Mathematics 2

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Lecture 6a: Differentiability (Partial Derivatives and an Application to Environmental Economics)

We now put together all of the concepts introduced so far into a relevant example.

For pedagogical purposes, the model presented here is a simplified version of the ones explored in [3] or [1].[1]

The overall goal of the example is to understand how small changes in economic decisions affect utility, using partial derivatives and differentiability.

[1] See also Sections 1.1 and 1.2 of [2].

Example 1 (An Economic Model of Capital and Pollution)

We consider a stylized economy, which produces a single composite good under constant returns to scale. Production relies on the aggregate capital stock K, wich relies on so-called *old technology*, includes physical capital, human capital, and intangible assets.

The production process K generates pollution which that is later released into the air in the form of *Fine particle matter*,[2] or PM2.5. These emissions accumulate in the atmospheric pollution stock P, negatively affecting the agents in the economy.

[2] See see Section 3.2 in [4].

A local planner within this economy is then tasked to manage an investment by controlling the amount of capital consumed for the upcoming period, denoted by c, and the amount dedicated to the abatement of pollution (reduction) in the locality, denoted by θ.

As a metric of success, the planner is provided with the next utility function ***conditioned to a positive capital at the end of the term***:

$$u = c - \beta P(1), \tag{1}$$

where $\beta \geq 0$ represents the monetary cost per microgram by cubic meter of PM2.5 in the air, and $P(1)$ is the pollution stock after one term. For simplicity, we consider $\beta = 1$.

To discourage a default situation, the managers of the planner opted to give them an utility of

$$u = -\infty \tag{2}$$

should the capital goes to or below 0 at the end of the term, i.e. if $K(1) \leq 0$.

In order to determine the adequate amount of capital to allocate in consumption and in environmental abatement, the planner posses the following data:

- An initial capital of \$100, or $K(0) = 100$.
- An initial pollution of $P(0) = 28\ \mu g/m^3$ of PM2.5 in the air.
- An interest rate of 3.5%; i.e. $r = 0.035$.

The planner is not allowed to spend more than the current capital; that is to say

- the consumption c is bounded:

$$0 \leq c \leq K(0) = 100;$$

- the abatement expenditure θ is bounded:

$$0 \leq \theta \leq K(0) = 100;$$

- the total expenditure $c + \theta$ is also bounded:

$$0 \leq c + \theta \leq K(0) = 100.$$

It is assumed that any unallocated amount

$$K(0) - c - \theta$$

is automatically reinvested in old technology, i.e. on the production stock K.

This means that after one period, the current capital is

$$K(1) = (1 + r)\big[K(0) - c - \theta\big] - (c + \theta).$$

Therefore

$$\boxed{K(1) = 103.5 - 2.035(c + \theta).} \tag{3}$$

Regarding the pollution stock, the following parameters are assumed:

- emissions from old technology are measured as $\epsilon = 16\mu g/m^3$ of PM2.5 for each monetary unit invested;
- a 5% rate, or $\alpha = 0.05$, of natural absorption of PM2.5 by the environment in each period;
- an efficiency of *new technologies* of $\sigma = 12\mu g/m^3$ reduced for each monetary unit invested in abatement θ.

Then:

$$P(1) = \epsilon\big[K(0) - (c + \theta)\big] - \alpha P(0) - \sigma\theta;$$

equivalently,

$$\boxed{P(1) = 1598.6 - 16c - 28\theta.} \tag{4}$$

Question 1: What strategies should the planner avoid at all costs?

Assume that it is in the planner's best interests to avoid a situation in which the utility is $-\infty$.

Mathematically, from equation (1)-(2), this means that we are interested in finding the (maximal) domain D_U of the function

$$U(c,\theta) = \begin{cases} c - P(1), & \text{if } K(1) > 0, \\ -\infty, & \text{otherwise.} \end{cases}$$

From (3) and (4), this is the same as

$$U(c,\theta) = \begin{cases} c - (1598.6 - 16c - 28\theta), \\ -\infty, \end{cases}$$

$$\implies \boxed{U(c,\theta) = \begin{cases} 17c + 28\theta - 1598.6, & \text{if } \frac{103.5}{2.035} > c + \theta, \\ -\infty, & \text{otherwise.} \end{cases}}$$

Hence,

$$\begin{aligned} &D_U\{(c,\theta) : 0 \le c, \theta \le 100,\ K(1) > 0\} \\ &\quad = \left\{(c,\theta) : 0 \le c, \theta \le 100,\ \frac{103.5}{2.035} > c + \theta\right\}. \end{aligned}$$

Observe: There is a discontinuity at the boundary $\text{bd}(D_U)$.

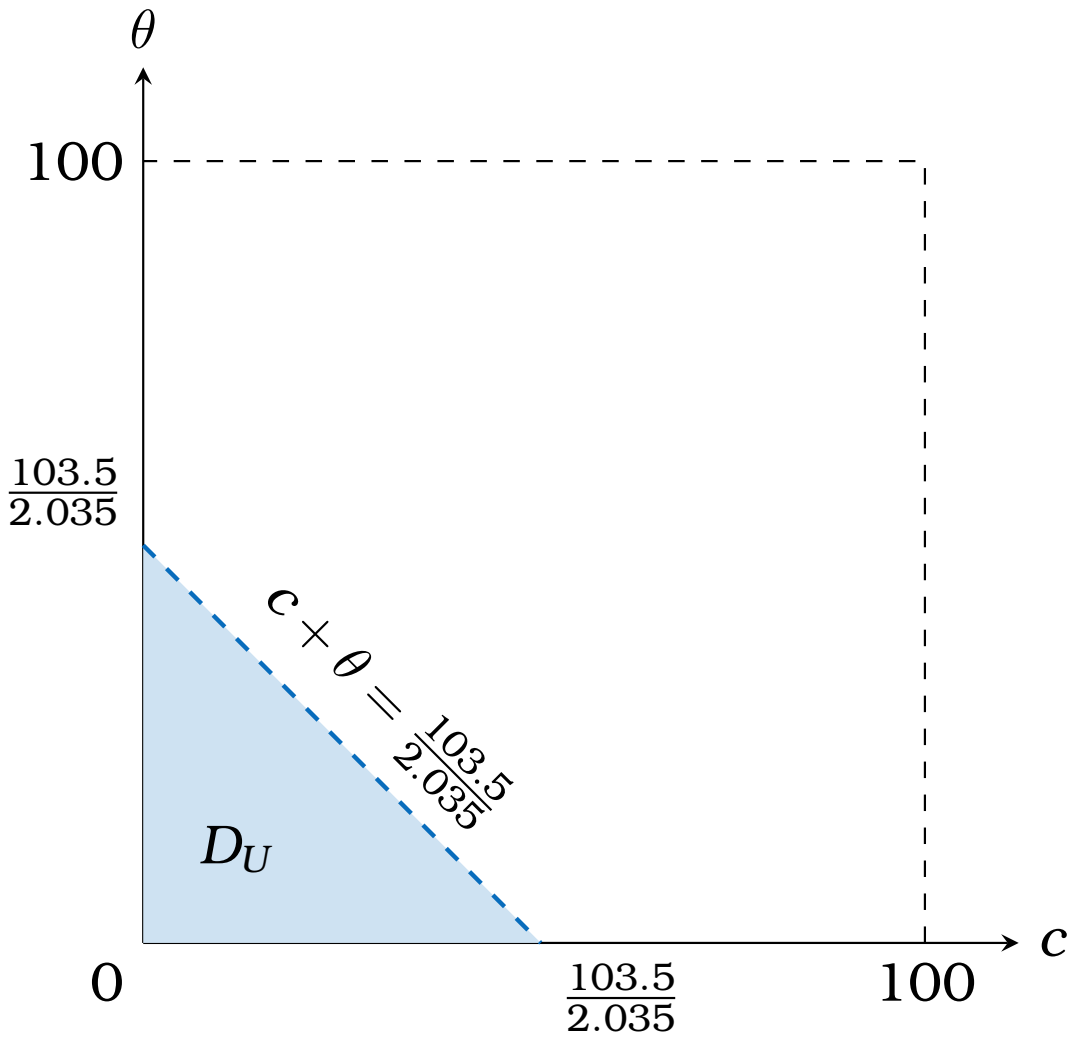


Figure: Plot of D_u.

Question 2: Assuming the planner has a strategy of investing one fourth of the consumption in abatement, what is the utility they can expect to obtain before defaulting?

The planner's strategy is to invest one quarter of the amount spent in consumption into abatement:

$$\boxed{\theta = \frac{c}{4}.} \tag{5}$$

From our previous computations, we know that the boundary $\mathrm{bd}(D_U)$ is characterized by equation (3):

$$0 = 103.5 - 2.035(c + \theta).$$

Thus, the point in the boundary corresponding to the strategy (5) is determine as follows:

$$0 = 103.5 - 2.035(c_0 + \frac{1}{4}c_0)$$

$$\iff \boxed{c_0 = \frac{103.5}{1.25 \cdot 2.035} \approx 40.69}$$

$$\implies \boxed{\theta_0 = \frac{c_0}{4} \approx 10.17.}$$

Hence, our goal now is to determine the limit

$$\lim_{\substack{(c,\theta)\to(c_0,\theta_0) \\ \{(c,\theta):0\leq c,\theta\leq 100,\, \theta=\frac{c}{4}\}}} U(c,\theta).$$

But from the definition of U,

$$\lim_{\substack{(c,\theta)\to(c_0,\theta_0)\\ \{(c,\theta):0\le c,\theta\le 100,\ \theta=\frac{c}{4}\}}} U(c,\theta)$$

$$= \lim_{c\to c_0^+} U\big(c,\frac{c}{4}\big)$$

$$= \lim_{c\to c_0^+} \big(17c + 28\frac{c}{4} - 1598.6\big)$$

$$= \lim_{c\to c_0^+} \big(24c\big) - 1598.6$$

$$\boxed{= 24c_0 - 1598.6 \approx -622.09.}$$

Question 3: What is the instantaneous increase (or decrease) in utility per monetary unit spent in consumption?
What about the change in abatement?

Definition 2 (Partial Derivatives)

Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a function of two variables.

The partial derivative with respect to c measures the instantaneous change in f when c changes and θ is held constant:

$$\boxed{\frac{\partial f}{\partial c}(c, \theta) = \lim_{h \to 0} \frac{f(c + h, \theta) - f(c, \theta)}{h}.}$$

The partial derivative with respect to θ measures the instantaneous change when θ changes and c is held constant:

$$\boxed{\frac{\partial f}{\partial \theta}(c, \theta) = \lim_{h \to 0} \frac{f(c, \theta + h) - f(c, \theta)}{h}.}$$

In each calculation, the other variable is treated as a constant.

Observe: By definition, the partial derivatives $\frac{\partial U}{\partial c}$ and $\frac{\partial U}{\partial \theta}$ describe the ***marginal changes*** we are looking for in Question 3.

Thus, in the context of the problem, we refer to $\frac{\partial U}{\partial c}$ and $\frac{\partial U}{\partial \theta}$ as the ***Marginal Utility of Consumption*** and ***Marginal Utility of Abatement***, respectively.

Computing the partial derivatives[3], observe that for any $(c, \theta) \in \text{int}(D_u)$,

$$\frac{\partial}{\partial c} U(c, \theta) = \frac{\partial}{\partial c} \big(17c + 28\theta - 1598.6\big)$$

Holding θ constant,

$$\boxed{\frac{\partial U}{\partial c} = 17.}$$

[3]For readability, we have included a list of the

Interpretation:

- An additional dollar of consumption directly increases utility by 1.
- It reduces old-technology investment and increases pollution by $\epsilon = 16$ units (factor due to emisions).
- The resulting pollution cost changes utility by $16 = 16\beta$ (monetary cost of emisions).

Thus the total marginal utility of consumption is $1 + \epsilon\beta = 17$.

Following a similar procedure for abatement, we obtain the following:

$$\frac{\partial}{\partial \theta} U(c, \theta) = \frac{\partial}{\partial \theta} \big(17c + 28\theta - 1598.6\big)$$

Holding c constant,

$$\boxed{\frac{\partial U}{\partial \theta} = 28.}$$

Comparing to the previous case, an additional dollar of abatement reduces:

- investment in old technology by one dollar;
- emissions from old technology by $\epsilon = 16$ units;
- pollution through new technology by another $\sigma = 12$ units.

The total reduction in pollution is 28β units, generating a utility increase of 28 units.

Suggested exercises:[4] **What is the marginal value of efficiency?**
What is the marginal value of consumption/abatment when subjected to specific trajectories?

[4] We shall go in depth for this questions in later notes.

Addendum: Common Derivative Formulas

$$\frac{d}{dx}c = 0 \qquad (c \text{ constant}),$$

$$\frac{d}{dx}x^n = nx^{n-1},$$

$$\frac{d}{dx}e^x = e^x,$$

$$\frac{d}{dx}a^x = a^x \ln a,$$

$$\frac{d}{dx}\ln x = \frac{1}{x},$$

$$\frac{d}{dx}\sin x = \cos x,$$

$$\frac{d}{dx}\cos x = -\sin x,$$

$$\frac{d}{dx}\tan x = \frac{1}{\cos^2 x}.$$

Addendum: Rules of Differentiation

$$
\begin{aligned}
(f+g)' &= f' + g', \\
(af)' &= af', \\
(fg)' &= f'g + fg', \\
\left(\frac{f}{g}\right)' &= \frac{f'g - fg'}{g^2}, \qquad g \neq 0, \\
(f \circ g)'(x) &= f'(g(x))g'(x), \\
\frac{d}{dx} f(g(x)) &= f'(g(x))g'(x).
\end{aligned}
$$

For partial derivatives, differentiate with respect to the chosen variable and treat all other variables as constants.

Addendum: Common Partial Derivatives

For $f = f(x, y)$,

$$\frac{\partial}{\partial x}(x^a y^b) = a x^{a-1} y^b,$$
$$\frac{\partial}{\partial y}(x^a y^b) = b x^a y^{b-1},$$
$$\frac{\partial}{\partial x}(x + y) = 1,$$
$$\frac{\partial}{\partial y}(x + y) = 1,$$
$$\frac{\partial}{\partial x} e^{g(x,y)} = e^{g(x,y)} \frac{\partial g}{\partial x},$$
$$\frac{\partial}{\partial y} e^{g(x,y)} = e^{g(x,y)} \frac{\partial g}{\partial y}.$$

If f is differentiable, then

$$df = f_x\, dx + f_y\, dy.$$

References

[1] Alexandra Brausmann and Lucas Bretschger. "Escaping Damocles' Sword: Endogenous Climate Shocks in a Growing Economy". en. In: *Environmental and Resource Economics* 87.6 (June 2024), pp. 1545–1592. DOI: 10.1007/s10640-023-00835-w.

[2] Daniel Hernández-Hernández and Joshué Helí Ricalde-Guerrero. *Optimal Control in Random Environments: Pontryagin Maximum Principle with Environment-dependent Jumps and Mean-Field Games with Common Poissonian Noise*. Submitted to Springer Briefs Series. 2025.

[3] Daria Sakhanda and Joshué Helí Ricalde-Guerrero. *Infinite-Horizon Optimal Control of Jump-Diffusion Models for Pollution-Dependent Disasters*. arXiv:2511.13568 [math]. Nov. 2025. DOI: 10.48550/arXiv.2511.13568.

[4] *WHO Global Air Quality Guidelines: Particulate Matter (PM2. 5 and PM10), Ozone, Nitrogen Dioxide, Sulfur Dioxide and Carbon Monoxide*. eng. 1st ed. Geneva: World Health Organization, 2021.