

ALM - Stochastic Term Structure Models

Tiago Fardilha and Walther Neuhaus

Course Program

- Basic interest rate theory
- Interest rate risk management
- Stochastic term structure models
- Risk measurement
- Reinsurance and insurance-linked securities
- Mean-variance analysis for ALM

Contents of the chapter

- Stochastic term structure models (a.k.a. **short rate** models).
 - Cox-Ingersoll-Ross model
 - Vasicek model
 - Hull-White model
 - Ho-Lee model
- Simulation of stochastic evolution of the yield curve.

Introduction 1

- We have seen that
 - **Matching** protects the surplus against all changes in yield curve, but is usually not useful in practice.
 - **Immunization** protects the surplus against parallel shifts in the yield curve, but not necessarily against other kinds of movement.
 - To be able to assess what changes one can *reasonably* expect, one must model the stochastic evolution of the yield curve - the **term structure**.

Introduction 2

- The purpose of **stochastic term structure models** is to represent the evolution of the yield curve as a stochastic process.
- At time t , the market price of €1 payable at time $T > t$ is $P(t, T)$. This price varies **randomly** from day to day.
- On any given day t , the function $\{P(t, T) : T > t\}$ can be determined, at least in principle, from the observed market prices of bonds and bills.

Introduction 3

- We write $P(t, T) = \exp(-R(t, T)(T - t))$, so that $R(t, T)$ is the random spot rate (zero rate) at time t for a term of $T - t$.
- The short rate is the interest rate at which one can borrow money for an infinitesimal period of time from time t :

$$r(t) \stackrel{\text{def}}{=} \lim_{T \rightarrow t} R(t, T).$$

Introduction 4

- If we write

$$P(t, T) = E \left(e^{-\int_t^T r(s) ds} \right)$$

- Then

$$R(t, T) = -\frac{1}{T-t} \ln E \left(e^{-\int_t^T r(s) ds} \right)$$

(remember how to compute the yield of a zero-coupon bond).

Term structure equilibrium models

- A **stochastic model** of the evolution of the short rate $r(t)$ together with a no-arbitrage assumption implies the entire yield curve at time t .
- **Vasicek** model
$$dr(t) = a(b - r(t))dt + \sigma dz(t)$$
- Cox-Ingersoll-Ross model - **CIR**
$$dr(t) = a(b - r(t))dt + \sigma\sqrt{r(t)}dz(t)$$
- The term $z(t)$ is a standard Brownian motion (white noise).

Term structure equilibrium models

- Both the Vasicek and CIR model have **mean reversion**.
- The parameter b describes the long-term average rate.
- The parameter a describes the strength of the gravitation back to the average rate.
- The Vasicek model **allows negative** interest rates and CIR model does not.

Affine term structure

- The term structure of the Vasicek and CIR model has the affine form

$$P(t, T) = A(t, T)e^{-B(t, T)r(t)}$$

- Equivalently,

$$R(t, T) = \frac{1}{T - t}(-\ln A(t, T) + B(t, T)r(t))$$

with functions A and B that depend on a , b and σ .

Term structure - Vasicek model

- The term structure is easy to calculate and simulate once the parameters are given or estimated.

- $P(t, T) = A(t, T)e^{-B(t, T)r(t)}$

- $B(t, T) = \frac{1 - e^{-a(t-T)}}{a}$

- $A(t, T) = e^{\frac{(B(t, T) - (T-t))(a^2 b - \frac{1}{2} \sigma^2)}{a^2}} \cdot e^{-\frac{\sigma^2 B^2(t, T)}{4a}}$

Term structure - CIR model

- The term structure is easy to calculate and simulate once the parameters are given or estimated.
- $P(t, T) = A(t, T)e^{-B(t, T)r(t)}$
- $A(t, T) = \left(\frac{2\gamma e^{(\gamma+a)(T-t)/2}}{(\gamma+a)(e^{\gamma(T-t)} - 1) + 2\gamma} \right)^{2ab/\sigma^2}$
- $B(t, T) = \frac{2(e^{\gamma(T-t)} - 1)}{(\gamma+a)(e^{\gamma(T-t)} - 1) + 2\gamma}$
- $\gamma = \sqrt{a^2 + 2\sigma^2}$

Example 1

- In script 6, we will fit the two models to the yield curve - CIR and Vasicek models.

Example 2

Present Value of liabilities, assets and surplus of our portfolio of 3 bonds, under yields approximated by CIR and Vasicek models:

	PV.liab.CIR	PV.assets.CIR	PV.surplus.CIR
1	93150030	94851023	1700993

	PV.liab.vsk	PV.assets.vsk	PV.surplus.vsk
1	93188994	94856641	1667647

- The surplus PV is different from 0 because the CIR model is **not a perfect replication of the empirical yield curve** that was used to find the immunising portfolio, and neither is the Vasicek model.

Simulation - CIR

- Discrete time approximation of the CIR model:

$$dr(t) = a(b - r(t))dt + \sigma\sqrt{r(t)}dz(t)$$

- If Δt is small, this is approximated by

$$r(t + \Delta t) = r(t) + a(b - r(t))\Delta t + \sigma\sqrt{r(t)} \cdot [\sqrt{\Delta t} \cdot N(0, 1)]$$

Simulation - Vasicek

- Discrete time approximation of the Vasicek model:

$$dr(t) = a(b - r(t))dt + \sigma' dz(t)$$

- If Δt is small, this is approximated by

$$r(t + \Delta t) = r(t) + a(b - r(t))\Delta t + \sigma' \cdot [\sqrt{\Delta t} \cdot N(0, 1)]$$

- Note that $\sigma' \neq \sigma$ (from CIR).

Simulation - Central results

Simulation of a univariate random variable by *inverting the cumulative distribution function*:

- If $F^{-1}(u) = \inf\{x : F(x) \geq u\}$ and $U \sim \text{Unif}[0, 1]$, then $F^{-1}(U) \sim F$.
- One can simulate any random variable $X \sim F$ by $X = F^{-1}(U)$, with $U \sim \text{Unif}[0, 1]$. In **R**, we may use the function `runif()`.
- If F has an explicit inverse or an available **R** function for the inverse, all becomes easy, e.g. `X = qnorm(runif())` will be normal distributed with mean `0` and s.d. `1`.

Example - I

- In [script 7](#), we will compute 5000 six month year ahead simulations, using CIR and assuming all payments are made at year end. [Homework](#): The same using Vasicek model.

```
# A tibble: 60,000 × 9
  sim   r_0  drift  unif    dz rand.term new.r `T-t` `P(t,T)`
  <int> <dbl>  <dbl>  <dbl>  <dbl>    <dbl>  <dbl> <dbl>  <dbl>
1     1 0.052 -0.0027 0.989  2.29  0.0184  0.0677  0.5  0.968
2     2 0.052 -0.0027 0.398 -0.259 -0.00209  0.0472  0.5  0.977
3     3 0.052 -0.0027 0.116 -1.20 -0.00965  0.0397  0.5  0.981
4     4 0.052 -0.0027 0.0697 -1.48 -0.0119  0.0374  0.5  0.982
5     5 0.052 -0.0027 0.244 -0.694 -0.00560  0.0437  0.5  0.979
6     6 0.052 -0.0027 0.792  0.813  0.00656  0.0559  0.5  0.973
7     7 0.052 -0.0027 0.340 -0.412 -0.00332  0.0460  0.5  0.978
8     8 0.052 -0.0027 0.972  1.91  0.0154  0.0647  0.5  0.969
9     9 0.052 -0.0027 0.166 -0.971 -0.00783  0.0415  0.5  0.980
10    10 0.052 -0.0027 0.459 -0.103 -0.000828  0.0485  0.5  0.977
# i 59,990 more rows
```

Example - II

- First of the 5000 one-half year ahead simulations

```
# A tibble: 5,000 × 4
  sim PV.liab.CIR PV.assets.CIR PV.surplus.CIR
  <int>      <dbl>      <dbl>      <dbl>
1     1    91310374.    96981389.    5671015.
2     2    96056924.    97326268.   1269344.
3     3    97875784.    97424721.  -451063.
4     4    98428393.    97451101.  -977293.
5     5    96896122.    97373931.    477809.
6     6    94023693.    97194496.   3170803.
7     7    96351264.    97343426.    992161.
8     8    91992501.    97039080.   5046579.
9     9    97433566.    97402436.   -31131.
10    10    95757137.    97308299.   1551162.
# i 4,990 more rows
```

Example - Simulated PV of Surplus

Stochastic term structure models - no arbitrage

- The term structure of a Vasicek or CIR model can never perfectly replicate the actual (observed) term structure.
- Sometimes one needs a model that starts with **today's exact term structure** and evolves stochastically from there.
- Models with that property are called **no-arbitrage models**.

Ho-Lee and Hull-White models 1

- Like the Vasicek and CIR models, the **Ho-Lee** and **Hull-White** models have an affine term structure of the form

$$P(t, T) = A(t, T)e^{B(t, T)r(t)},$$

with different functions A and B .

- In the Ho-Lee and Hull-White models, the function $A(t, T)$ depends not only on model parameters but also on the initial term structure.
- The Ho-Lee and Hull-White models permit perfect fit to today's empirical term structure.

Ho-Lee and Hull-White models 2

- Ho-Lee model

$$dr(t) = \theta(t)dt + \sigma dz(t)$$

- Hull-White model

$$dr(t) = (\theta(t) - ar(t))dt + \sigma dz(t)$$

- The term $z(t)$ is standard Brownian motion (white noise).

Speaker notes