# Mathematics 2

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# Lecture 5b: Limits and continuity

Let $D_f \subseteq \mathbb{R}^d$ and let $a \in \mathrm{cl}(D_d)$. We study the behaviour of a function

$$f : D_f \subseteq \mathbb{R}^d \longrightarrow \mathbb{R}$$

near the point $a$.

In several variables, the limit must be the same regardless of the way the input approaches the point.

## Definition 1 (Limit according to Heine[1])

Let $D_f \subseteq \mathbb{R}^d$, $a \in \mathrm{cl}(D_f)$, and $\ell \in \mathbb{R}$. We say that

$$\lim_{x \to a} f(x) = \ell$$

if for every sequence $(x(n))_{n \geq 1}$ in $D_f \setminus \{a\}$ such that $x(n) \to a$, we have

$$\lim_{n \to \infty} f(x(n)) = \ell.$$

[1] Eduard Heine. "Die Elemente der Functionenlehre". ger. In: *Journal für die reine und angewandte Mathematik* 74 (1872), pp. 172–188

## Definition 2 (Limit according to Cauchy[2])

Let $D_f \subseteq \mathbb{R}^d$, $a \in \mathrm{cl}(D_f)$, and $\ell \in \mathbb{R}$. We say that

$$\lim_{x \to a} f(x) = \ell$$

if for every $\varepsilon > 0$ there exists $\delta > 0$ such that, for all $x \in D_f$,

$$0 < \|x - a\| < \delta \qquad \Longrightarrow \qquad |f(x) - \ell| < \varepsilon.$$

[2] Augustin-Louis Cauchy. *Analyse algébrique*. Vol. I. Analyse algébrique Cours d'analyse de l'École Royale Polytechnique. Paris: De l'Imprimerie Royale, 1821

See also the work of Weierstrass [3] on the rigorous, modern definition of limit.

**Observe:** In both Definition 1 and Definition 2, the limit of a function depends on values of $f$ near $a$, **not necessarily on the value of $f$ at $a$.**

In fact, the **function $f$ need not be defined at $a$ for its limit to exist**.

## Theorem 3

*The Heine and Cauchy definitions of the limit are equivalent.*

## Proof.

Sketch presented in the whiteboard. □

## Why do we need more than one definition?

In practice, each definition is sometimes better suited for a different purpose:

- With Heines's (Definition 1), we can disprove the existence of a limit by finding two sequences converging to different values.
- With Cauchy's (Definition 2), we can prove the existence of a limit (and find its value) by using estimates.

With the notion of limit established, we now present a collection of properties that will prove useful when dealing with limits.

## Proposition 4 (Uniqueness of the limit)

*If the limit*

$$\lim_{x \to a} f(x)$$

*exists (i.e. its value equals a real number* $\ell \in (-\infty, +\infty)$*), then its value is unique.*

## Proposition 5 (Composition of limits)

*Let* $f : D_f \subseteq \mathbb{R}^d \to \mathbb{R}^n$ *and* $g : D_g \subseteq \mathbb{R}^n \to \mathbb{R}$ *be two functions such that*

$$f(x) \in D_g \qquad \text{for all } x \in D_f.$$

*If it holds that*

$$\lim_{x \to a} f(x) = b \qquad \text{and} \qquad \lim_{y \to b} g(y) = \ell$$

*for some* $b \in \mathrm{cl}(D_g)$*, then*

$$\lim_{x \to a} g(f(x)) = \ell.$$

## Proposition 6 (Algebraic properties of limits)

*Suppose the limits*

$$\lim_{x\to a} f(x) = \ell \qquad \textit{and} \qquad \lim_{x\to a} g(x) = m$$

*exist. Then*

$$\lim_{x\to a} \big(f(x) + g(x)\big) = \big(\lim_{x\to a} f(x)\big) + \big(\lim_{x\to a} g(x)\big) = \ell + m,$$
$$\lim_{x\to a} \big(f(x)\, g(x)\big) = \big(\lim_{x\to a} f(x)\big)\,\big(\lim_{x\to a} g(x)\big) = \ell\, m.$$

*If, in addition,* $m \neq 0$,

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)} = \frac{\ell}{m}.$$

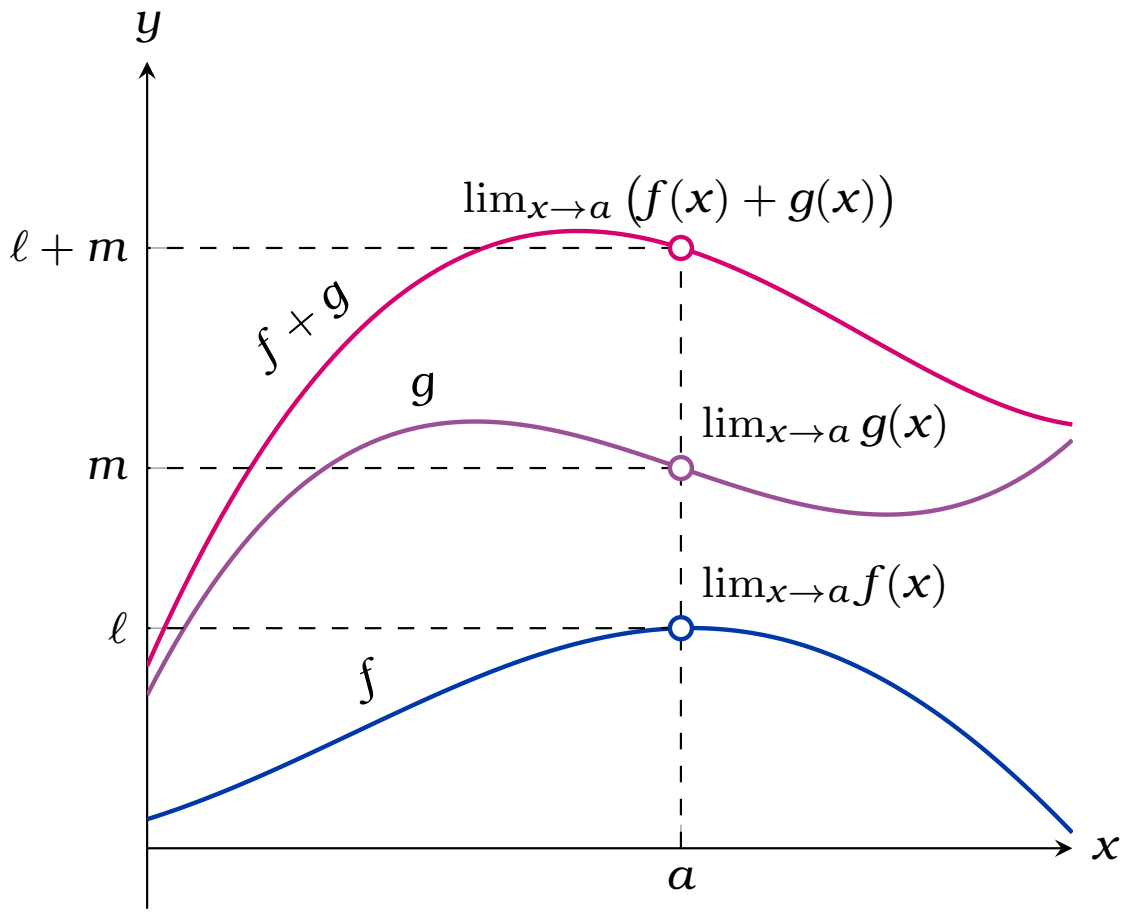



Figure: Plot illustrating the sum of limits. Product and division follow an similar procedure.

## Theorem 7 (Squeeze theorem)

*Let $f, g, h$ real-valued functions such that*

$$f(x) \leq g(x) \leq h(x),$$

*for all $x$ in some neighborhood $B_\varepsilon(a)$ of $a \in \mathrm{cl}(D_f) \cap \mathrm{cl}(D_g) \cap \mathrm{cl}(D_h)$.*

*If it holds that the limits of $g$ and $h$ as $x \to a$ both exist and coincide, i.e.*

$$\lim_{x \to a} f(x) = \lim_{x \to a} h(x) = \ell,$$

*then*

$$\lim_{x \to a} g(x) = \ell.$$

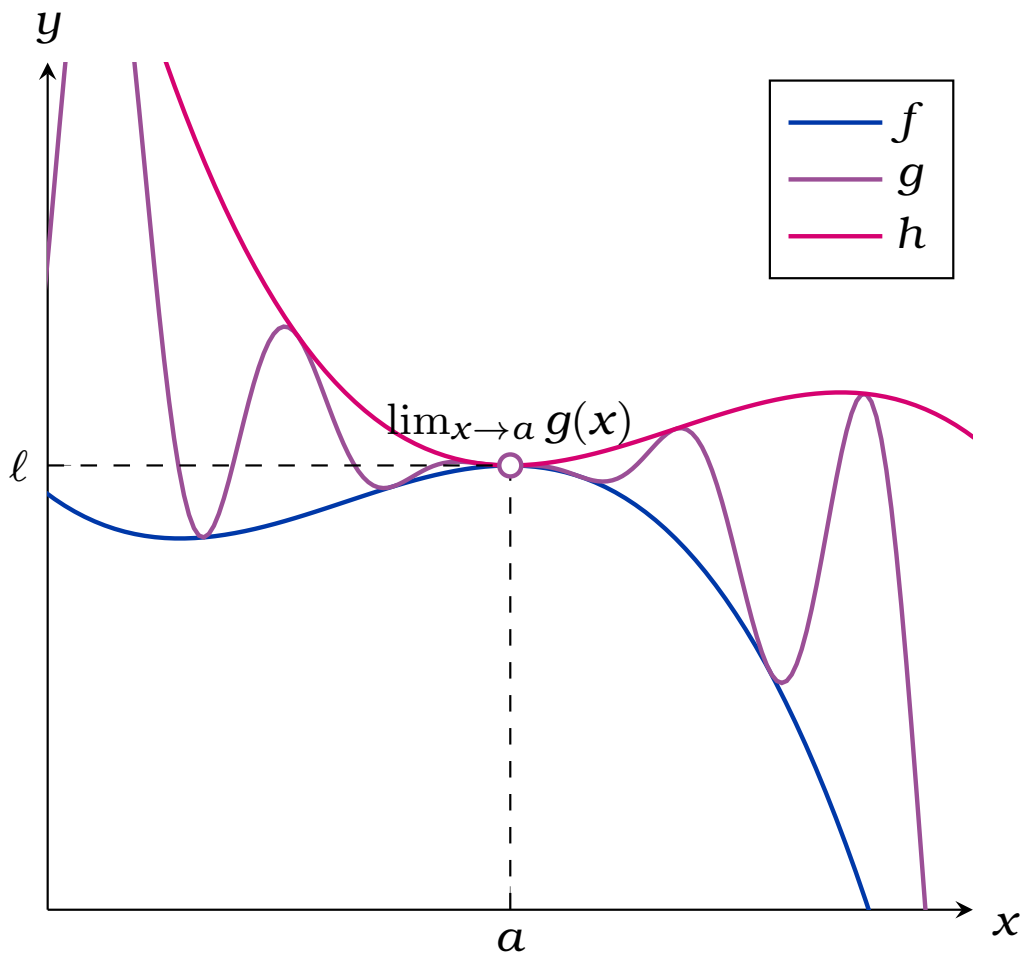



Figure: Plot illustrating the squeeze theorem.

## Example 8

We are interested in finding wether or not the limit

$$\lim_{(x,y)\to(0,0)} \frac{xy}{\sqrt{x^2+y^2}}$$

exists.

From the binomial product, we know that

$$(x+y)^2 = x^2+2xy+y^2 \quad \text{and} \quad (x-y)^2 = x^2-2xy+y^2$$

for any two real numbers $x, y$.

## Example 8 (cont.)

Since $(x+y)^2 \geq 0$ and $(x-y)^2 \geq 0$, by adding and reordering some terms we obtain that

$$-x^2 - y^2 \leq 2xy \leq x^2 + y^2. \tag{1}$$

Now, recall that the definition of limit does not require us to evaluate the function at the limit, in this case at $(0,0)$. Thus, we can multiply the terms in (1) by

$$\frac{1}{2\sqrt{x^2+y^2}} > 0$$

without reversing the order of the inequality. Indeed: since $(x,y) \neq (0,0)$, the fraction is well defined.

## Example 8 (cont.)

As a result:

$$\underbrace{-\frac{x^2+y^2}{2\sqrt{x^2+y^2}}}_{=-\frac{1}{2}\sqrt{x^2+y^2}} \leq \frac{xy}{\sqrt{x^2+y^2}} \leq \underbrace{\frac{x^2+y^2}{2\sqrt{x^2+y^2}}}_{=\frac{1}{2}\sqrt{x^2+y^2}}.$$

Lastly, observe both bounds tend to 0 as $(x, y) \to (0, 0)$. By the squeeze theorem, we conclude that the limit exists and equals zero:

$$\boxed{\lim_{(x,y)\to(0,0)} \frac{xy}{\sqrt{x^2+y^2}} = 0.}$$

## Example 9

Let $f : \mathbb{R}^2 \setminus \{(0,0)\} \to \mathbb{R}$ be defined as

$$f(x,y) = \frac{x^2 y}{x^2 + y^2}.$$

Using the squeeze theorem, we can verify that the limit

$$\lim_{(x,y)\to(0,0)} f(x,y).$$

exists.

## Proof.

**Suggested exercise.**[3] □

[3] *Hint:* observe that $0 \leq x^2 \leq x^2 + y^2$ and $-\sqrt{x^2+y^2} \leq y \leq \sqrt{x^2+y^2}$.

## Definition 10 (Limit along a subset)

Let $f$ be a function, and let $A \subseteq D_f$ be a given subset of the domain. Let $a \in \mathrm{cl}(D_f) \cap \mathrm{cl}(A)$, and $\ell \in \mathbb{R}$. We say that $\ell$ ***is the limit of*** $f$ ***as*** $x \to a$ ***along the subset*** $A$, denoted by

$$\lim_{\substack{x \to a \\ x \in A}} f(x) = \ell$$

if for every sequence $(x(n))_{n \geq 1}$ in $A \setminus \{a\}$ such that $x(n) \to a$, we have

$$\lim_{n \to \infty} f(x(n)) = \ell.$$

**Important:** Agreement along a few subsets does not by itself prove that the full limit exists. For the limit to exists, *all of the limits along any given partition of the domain must coincide*:

$$\lim_{x\to a} f(x) = \ell \qquad \Longleftrightarrow \qquad \lim_{\substack{x\to a \\ x\in A_i}} f(x) = \ell, \text{ for all } i = 1, \ldots, n,$$

where $A_1, A_2, \ldots, A_n$ are pairwise disjoint subsets of $D_f \setminus \{a\}$ (no common elements between any pair of them, $A_i \cap A_j = \varnothing$), such that

$$A_1 \cup A_2 \cup \ldots \cup A_n = D_f \setminus \{a\}.$$

Thus, if the full limit exists, then every limit along a subset approaching $a$ must have the same value.

## Example 11

Consider the function

$$f(x,y) = \frac{x^2 y}{x^4 + y^2}$$

over the set

$$\{(x,y) \in \mathbb{R}^2 : x \neq 0,\ y \neq 0\}$$

Notice that along the line $y = x$,

$$\lim_{\substack{(x,y)\to(0,0)\\ \{(x,y):y=x\}}} f(x,y) = \lim_{\substack{(x,y)\to(0,0)\\ \{(x,y):y=x\}}} f(x,x)$$

$$= \lim_{x\to 0} \frac{x^3}{x^4 + x^2} = \lim_{x\to 0} \frac{x}{x^2 + 1} = 0.$$

## Example 11 (cont.)

However, if we take a different curve (namely, $y = x^2$) we obtain a different limit:

$$\lim_{\substack{(x,y)\to(0,0)\\ \{(x,y):y=x^2\}}} f(x,y) = \lim_{\substack{(x,y)\to(0,0)\\ \{(x,y):y=x^2\}}} f(x,x^2)$$

$$= \lim_{x\to 0} \frac{x^4}{x^4+x^4} = \frac{1}{2}.$$

Since the limits along different subsets do not coincide, the full limit does not exist.

## Definition 12 (Continuity)

Let $f : D_f \subseteq \mathbb{R}^d \to \mathbb{R}$ and let $a \in D_f$. We say that ***the function*** $f$ ***is continuous at*** $a$ if

$$\lim_{x \to a} f(x) = f(a).$$

Consequently, we say that $f$ ***is continuous over a subset*** $A \subseteq D_f$ if for every $a \in A$, it holds that $f$ is continuous at the point $a$.

In topological terms (i.e. in terms of neighborhoods and open sets), $f$ is continuous at $a$ if for every $\varepsilon > 0$ there exists $\delta > 0$ such that for any $x \in D_f$,

$$\|x - a\| < \delta \qquad \Longrightarrow \qquad |f(x) - f(a)| < \varepsilon.$$

As the idea of continuity is build upon the notion of limits, it is natural that some of the properties of the latter are inherited by the former. We now present some of them.

## Proposition 13 (Algebraic properties of continuous functions)

*Let $f : D_f \to \mathbb{R}$ and $g : D_g \to \mathbb{R}$ be two functions, and let $a \in D_f \cap D_g$. If $f$ and $g$ are continuous at $a$, then the following functions are also continuous at $a$:*

- *the sum: $(f+g)(x) = f(x) + g(x)$;*
- *the product: $(f\,g)(x) = f(x)\,g(x)$;*
- *particularly, the product by a constant: $\lambda f(x)$, for any $\lambda \in \mathbb{R}$.*

*If in addition $g(a) \neq 0$, then*

$$\frac{f(x)}{g(x)}$$

*is also continuous at $a$.*

## Proposition 14 (Composition of continuous functions)

*Let* $f : D_f \subseteq \mathbb{R}^d \to \mathbb{R}$ *and* $g : D_g \subseteq \mathbb{R} \to \mathbb{R}$ *be two functions such that*

$$f(x) \in D_g \qquad \text{for all } x \in D_f.$$

*If it holds that* $f$ *is continuous at* $a \in D_f$ *with*

$$f(a) = b,$$

*and* $g$ *is continuous at* $b \in D_g$*, then the composition*

$$(g \circ f)(x) = g(f(x))$$

*is also continuous at* $a$*.*

## Example 15

The function

$$f(x, y) = x^2 + 3xy + \sin y$$

is continuous on $\mathbb{R}^2$.

Indeed: it is a known fact that polynomial functions and the sine function are continuous. From the algebraic properties of continuous functions, the result is obtained.

## Example 16

Similarly, the function

$$f(x,y) = \frac{1}{x^2 + y^2 - 1}$$

is continuous at every point of its domain

$$D_f = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 \neq 1\}.$$

We now combine the ideas of continuity and limits to provide a way to extend the original domain of a given function, provided that the function itself “behaves well” at its boundaries.

## Definition 17 (Extension by continuity)

Let $f : D_f \to \mathbb{R}$ be given, and let $a \in \mathrm{bd}(D_f)$ with $a \notin D_f$. If

$$\lim_{x \to a} f(x) = \ell$$

exists and is finite, we define the ***continuous extension*** $\widetilde{f}$ ***of*** $f$ by setting

$$\widetilde{f}(x) = \begin{cases} f(x), & \text{if } x \in D_f, \\ \ell, & x = a. \end{cases}$$

## Example 18

Let $f : \mathbb{R}^2 \setminus \{(0,0)\} \to \mathbb{R}$ be defined as

$$f(x,y) = \frac{\sin(x^2+y^2)}{x^2+y^2}.$$

In order to extend the domain of the function, we recall the properties of limits. More precisely, we call back to Proposition 5 on the composition of limits.

## Example 18 (cont.)

Let $z : \mathbb{R}^2 \to \mathbb{R}$ be defined as

$$z(x, y) = x^2 + y^2,$$

and notice that

$$\lim_{(x,y)\to(0,0)} z(x, y) = 0.$$

Thus, from Proposition 5,

$$\lim_{(x,y)\to(0,0)} \frac{\sin(x^2 + y^2)}{x^2 + y^2} = \lim_{(x,y)\to(0,0)} \frac{\sin(z(x, y))}{z(x, y)} = \lim_{z\to 0} \frac{\sin(z)}{z} = 1.$$

As a result, the function $f$ can be extended continuously by defining

$$\widetilde{f}(0, 0) = 1.$$

We conclude this part of the notes by introducing a well-known result that connects to many of the ideas we have introduced in the lectures, and is fundamental in many real-world applications.

## Theorem 19 (Weierstrass)

*Let $f : K \subseteq D_f \to \mathbb{R}$ be a* ***continuous function*** *defined over some* ***compact, non-empty subset*** *$K \subseteq \mathbb{R}^d$. Then, $f$ attains both a maximum and a minimum on $K$.*

*That is to say, there exist points $x_{\min}, x_{\max} \in K$ such that*

$$f(x_{\min}) \leq f(x) \leq f(x_{\max})$$

*for all $x \in K$.*

# References


[1] Augustin-Louis Cauchy. *Analyse algébrique*. Vol. I. Analyse algébrique Cours d'analyse de l'École Royale Polytechnique. Paris: De l'Imprimerie Royale, 1821.

[2] Eduard Heine. "Die Elemente der Functionenlehre". ger. In: *Journal für die reine und angewandte Mathematik* 74 (1872), pp. 172–188.

[3] K. Weierstrass, J. Knoblauch, G. Hettner, Deutsche Akademie der Wissenschaften zu Berlin, and R.E. Rothe. *Mathematische werke von Karl Weierstrass: bd. Abhandlungen I. 1894*. Mathematische werke von Karl Weierstrass. Mayer & Müller, 1894.