STATISTICAL LABORATORY

Lecture 9: Link Indices and Chain Indices (Theory and Practice)

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Program

1. Fundamental Concepts of Statistics
2. Exploratory Data Analysis
3. Organizing and Summarizing Data
4. Association and Relationships Between Variables
5. **Index Numbers**
6. **Time Series Analysis**

Link Index (Year-to-Year Index)

Purpose:

- To compare each period with the one immediately before it.
- This index measures short-term changes and avoids having one fixed base year.

Formula:

$$I_{t|t-1} = \frac{p_t}{p_{t-1}} \times 100$$

where p_t = price in period t and p_{t-1} = price in the previous period.

Example: Suppose product prices were Year 1 = 4.00, Year 2 = 5.00.

$$I_{2|1} = \frac{5.00}{4.00} \times 100 = 125$$

$\rightarrow$ Prices increased by 25% from Year 1 to Year 2.

Chain Index (Chained Index)

Purpose:

- To measure the change from a base year (0) to any later year (t) by multiplying the sequence of link indices.
- Useful when changing product baskets over time.

Main Formula:

$$I_{t|0}^{\text{chain}} = I_{1|0} \times I_{2|1} \times \cdots I_{t-1|t-2} \times I_{t|t-1}.$$

(All index numbers in the main formula have to be in **ratio form**)

Alternative Notation:

$$I_{t|0}^{\text{chain}} = I_{1|0} \times \frac{I_{2|1}}{100} \times \cdots \times \frac{I_{t|t-1}}{100}$$

(This allows us to write $I_{t|0}^{\text{chain}}$ in percentage form)

Note

The link (moving-base) indices are not expressed as percentages, but as relative decimals when chained via alternative notation.

Fixed-Base and Moving-Base Indices

Fixed-base Index:

- Compares each period to a single base period (t_0).
- $I_{t|0}^{\text{fixed}} = \frac{x_t}{x_0} \times 100$

Moving-base Index:

- Compares each period to a changing base ($t-1$).
- $I_{t|t-1}^{\text{moving}} = \frac{x_t}{x_{t-1}} \times 100$

Type	Definition	Purpose
Link Index	Base is previous period ($t-1$)	Captures period -to-period variation
Homologous Index	Base is same period prev. year ($t-12$)	Controls for seasonal patterns

Chained Index: Fixed-Base or Moving-Base?

According to Silvestre (2007):

- Chained indices are obtained from link relatives (moving-base indices).
- These link indices are multiplied in sequence to show the accumulated change since a fixed initial period.
- Therefore, a chained index is considered a **fixed-base index** because it shows evolution relative to an initial period.

Index Type	Comparison	Note
Fixed base	Always with t_0	e.g., Laspeyres, Paasche
Moving base	Each period with $t-1$	e.g., link index
Chained index	Calculated from link relatives	Final result relative to t_0

Changing the Base of an Index

Formula:

$$i_{t|0} = \frac{x_t}{x_0} = \frac{x_t}{x_0} \cdot \frac{x_b}{x_b} = \frac{x_t}{x_b} \cdot \frac{x_b}{x_0} = i_{t|b} \cdot i_{b|0}$$

$$i_{t|b} = \frac{i_{t|0}}{i_{b|0}} \cdot 100$$

Notation:

- $i_{t|0} \rightarrow$ Index of period t with original base 0
- $i_{t|b} \rightarrow$ Index of period t with new base b
- $i_{b|0} \rightarrow$ Index of new base b relative to original base 0

Usage: Allows rescaling of index series without recalculating all values from scratch.

Changing the Base of an Index: Example

Suppose we have a price series:

Period (t)	Price x_t	Index $x_{t\|0} = (x_t/x_0) \times 100$
0	100	100
1	120	120
2	150	150

We want to change the base from period 0 to period 1 ($b = 1$). Compute $i_{t|b}$ using the formula $i_{t|b} = \frac{i_{t|0}}{i_{b|0}} \cdot 100$:

- $i_{1|1} = \frac{120}{120} \cdot 100 = 100$
- $i_{2|1} = \frac{150}{120} \cdot 100 = 125$

When should we change the base?

- The further away from the reference period that the series gets, the more difficult it is to interpret movements in the series.

Table 9.5 A simple series.

	February	March	April	May	June	July	August
Original	425.5	426.2	432.0	428.9	430.2	442.3	443.5
Re-referenced	100.0	100.2	101.5	100.8	101.1	103.9	104.2

Source: A Practical Introduction to Index Numbers (Ralph, O'Neill and Winton 2015)

- Can you see the clear advantage of changing the base here?

When should we change the base?

- The second reason for re-referencing is to enable comparisons between different series.

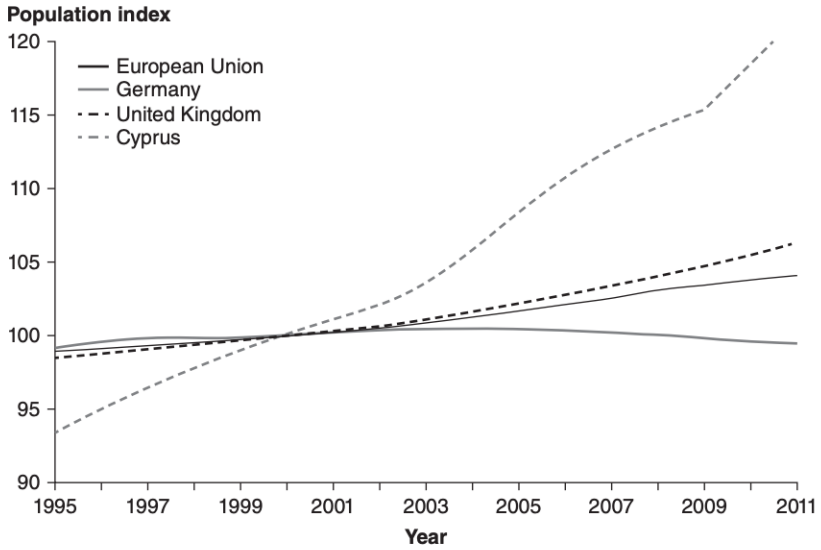


Figure 2.2 Index of population on 1 January (2000 = 100), European Union, 1995–2011. Source: Eurostat. These graphs have been created using data from the Eurostat Statistics Database: http://epp.eurostat.ec.europa.eu/portal/page/portal/statistics/search_database. Eurostat is the statistical office of the European Union and it is a Web site that provides access to all Eurostat's statistical databases and the associated electronic publications.

Source: A Practical Introduction to Index Numbers (Ralph, O'Neill and Winton 2015)

Changing the Base: Interpretation

Period (t)	Price x_t	$x_{t\|0}$ (Base 0)	$x_{t\|1}$ (Base 1)
0	100	100	83.33
1	120	120	100
2	150	150	125

- $i_{t|0}$ = Index with original base ($0 = 100$)
- $i_{t|1}$ = Index with new base ($1 = 100$)
- Period 0 relative to new base 1: $i_{0|1} = \frac{i_{0|0}}{i_{1|0}} \cdot 100 = 83.33$
- Period 2 relative to new base 1: $i_{2|1} = \frac{i_{2|0}}{i_{1|0}} \cdot 100 = 125$

Reconciliation of Indices: Base Uniformization

Base uniformization consists in expressing all index series on the same reference base, so that they can be compared.

This may occur in two situations:

1. The indices already share the same base, but we want to change to a new common base.
2. The indices have different bases, so the first step is to convert them to a common base.

Reconciliation: Situation 1

Consider two index series:

- Index A defined for periods $0, 1, \ldots, b$ with base 0
- Index B defined for periods $b, b+1, \ldots, T$ with base b

Goal: Obtain one single index series over the whole period.

Final Series Expressed in Base b:

$$I^C_{t|b} = \begin{cases} \frac{I^A_{t|0}}{I^A_{b|0}} \times 100, & t = 0, 1, \ldots, b \\ I^B_{t|b}, & t = b+1, \ldots, T \end{cases}$$

Note on $\times 100$ and $\div 100$

If base $= 100$, keep scaling factors; if base $= 1$ (ratios), remove them.

Reconciliation: Situation 1

1. For the Earlier Period ($t = 0, 1, \ldots, b$)

$$I^C_{t|b} = \frac{I^A_{t|0}}{I^A_{b|0}} \times 100$$

   - Index A originally uses period 0 as its base ($I^A_{b|0}$ represents the value of Index A at period b).
   - To shift the base from 0 to b, you divide each term by the value of the index at the new base period (b) and scale by 100.
   - This successfully rebases the historical series so that period b equals 100 (or 1 if using ratios).

2. For the Later Period ($t = b + 1, \ldots, T$)

$$I^C_{t|b} = I^B_{t|b}$$

   - Index B is already constructed using period b as its base. Therefore, no transformation or rebasing is required.

Reconciliation: Situation 1

Period (t)	**Index A (Base 0)**	**Index B (Base 2)**	**Index C**	**Calculation Methodology**
0	100.00	—	**83.33**	$(100.00/120.00) \times 100$
1	110.00	—	**91.67**	$(110.00/120.00) \times 100$
2 (b)	**120.00**	**100.00**	**100.00**	$(120.00/120.00) \times 100$
3	—	105.00	**105.00**	Direct from Index B ($I^B_{3\|2}$)
4	—	112.00	**112.00**	Direct from Index B ($I^B_{4\|2}$)

Reconciliation: Situation 2

Consider Index A (base 0) and Index B (base b). Goal: Obtain one single index series expressed in **Base 0**.

Final Series Formula:

$$I^C_{t|0} = \begin{cases} I^A_{t|0}, & t = 0, 1, \ldots, b \\ I^B_{t|b} \cdot \frac{I^A_{b|0}}{100}, & t = b+1, \ldots, T \end{cases}$$

- Keep Index A as it is.
- Convert Index B from base b to base 0.

Reconciliation: Situation 2

Period (t)	**Index A (Base 0)**	**Index B (Base 2)**	**Index C (Base 0)**	**Calculation Methodology**
0	100.00	—	**100.00**	Direct from Index A ($I^A_{0\|0}$)
1	110.00	—	**110.00**	Direct from Index A ($I^A_{1\|0}$)
2 (b)	**120.00**	**100.00**	**120.00**	Direct from Index A ($I^A_{2\|0}$) (Bridge Point)
3	—	105.00	**126.00**	$105.00 \times (120.00/100)$
4	—	112.00	**134.40**	$112.00 \times (120.00/100)$

Why Division in Case 1 and Multiplication in Case 2?

Case 1 (Base $0 \rightarrow b$): Divide

$$I^A_{t|b} = \frac{I^A_{t|0}}{I^A_{b|0}} \times 100$$

- Mathematically, dividing by $I^A_{b|0}$ strips out the accumulated growth factor that occurred between period 0 and period b.
- It removes the cumulative change from period 0 to b.

Why Division in Case 1 and Multiplication in Case 2?

Case 2 (Base $b \to 0$): Multiply

$$I^B_{t|0} = I^B_{t|b} \times \frac{I^A_{b|0}}{100}$$

- To translate Index B into base 0 scale, every value must be scaled upward (or downward) by the historical growth factor that happened between period 0 and period b (which is $\frac{I^A_{b|0}}{100}$).
- It adds the cumulative change from period 0 to b.

Direction of Base Change

Direction of Base Change	**Operation**	**Why**
From an older base to a more recent base	Divide	Removes the cumulative variation up to b
From a more recent base to an older base	Multiply	Adds the cumulative variation up to b

Base Uniformization: Example

Suppose we have two index series:

- **Index A (base year $= 0$):**

$$I^A_{0|0} = 100, \quad I^A_{1|0} = 110, \quad I^A_{2|0} = 120, \quad I^A_{3|0} = 150$$

- **Index B (base year $= 3$):**

$$I^B_{3|3} = 100, \quad I^B_{4|3} = 105, \quad I^B_{5|3} = 115$$

We want a single reconciled series over $t = 0, \ldots, 5$.

Example: Case A (Base = 3)

Formula: Converting older base to later base (divide).

$$I^C_{t|3} = \begin{cases} \frac{I^A_{t|0}}{I^A_{3|0}} \times 100, & t = 0, 1, 2, 3 \\ I^B_{t|3}, & t = 4, 5 \end{cases}$$

Calculations (since $I^A_{3|0} = 150$):

- $I^C_{0|3} = 100/150 \times 100 = 66.67$, $I^C_{1|3} = 110/150 \times 100 = 73.33$
- $I^C_{2|3} = 120/150 \times 100 = 80.00$, $I^C_{3|3} = 150/150 \times 100 = 100.00$
- $I^C_{4|3} = 105$, $I^C_{5|3} = 115$ (from B)

Example: Case B (Base = 0)

Formula: Converting later base back to earlier base (multiply).

$$I^C_{t|0} = \begin{cases} I^A_{t|0}, & t = 0, 1, 2, 3 \\ I^B_{t|3} \times \frac{I^A_{3|0}}{100}, & t = 4, 5 \end{cases} \qquad (b = 3)$$

Calculations:

- For $t = 0, 1, 2, 3$ keep A: $100, 110, 120, 150$
- $I^C_{4|0} = 105 \times 150/100 = 157.50$
- $I^C_{5|0} = 115 \times 150/100 = 172.50$

Time Series: Introduction

- **Data observed over time:** measurements or values recorded at successive time points.
- **Observations collected in time order:** the sequence matters; order affects analysis.
- **Useful for studying trends, cycles, and temporal patterns:** e.g., stock prices, temperature, monthly sales.
- **Graphical representation:** often visualized in a time plot (chronogram) to detect patterns.
- **Different from cross-sectional data:** cross-sectional data are collected at a single point in time for different individuals, firms, or regions, while chronological data track one variable over time.

Note

Time is the essential factor in chronological data, unlike cross-sectional data where time does not vary.

Objectives of Studying Time Series

- **Description:** To summarize and illustrate the behavior of a variable over time.
- **Explanation:** To identify the causes behind observed patterns and fluctuations.
- **Forecasting:** To predict future values based on past trends.
- **Control:** To monitor and manage processes by detecting deviations from expected behavior.

Definition of a Time Series

Let x be a variable observed over time.

Suppose we collect n observations:

$$x_1, x_2, x_3, \ldots, x_n$$

Abbreviated notation: $\{x_t\}_{t=1}^{n}$

Graphical representation: displayed in a time plot (chronogram).

Time Series: Example

Example: Monthly sales of a company over one year:

$$x_1 = 1200, \quad x_2 = 1350, \quad x_3 = 1280, \quad \ldots, \quad x_{12} = 1500$$

Summary of Interpretation:

- Sales increased from January to February, then decreased in March.
- Rose steadily until May, declined until June, and then rose again, peaking in November before declining in December.
- Shows a fluctuating trend with two main growth periods and a November peak, indicating possible seasonal effects or promotions.

Components and Models of a Time Series

Models:

- **Additive Model:** $x_t = T_t + s_t + c_t + e_t$
- **Multiplicative Model:** $x_t = T_t \cdot s_t \cdot c_t \cdot e_t$
- **Mixed Models (Silvestre):**
  1. Type 1: $x_t = (T_t + c_t) \cdot s_t + e_t$
  2. Type 2: $x_t = T_t \cdot c_t \cdot s_t + e_t$

Components: Trend (T_t), Seasonality (s_t), Cycle (c_t), Residual (e_t).

Components and Models: Details

- **Additive Model ($x_t = T_t + s_t + c_t + e_t$):** Assumes components add together to form the series.
- **Multiplicative Model ($x_t = T_t \cdot s_t \cdot c_t \cdot e_t$):** Assumes components multiply together.
- **Mixed Models (Silvestre):**
  - **Type 1:** Trend and cycle add, then multiplied by seasonality, plus residual.
  - **Type 2:** Trend, cycle, and seasonality multiply, plus residual.

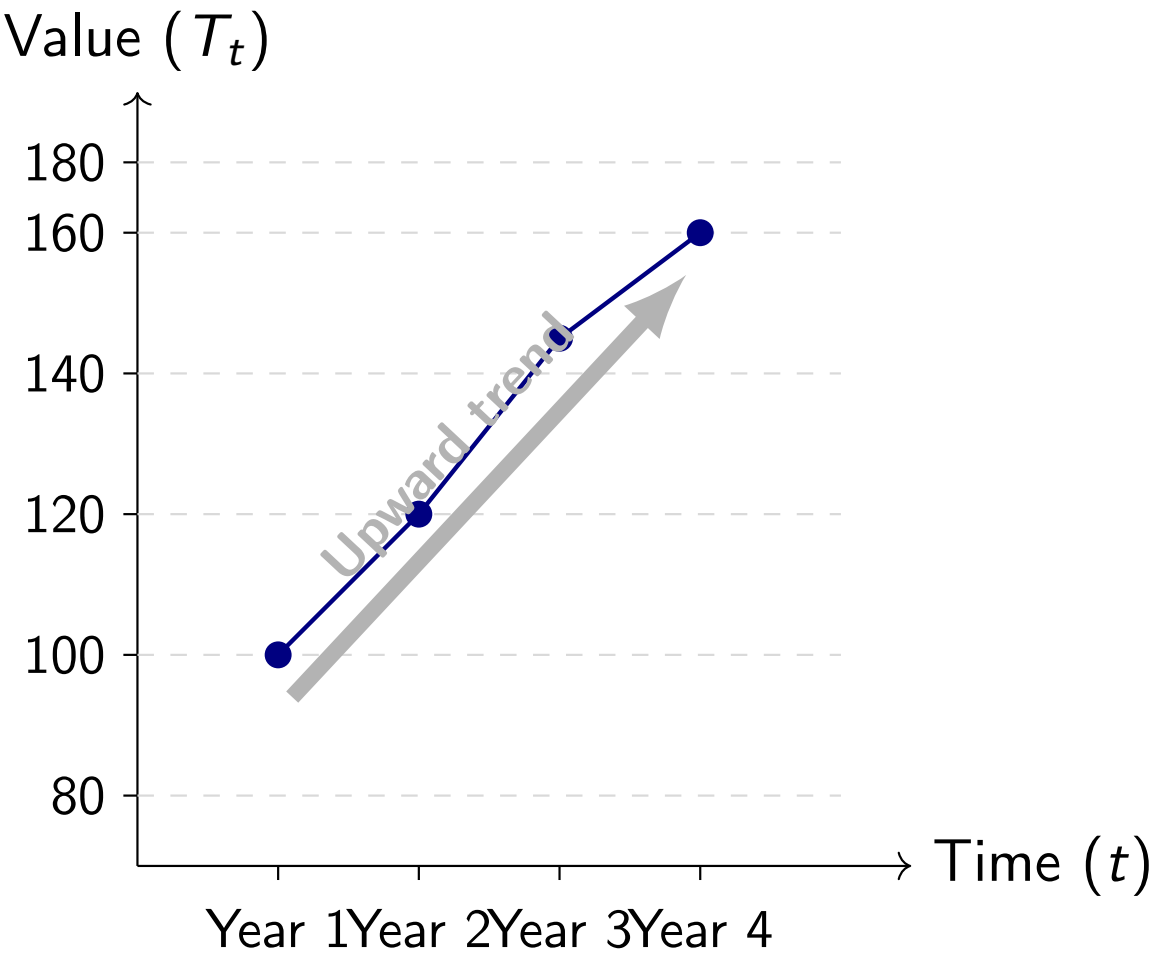
Components of a Time Series: Trend

1) Trend (T_t)

- The long-term movement or general direction of the series.
- **Key question:** "Is the series generally increasing, decreasing, or remaining stable over time?"
- If sales grow year after year $\rightarrow$ Upward trend.
- If they decrease $\rightarrow$ Downward trend.

Components of a Time Series: Trend

Example: Year 1: 100, Year 2: 120, Year 3: 145, Year 4: 160 $\rightarrow$ Overall direction is increasing (upward trend).



Components of a Time Series: Seasonality

2) Seasonality (S_t)

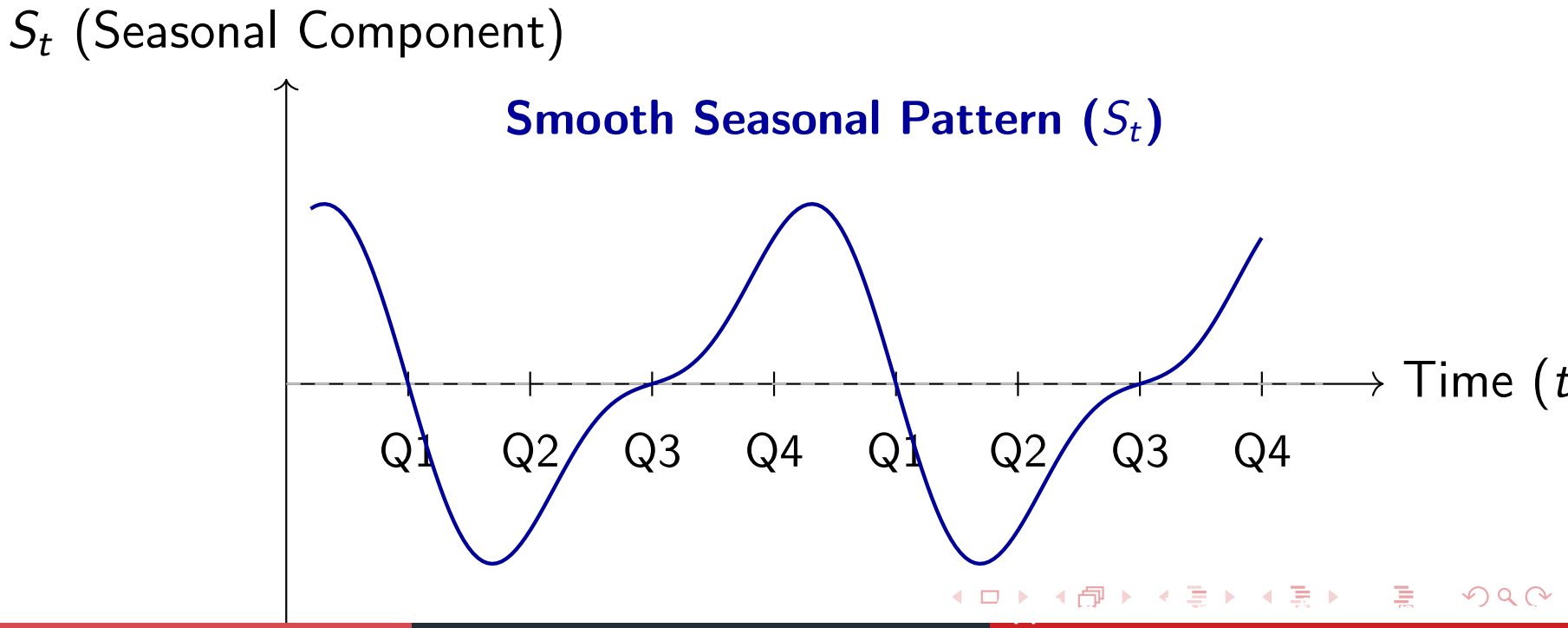
- A regular and repeating pattern within a fixed period (usually within each year).
- **Key question:** "Does the series show a pattern that repeats at the same times each year?"
- **Examples:** Higher sales every Christmas, more restaurant demand every summer, higher electricity usage every winter.

Mental image: Small repeating waves always occurring in the same months.

Components of a Time Series: Seasonality

In practice, there are multiple ways to measure seasonality.

1. Seasonal Dummy Variables (use Spring, Summer, Autumn and Winter as regressors)
2. Regression model with past values as regressors
3. Impose smooth seasonal patterns using sine and cosine pairs (Fourier seasonal components)



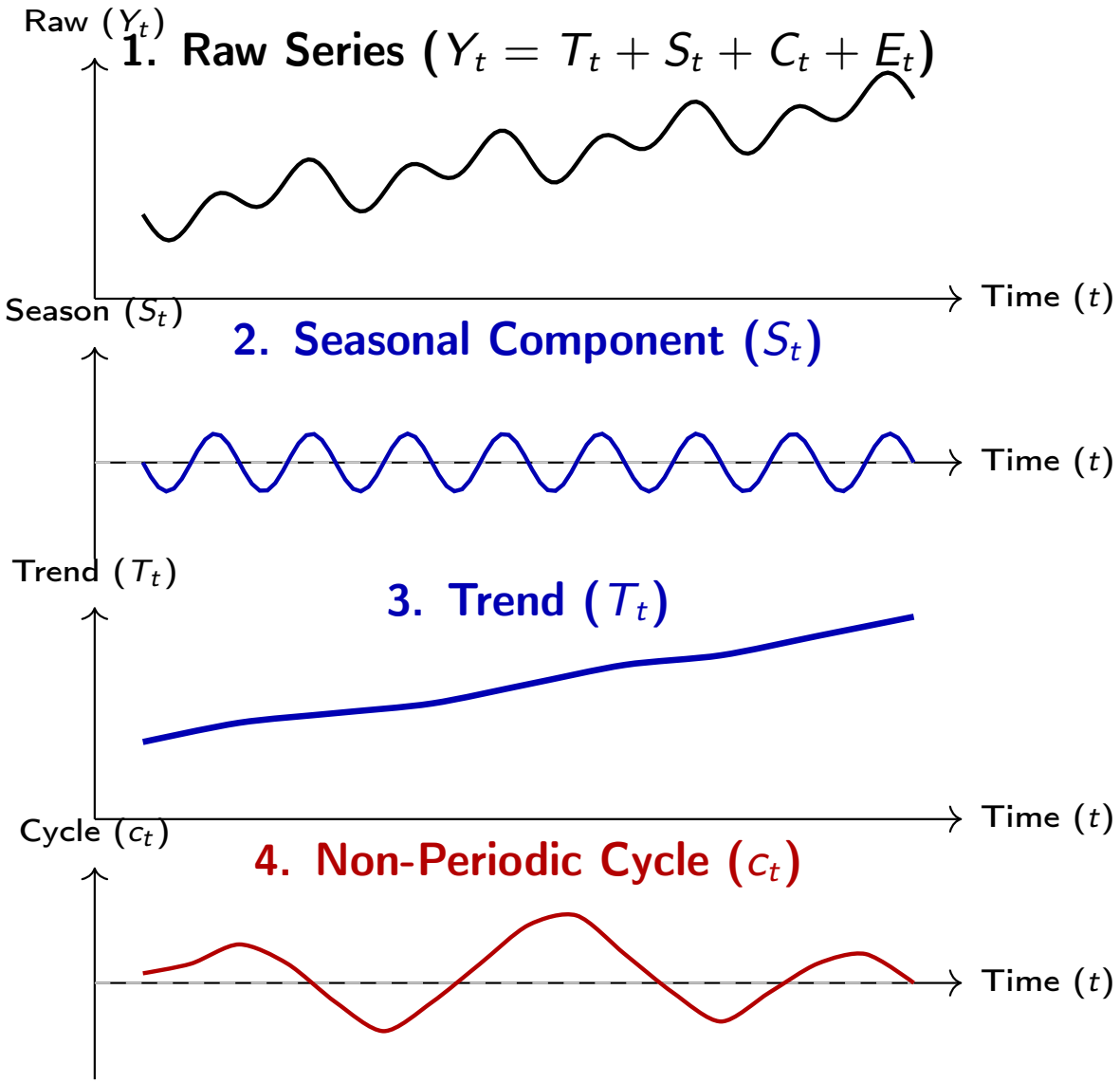
Components of a Time Series: Cycle

3) Cycle (C_t)

- Medium-to-long-term fluctuations that do not follow a fixed calendar pattern.
- **Key question:** "Are there expansions and contractions that happen, but not at regular intervals?"

Seasonality	**Cycle**
Repeats on a fixed schedule (e.g., December)	No fixed period; timing varies
Short period (within the year)	Long period (several years)

Components of a Time Series: Cycle



Components of a Time Series: Irregular or Residual

4) Irregular or Residual Component (E_t)

- The unpredictable part of the series; random variation.
- **Includes:**
  - Unexpected events (strikes, storms, pandemics)
  - Measurement errors
  - Accidents and non-systematic shocks
- **Mental image:** Noise that cannot be explained by Trend, Seasonality, or Cycle.

Components of a Time Series: Summary Table

Component	**Meaning**	**Example**
Trend (T_t)	Long-term direction	Sales gradually increase over years
Seasonality (S_t)	Repeating pattern within each year	Christmas demand spike every year
Cycle (C_t)	Long-term economic fluctuations	Periods of recession and recovery
Residual (E_t)	Unpredictable/random variation	Strikes, pandemic, shocks

Components and Models of a Time Series: Example

- Visualized components: Original Time Series, Trend Component, Seasonal Component, and Residual Component.
- Here is a real example, already fully decomposed with trend, seasonality, and residual components.

Components and Models: Example Analysis

- **Original Series:** Shows actual observed values over time.
- **Trend Component (T_t):** Smooth long-term movement, steadily increasing from 100 to about 160.
- **Seasonal Component (S_t):** Repeating pattern within each year.
- **Residual Component (E_t):** $E_t = Y_t - T_t - S_t$, representing random shocks and unpredictable variation.

Note

No cyclical component is considered since economic cycles require long time spans. The decomposition is simplified to: $Y_t = T_t + S_t + E_t$.

Identifying the Appropriate Time Series Model

Feature Observed	**Interpretation**	**Model Type**
Constant amplitude	Variations do not depend on level	Additive: $Y_t = T_t + S_t + C_t + E_t$
Amplitude grows proportionally	Variations are relative to level	Multiplicative: $Y_t = T_t \cdot S_t \cdot C_t \cdot E_t$
Grows slightly, not proportionally	Part constant, part depends on level	Mixed: $Y_t = (T_t + C_t) \cdot S_t + E_t$

Key Guiding Rule: Constant amplitude $\rightarrow$ Additive; grows with trend $\rightarrow$ Multiplicative; grows slightly $\rightarrow$ Mixed.

Identifying the Appropriate Time Series Model

- Additive Model ($Y_t = T_t + S_t + C_t + E_t$)
  - Use this when the magnitude of the seasonal swings and business cycle variations remains roughly constant regardless of whether the series level is high or low.

$$\frac{\partial X_t}{\partial T_t} = 1$$

- Multiplicative ($Y_t = T_t \cdot S_t \cdot C_t \cdot E_t$)
- Use this when the amplitude of the seasonal and cyclical fluctuations grows proportionally with the trend level.

$$\frac{\partial X_t}{\partial T_t} = S_t \cdot C_t \cdot E_t$$

- Applying the natural logarithm to both sides:

$$\ln(X_t) = \ln(T_t) + \ln(S_t) + \ln(C_t) + \ln(E_t)$$

- First difference of logs can be use to approximate growth rate.

Identifying the Appropriate Time Series Model

- Mixed Model ($Y_t = (T_t + C_t) \cdot S_t + E_t$)
  - When to choose: Use this when the data exhibits hybrid behavior–for instance, where seasonal variations scale proportionally with the level of the trend-cycle, but the irregular component or cyclical interactions behave differently.

$$\frac{\partial Y_t}{\partial T_t} = S_t \cdot \frac{\partial T_t}{\partial T_t} + 0 + 0 = S_t$$

THANKS!

Questions?