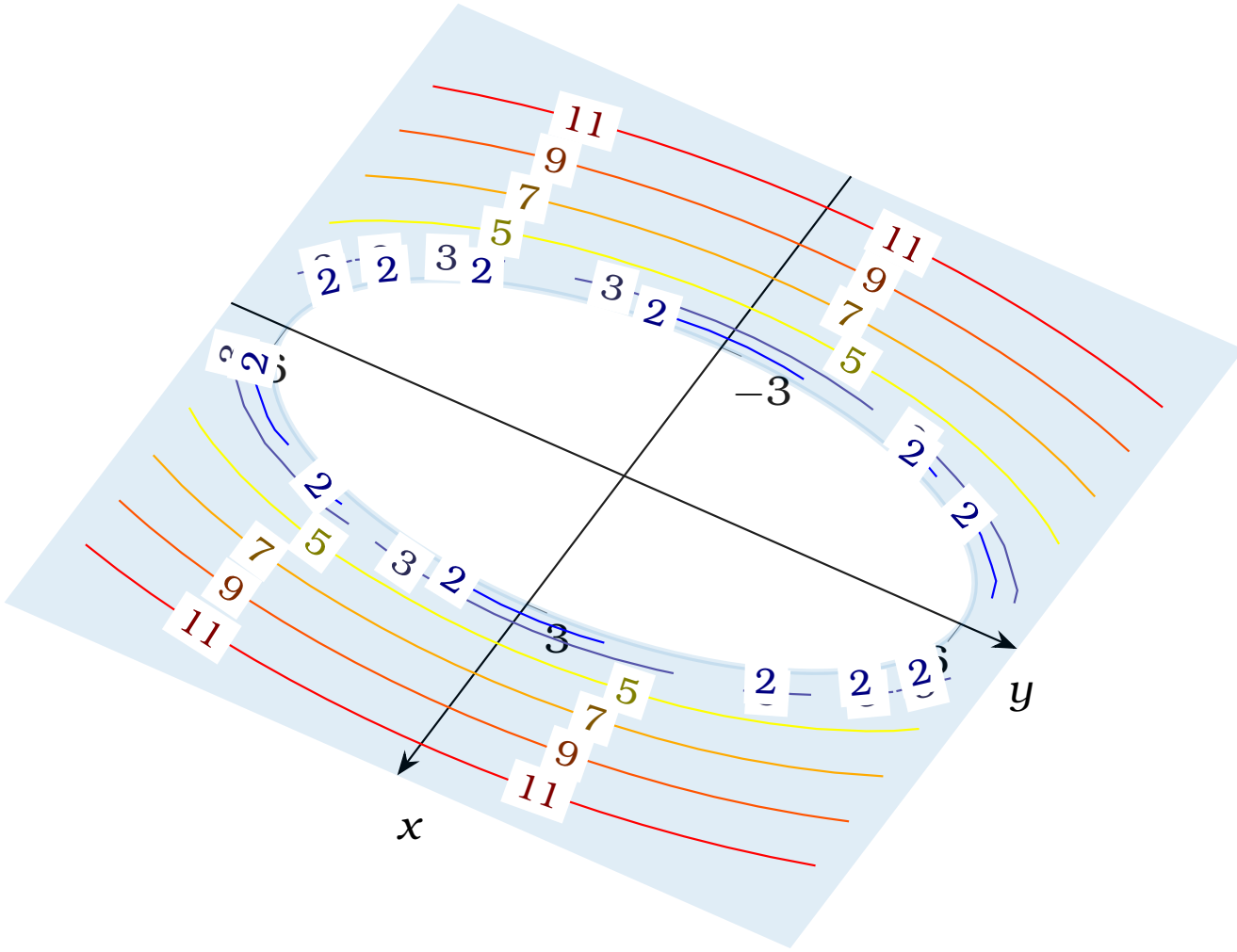


Figure: Different level sets of $f(x, y) = \sqrt{4x^2 + y^2 - 36}$

Example 7 (cont.)

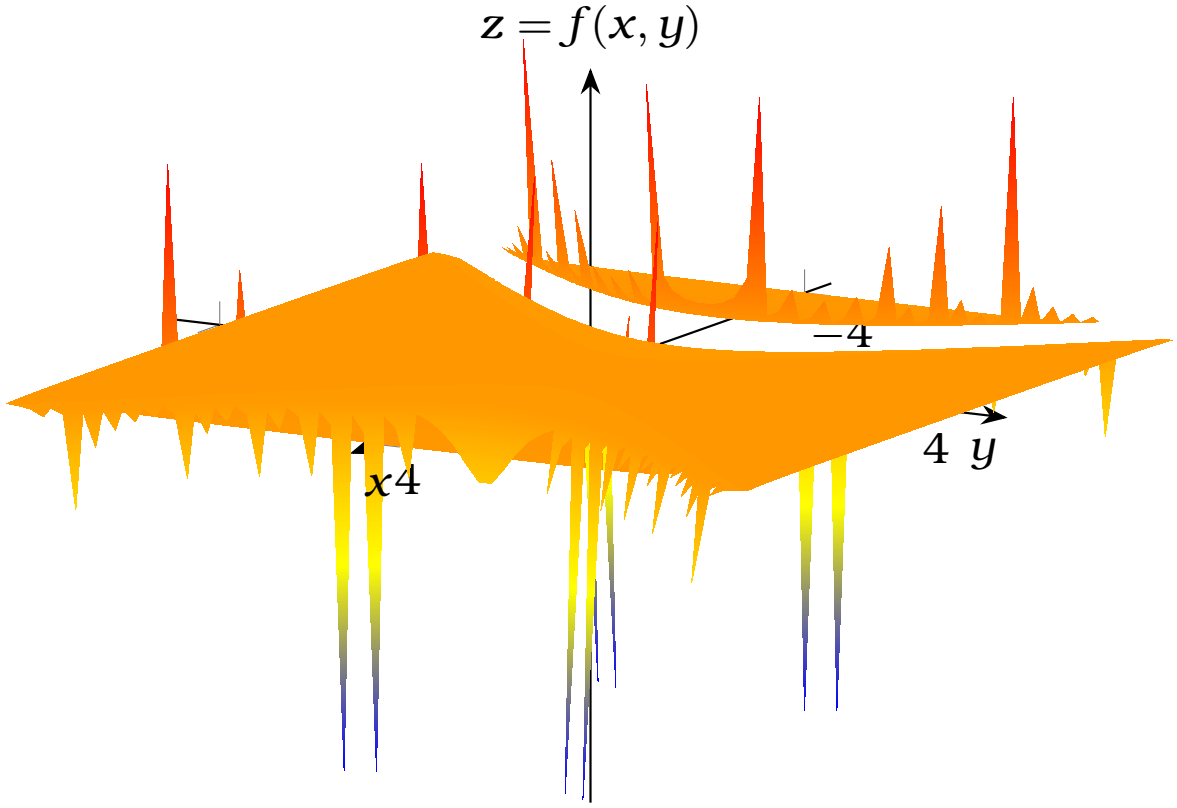


Figure: Graph of $f(x, y) = \frac{1}{x^2 - y^2 - 4}$

Example 7 (cont.)

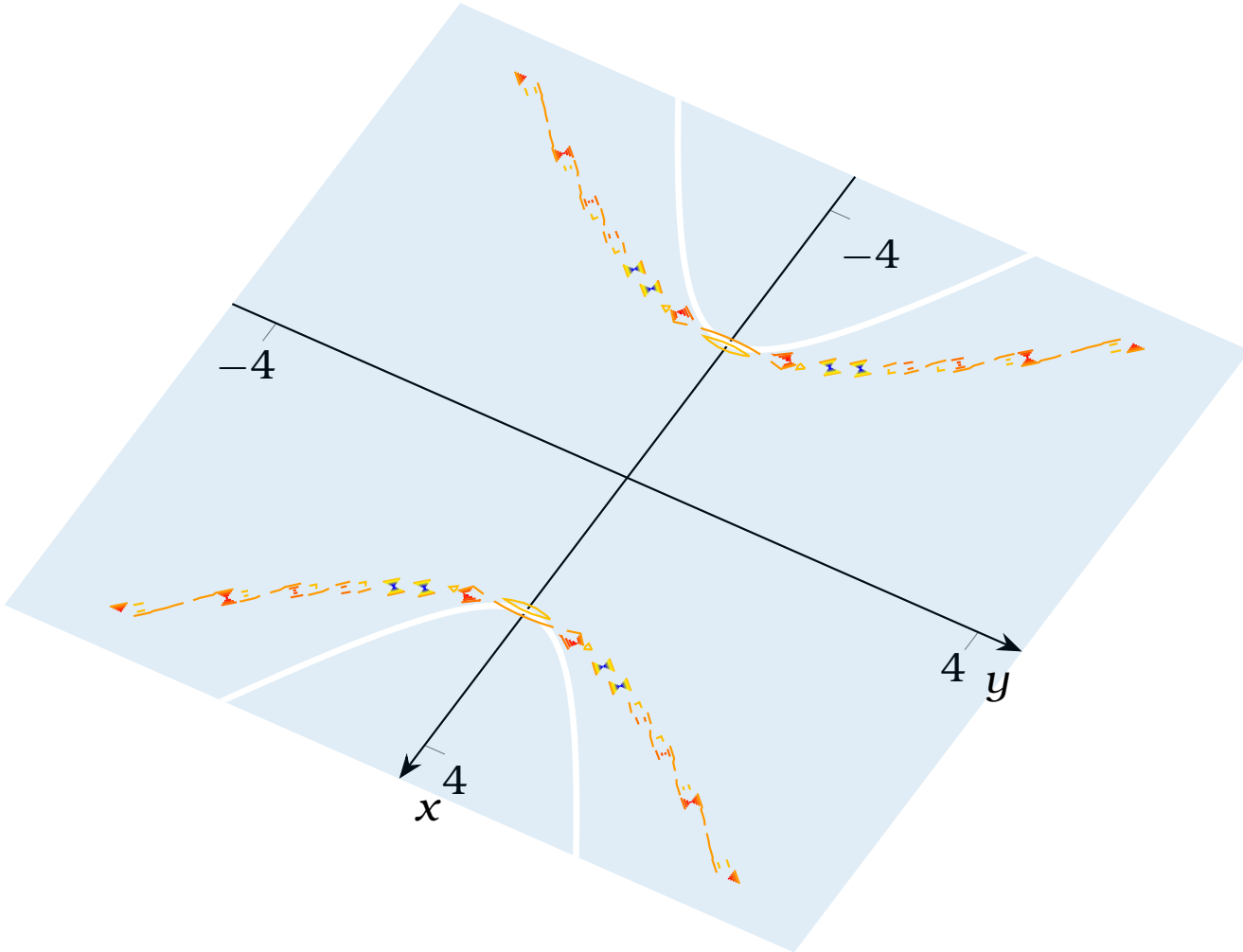


Figure: Different level sets of $f(x, y) = \frac{1}{x^2 - y^2 - 4}$

Definition 10

A ***vector-valued function*** is a map

$$F : D_F \subseteq \mathbb{R}^d \longrightarrow \mathbb{R}^n,$$
$$x \longmapsto F(x).$$

Moreover, its value can be written in coordinates as

$$F(x) = \begin{bmatrix} f_1(x) \\ f_2(x) \\ \vdots \\ f_n(x) \end{bmatrix},$$

where each individual $f_i : D_{f_i} \subseteq \mathbb{R}^d \to \mathbb{R}$, $i = 1, \ldots, n$, is a real-valued function.

Example 11

Let $F : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by

$$F(x, y) = \begin{bmatrix} x + y \\ x - y \end{bmatrix}.$$

The coordinate functions are

$$f_1(x, y) = x + y \qquad \text{and} \qquad f_2(x, y) = x - y.$$

For example,

$$F(2, 1) = \begin{bmatrix} 3 \\ 1 \end{bmatrix}.$$

Example 12

Let

$$F(x, y) = \begin{bmatrix} x + y \\ \ln(xy) \\ \sqrt{x - y} \end{bmatrix}.$$

For every component to be defined, we require both $xy > 0$ and $x - y \geq 0$; the domain is the intersection of these conditions.

Suggested exercise: What is the plot of the domain D_F? Can its graph be plotted? How?

Addendum: Classic functions and geometric objects

- Straight line of slope m crossing the y-axis at $y = b$: $y = mx + b$.
- Parabola with vertex (h, k): $y = a(x - h)^2 + k$.
- Circle centered at (h, k), radius r:

$$(x - h)^2 + (y - k)^2 = r^2.$$

- Ellipse centered at (h, k):

$$\left(\frac{x - h}{a}\right)^2 + \left(\frac{y - k}{b}\right)^2 = 1.$$

The order of a and b determines the orientation: if $a > b$, the ellipse is wider in the x–axis; if $a < b$, the ellipse is wider in the y–axis. If $a = b$, the ellipse collapses to a circle of radius $r = a = b$.