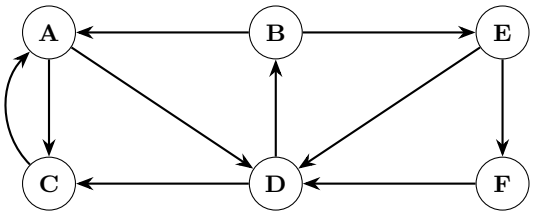
Mathematical Programming II

Chapter 3: Network Optimization

Academic Year 2026/2027

1. Consider the following directed graph G:

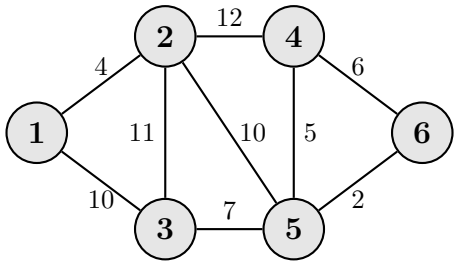


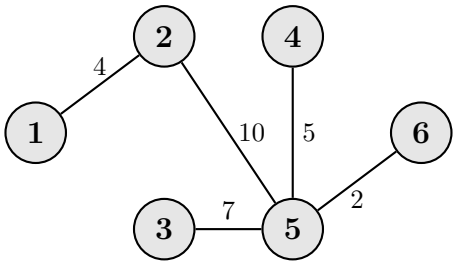
   (a) Find a simple, elementary path from node A to node F.

   (b) Ignoring arc directions, identify three distinct elementary chains from A to F and a cycle containing every node of the graph.

   (c) State, with justification, whether graph G is connected and, if so, whether it is strongly connected.

2. The following figure shows a graph G and its minimum spanning tree T^*:

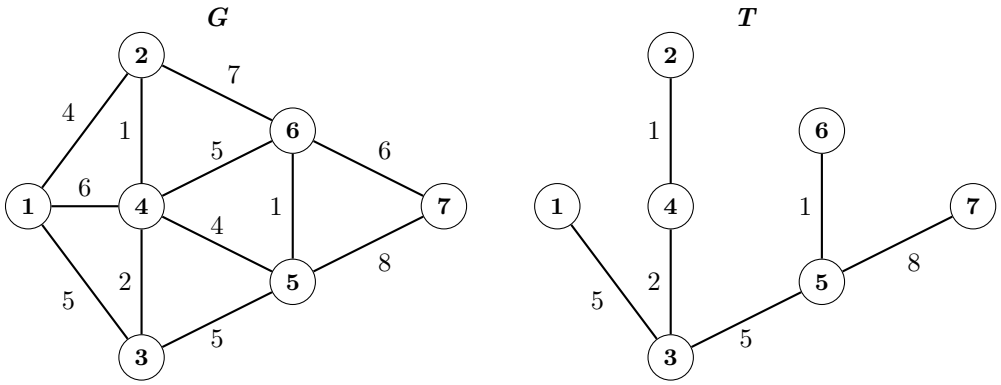




   (a) For each edge not belonging to the minimum spanning tree T^*, verify that T^* satisfies the cycle optimality condition.

   (b) For each edge not belonging to the minimum spanning tree T^*, verify that T^* satisfies the cut optimality condition.

3. The following figure shows a graph G and one of its spanning trees T:



(a) Using the cycle and cut optimality conditions, show that T is not optimal and propose a lower-cost spanning tree.

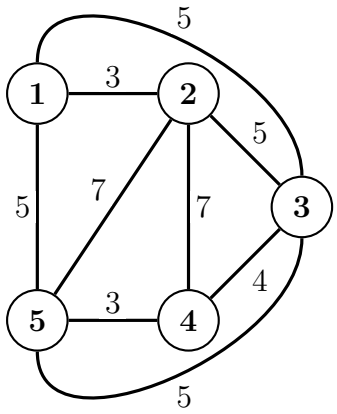
(b) How can you check whether alternative optimal solutions exist without enumerating all spanning trees?

(c) Find a minimum spanning tree:

i. Using Kruskal's algorithm.

ii. Using Prim's algorithm.

4. A city has five neighbourhoods experiencing theft and violence. The police have decided to install a computer network to facilitate communication between their respective police stations. The geographical arrangement of the neighbourhoods and the distances between them are shown in the following figure:



Underground cables are to be installed so that every neighbourhood can communicate with every other neighbourhood while minimising the total length of cable used.

(a) Solve this problem using an appropriate algorithm.

(b) Show that the solution is not unique and find an alternative optimal solution.

5. The *Wirehouse Lumber Company* will soon begin operations on eight forest plots in the same area. It therefore needs to develop a network of dirt roads allowing access from any plot to any other. The distances between pairs of plots, in distance units, are as follows:

	1	**2**	**3**	**4**	**5**	**6**	**7**	**8**
1	-	13	21	9	7	18	20	15
2	13	-	9	18	12	26	23	11
3	21	9	-	26	17	25	19	10
4	9	18	26	-	7	16	15	9
5	7	12	17	7	-	9	11	8
6	18	26	25	16	9	-	6	10
7	20	23	19	15	11	6	-	5
8	15	11	10	9	8	10	5	-

Management wishes to determine which roads should be built to connect all plots with minimum total length. Solve the problem using an appropriate algorithm and identify the roads to be built.

6. *Premiere Bank* will soon connect the computers at each of its branches to the central computer at headquarters. Dedicated telephone lines may connect branches directly or indirectly (via another branch). The cost of each line depends on the distance between locations and is given in the following table:

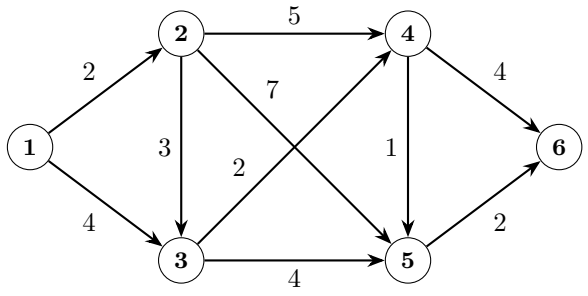
	Headquarters	**Branch 1**	**Branch 2**	**Branch 3**	**Branch 4**
Headquarters	–	190	70	115	270
Branch 1	190	–	100	110	215
Branch 2	70	100	–	140	120
Branch 3	115	110	140	–	175
Branch 4	270	215	120	175	–

Determine which lines should be installed to minimise the total connection cost.

7. Adapt Kruskal's and Prim's algorithms to find a maximum spanning tree of a graph.

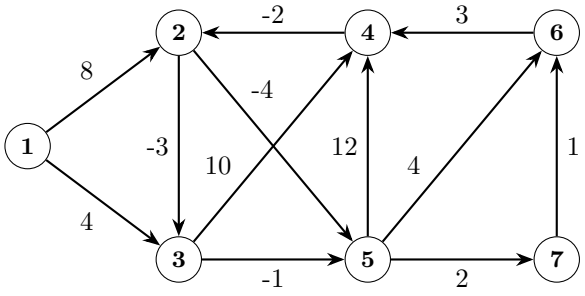
8. State whether the following statements are true or false, giving reasons:
   (a) In a spanning tree, there is a path between any nodes i and j; if the tree is a minimum spanning tree, then this path is a shortest path between i and j.
   (b) If a shortest path from node i to node j passes through node k, then the subpath from i to k is a shortest path from i to k.

9. Consider the following directed graph G where the values next to the arcs represent their lengths:



   (a) Use Dijkstra's algorithm to find a shortest path from node 1 to node 6.
   (b) Ignore arc directions and identify a simple but non-elementary chain from node 1 to node 6.

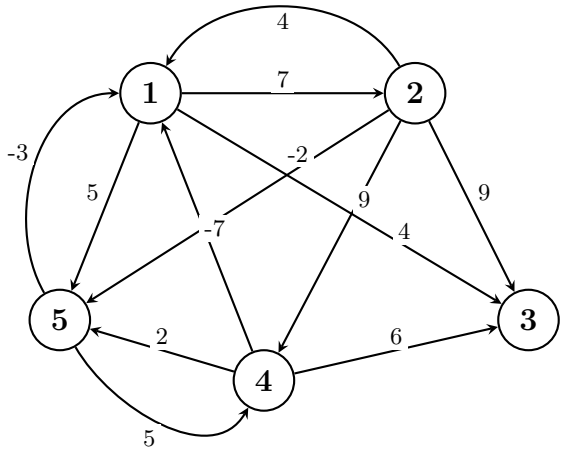
10. Consider the following graph:



   (a) Apply the Bellman–Ford algorithm to find shortest paths from node 1 to every other node.
   (b) Change the cost of arc $(2, 3)$ from -3 to -6 and repeat the exercise.

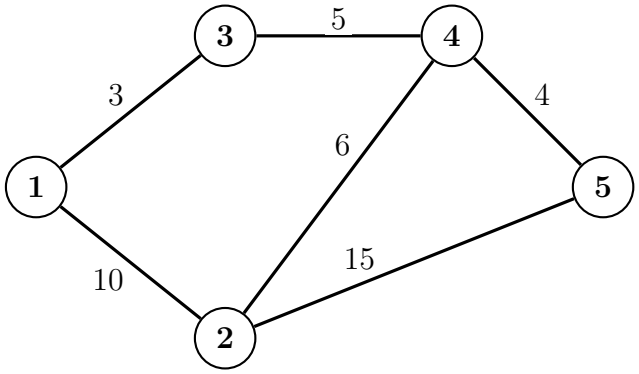
   In each case, identify any negative-cost directed cycles as soon as they are detected.

11. Now consider the following graph:



Apply Floyd's algorithm to find the shortest paths between all pairs of nodes. Give a shortest path from 1 to 4 and its cost.

12. Now consider the following graph:



(a) Perform two iterations of Floyd's algorithm.

(b) Suppose that applying Floyd's algorithm yields the following final matrices:

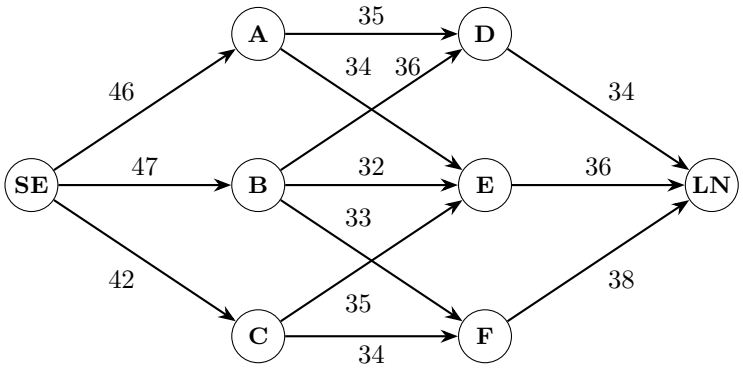
$$D^5 = \begin{bmatrix} - & 10 & 3 & 8 & 12 \\ 10 & - & 11 & 6 & 10 \\ 3 & 11 & - & 5 & 9 \\ 8 & 6 & 5 & - & 4 \\ 12 & 10 & 9 & 4 & - \end{bmatrix} \qquad P^5 = \begin{bmatrix} - & 1 & 1 & 3 & 4 \\ 2 & - & 4 & 2 & 4 \\ 3 & 4 & - & 3 & 4 \\ 3 & 4 & 4 & - & 4 \\ 3 & 4 & 4 & 5 & - \end{bmatrix}$$

Identify the following paths and their respective lengths:

i. Shortest path from 1 to 5.

ii. Shortest path from 1 to 5 passing through node 2.

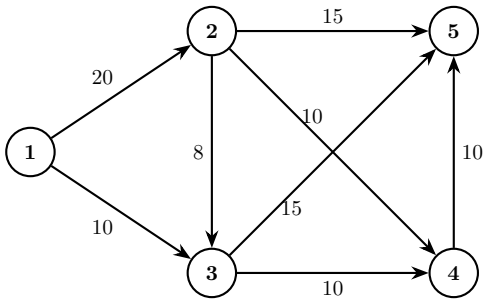
13. *Speedy Airlines* flights are about to depart from Seattle for London. There is some flexibility in route selection, which must account for weather conditions. The following network represents the routes under consideration, where *SE* and *LN* denote Seattle and London, respectively, and the other nodes represent intermediate locations.



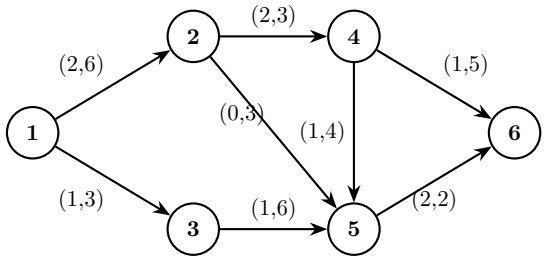
(a) What do the "distances" represent when modelling this as a shortest-path problem?

(b) Use an appropriate shortest-path algorithm to solve this problem.

14. Find the maximum flow from 1 to 5 in the following graph using the Ford–Fulkerson algorithm. The value next to each arc represents its capacity.



15. Consider the following graph, where the pair (x_{ij}, u_{ij}) next to each arc gives its current flow (x) and capacity (u).



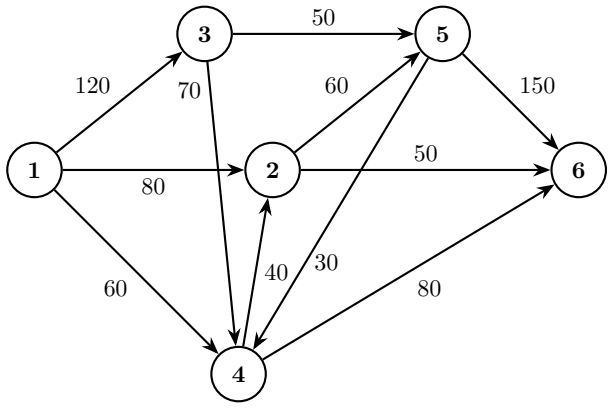
(a) Starting from the given flow, use the Ford–Fulkerson algorithm to find a maximum flow from node 1 to node 6 and identify a minimum-capacity cut separating them.

(b) Formulate this problem as a linear programme and formulate its dual.

16. Two paths in a graph are arc-disjoint if they share no arcs. Prove that the maximum number of arc-disjoint paths between two nodes equals the minimum number of arcs that must be removed to separate those nodes.

17. Consider a flow network with m nodes and n arcs. Let u_{ij} be the capacity of arc (i, j). Show that the value of any feasible flow from 1 to m does not exceed the capacity of any cut separating 1 from m.

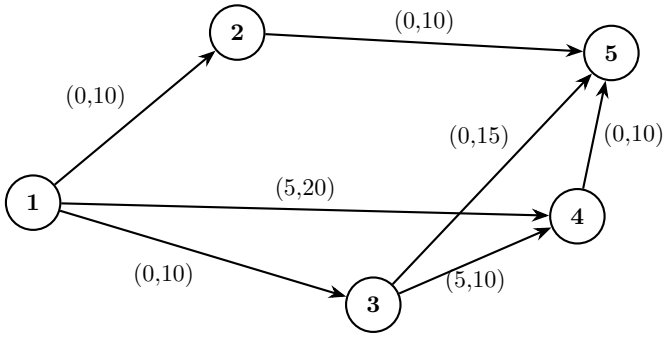
18. Consider the following graph, with the capacity indicated next to each arc.



(a) Use the Ford–Fulkerson algorithm to find a maximum flow from node 1 to node 6.

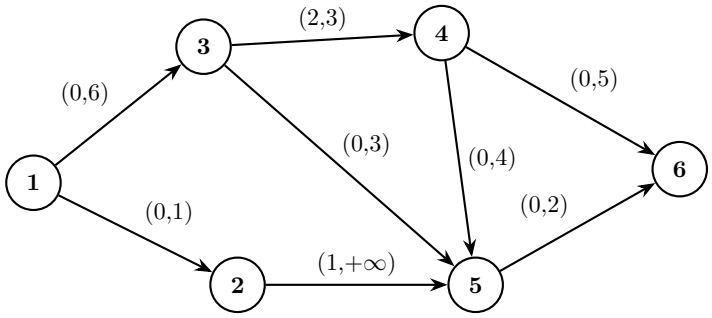
(b) Identify a minimum-capacity cut separating nodes 1 and 6.

19. Consider the following graph, where the lower and upper flow bounds (l_{ij}, u_{ij}) are shown next to each arc:



Taking as the initial flow $x_{12} = 0$, $x_{25} = 0$, $x_{14} = 5$, $x_{13} = 5$, $x_{34} = 5$, $x_{35} = 0$ and $x_{45} = 10$, use the Ford–Fulkerson algorithm to find a maximum flow from node 1 to node 5. What is its value? Identify a minimum-capacity cut separating 1 and 5. Show that the capacity of this cut equals the maximum flow value.

20. Consider the following graph, where the values next to each arc give its minimum and maximum flow capacities, in that order.



(a) Using the method presented in class, find a feasible flow from node 1 to node 6.

(b) Consider the following feasible flow from node 1 to node 6:

$$x_{12} = 1, \quad x_{13} = 3, \quad x_{25} = 1, \quad x_{34} = 3, \quad x_{35} = 0, \quad x_{45} = 1, \quad x_{46} = 2, \quad x_{56} = 2$$

use the Ford–Fulkerson algorithm to find a maximum flow from 1 to 6. Identify the optimal solution.

(c) Identify a minimum-capacity cut separating 1 from 6.

(d) Comment on the following statement: *"To increase the maximum flow from 1 to 6 by two units, it is sufficient to increase the capacity of arc (1,2) by two units."*

21. Determine the maximum number of daily flight connections possible between Juneau (Alaska) and Dallas (Texas).

There are no direct flights between the two cities, but connections are possible via Seattle and Denver or via Los Angeles. The available numbers of daily flights are as follows:

From	To	Number of daily flights
Juneau	Seattle	3
Seattle	Los Angeles	2
Seattle	Denver	3
Denver	Dallas	2
Los Angeles	Dallas	1

Model this as a network optimisation problem and explain how to determine an optimal solution.

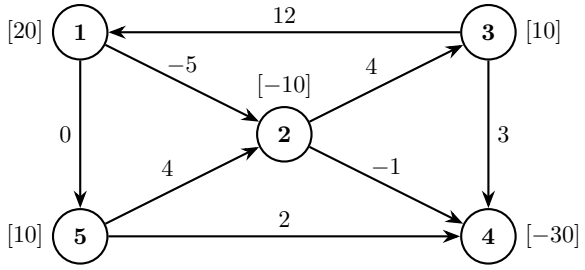
22. A hospital transplant service has information on five kidney donors (**1**, **2**, **3**, **4**, **5**) and five patients (**A**, **B**, **C**, **D**, **E**) awaiting transplantation. The following table shows donor–patient compatibility.

	A	**B**	**C**	**D**	**E**
1		X			
2	X				
3	X	X			
4	X	X			X
5			X	X	X

Formulate a network optimisation problem to determine the maximum number of transplants that can be performed for patients **A** to **E** using donors **1** to **5**.

23. Formulate the problem presented in Exercise 14 as a minimum-cost flow problem.

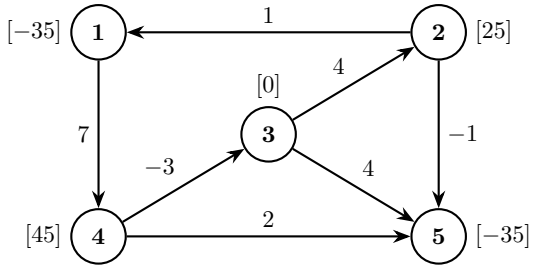
24. Consider the following instance of a minimum-cost flow problem, with the unit cost indicated next to each arc.



(a) Take the following arcs as basic arcs: $(1, 2)$, $(2, 4)$, $(5, 4)$ and $(3, 1)$.
   i. Construct the basic feasible solution (BFS) associated with these arcs.
   ii. Show that this BFS is not optimal.

(b) Starting from the given BFS, use the Network Simplex method to find a lower-cost solution.

25. Consider the following instance of a minimum-cost flow problem, with the unit cost indicated next to each arc:



(a) Take the following arcs as basic arcs: $(2,1)$, $(3,2)$, $(2,5)$ and $(4,3)$.

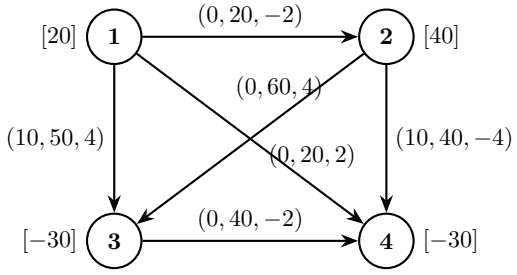
i. Construct the basic feasible solution (BFS) associated with these arcs.

ii. Show that this BFS is optimal. State whether alternative optimal solutions exist.

(b) Starting from the given solution, bring arc $(4,5)$ into the basis and determine the new BFS, indicating how much the solution cost increases.

26. Explain how the Network Simplex algorithm could be adapted to solve maximisation problems.

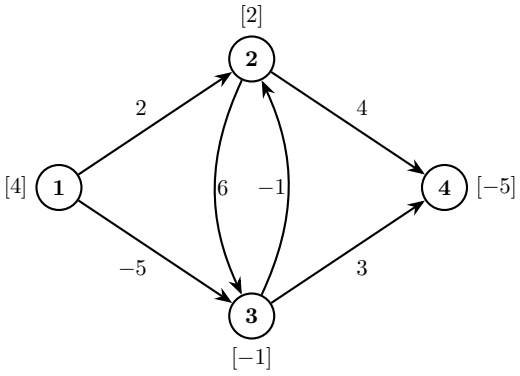
27. Consider the following problem with capacities and costs (l_{ij}, u_{ij}, c_{ij}):

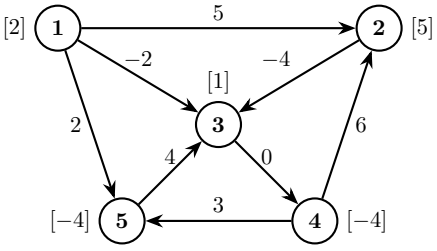


(a) Construct a BFS using the variables associated with arcs $(1,4)$, $(2,3)$ and $(3,4)$ as basic variables.

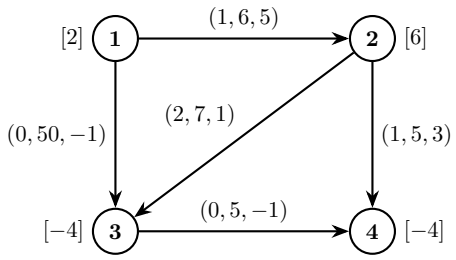
(b) Starting from the solution obtained in the previous part, find an optimal solution using the Network Simplex method for bounded variables.

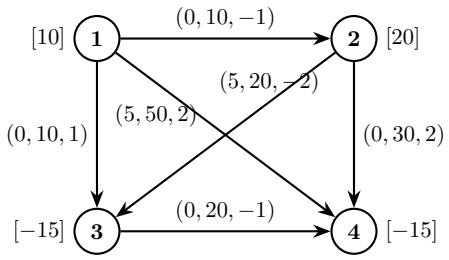
28. Solve the following problems using the Network Simplex method.





29. Solve the following problems using the Network Simplex method.





30. A factory has three production stations capable of producing up to 600, 650 and 750 units of an item, respectively, per production period. Maintenance requirements mean that each station must produce at least 30% of its maximum capacity. Because the stations have different characteristics, their unit production times are 10, 13 and 20 minutes for stations 1, 2 and 3, respectively. After production, items leave each station on one of two conveyor belts leading to one of two labelling machines. The maximum capacities of the belts (in units) and their transport times (in minutes per unit) are given in the following table.

Station	Machine 1 (max. cap. / time)	Machine 2 (max. cap. / time)
1	350 / 1.00	300 / 0.75
2	- / -	350 / 1.00
3	300 / 0.50	300 / 1.00

The two labelling machines are identical, each with a maximum capacity of 900 units per production period. Labelling takes 0.10 minutes per unit. The items then leave the factory and are sent to three distribution points, each requiring 400 units per production period. Formulate a network model to determine the production plan that minimises total time spent within the factory.

31. A television station is planning its coverage of the *Portugal-Best* festival at Parque da Paz in Almada. The main challenge concerns the final day, when several important events will take place. Their start times, expected durations and locations are known.

Events	Start time	Duration	Location
Sons da Terra Group	15:00	1:30	(L1)
Algarve Folk Group	17:00	1:00	(L2)
Alentejo Singing Group	16:30	1:00	(L3)
Celtas do Minho Group	21:00	3:00	(L4)
Northern Dance Group	18:15	1:00	(L5)

The travel times (in minutes) for a reporting crew between the various event locations are also known:

	L1	L2	L3	L4	L5
L1	0	5	10	2	-
L2	5	0	3	12	20
L3	10	3	0	-	-
L4	2	12	-	0	-
L5	-	20	-	-	0

(- means that the corresponding journey is not possible for a reporting crew)

The television manager wishes to determine the minimum number of reporting crews that must be sent to the *Portugal-Best* venue to ensure coverage of all events. Formulate this as a minimum-cost flow problem on a network.

32. Formulate each of the following problems as a minimum-cost flow problem. Give both the network formulation and the linear programming formulation.

(a) Transportation problem.

(b) Shortest-path problem.

(c) Minimum-cost flow problem.

33. Represent the following activity projects using activity-on-arc networks.

(a) Activities: A, B, C, D, E, F, G, H e I.
Precedence relationships: A precedes D and E; B precedes F; C precedes G and H; D precedes F; and F and G precede I.

(b) Activities: A, B, C, D, E e F.
Precedence relationships: A and B precede C; A precedes D; B precedes E; and D precedes F.

(c) Activities: A, B, C, D, E, F, G, H e I.
Precedence relationships: C and D follow A and B; E follows B; F and H follow D and E; G follows C and F; and I follows G and H.

(d) Activities: A, B, C, D, E, F, G, H, I, J, K, L, M, N, O e P.
Precedence relationships: A, B and C are the first tasks and may start simultaneously; D, E and F may start only after A is completed; I and G may start only after B and D are completed; H may start after C and G are completed; K and L follow I; J follows E and H; M and N follow F but may start only after E and H are completed; O follows M and I; P follows J, L and O; and K, N and P are the final tasks of the project.

34. Represent the following projects using activity-on-arc networks.

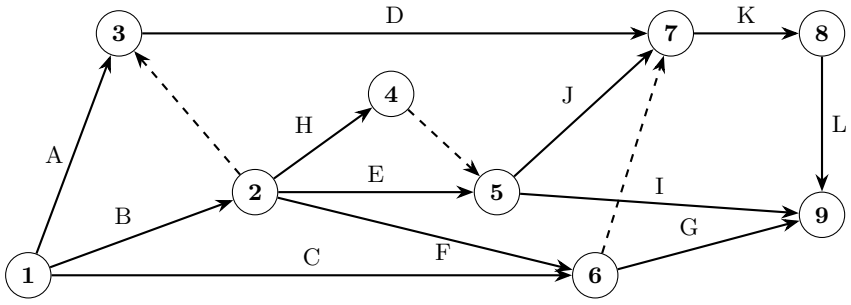
(a)

Activity	A	B	C	D	E	F	G	H
Predecessors	-	-	-	-	A,C,D	A,C,D	A,C,D	E,F,G

(b)

Activity	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O
Predecessors	-	A	A	C	C	C	D,E,F	G	H	H	I,J	K	L	L	B,M,N

35. Consider the representation of a project with nodes numbered from 1 to 9 and arcs representing activities A to L.



(a) For each activity, identify the activities that precede and follow it.

(b) Identify the new direct precedence relationships resulting from adding each of the following dummy activities:
   i. (3,5);
   ii. (3,4);
   iii. (5,6);
   iv. (3,6);

(c) Explain how the relationship "D precedes G" can be incorporated into the network.

36. A Mathematical Programming II student wants to make chocolate mousse. Faced with the difficulty of this task, the student decides to use project management techniques to ensure the best possible result. The following activities are identified:

P: break the eggs and separate the yolks from the whites (5 minutes).

D: melt a chocolate bar in a bain-marie (10 minutes).

B: beat the egg whites until stiff (6 minutes).

M: mix the yolks with the melted chocolate (3 minutes).

E: fold the egg whites into the yolk and chocolate mixture (4 minutes).

The chocolate must be melted before being mixed with the yolks. The whites must be beaten and the yolks mixed with the chocolate after the eggs have been broken and before the whites are folded into the mixture.

(a) Represent this project using an activity-on-arc network.

(b) Determine the minimum time required to prepare the mousse.

(c) Can the student complete this difficult project in the minimum time without help from another person?

(d) If help is needed, how should the work be divided so that the previously determined minimum time is not exceeded?

37. Consider a project with 10 activities. The duration (in days) of each activity is as follows:

Activity	A	B	C	D	E	F	G	H	I	J
Duration	10	5	3	4	5	6	5	6	6	4

The precedence relationships are: A and B are the initial activities; B precedes C; A and C precede D and E; E precedes F; D precedes G and H; H and F precede I; G precedes J; and I and J are the final activities of the project.

(a) Represent the project using an activity network.

(b) Determine the earliest and latest event times for each node, as well as the critical path and the minimum project duration.

(c) For each task, determine:

i. the earliest start time;

ii. the latest start time;

iii. the earliest finish time;

iv. the latest finish time;

v. the slack.

(d) Identify the tasks that can be delayed without delaying project completion.

(e) Identify the tasks that can be delayed (within their slack) without delaying the start of subsequent tasks.

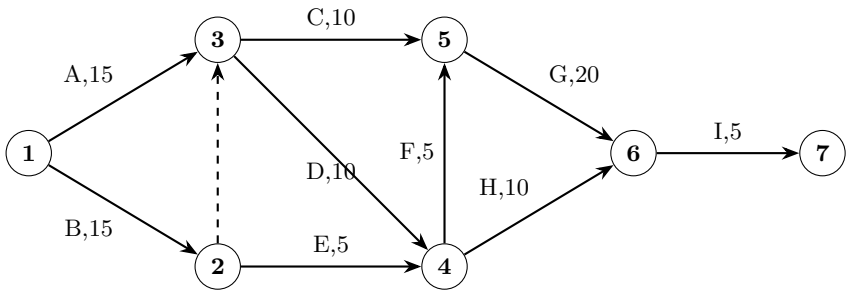
(f) If task E took 7 days, would the project completion date change?

(g) If task B were delayed by 1 day, would the project completion date change? What if the delay were 4 days?

(h) Draw the project schedule.

(i) Suppose each task is carried out by one (non-specialised) worker. What is the minimum number of workers needed to complete the project at the earliest possible date?

38. Consider a project represented by the following activity network:



(a) Determine the minimum project duration and identify the critical activities and critical path(s).

(b) Verify that all nodes have zero slack.

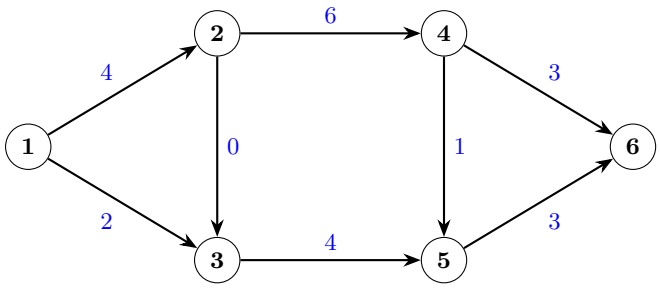
(c) Due to difficulties in obtaining skilled labour, activity C may be delayed by 5 time units. What are the consequences of this delay?

(d) Comment on the following statements:

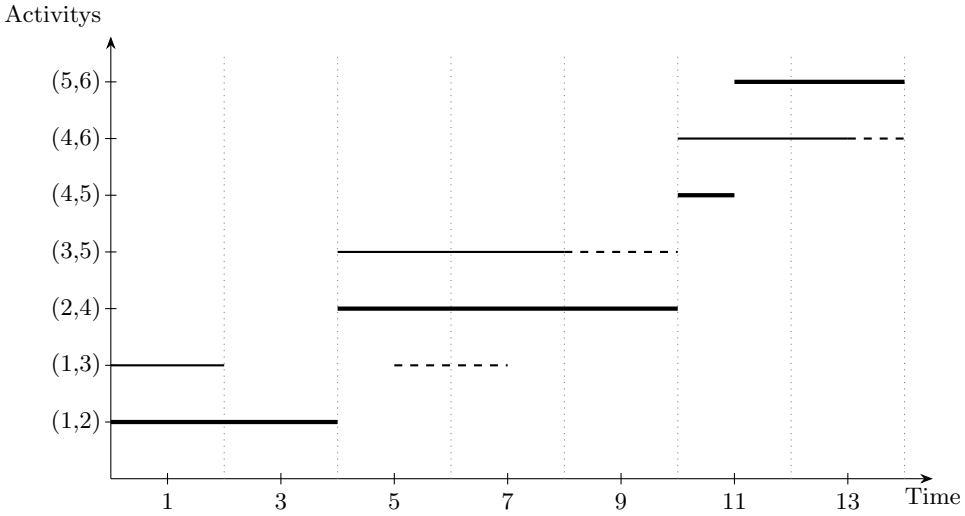
i. "Regardless of the activity durations, activity I would always be critical."

ii. "Since all nodes have zero slack, there must necessarily be at least two critical paths."

39. Assume that the graph is an activity network for a project, where activity (2,3) is a dummy activity (duration 0) and the remaining activities have the specified durations.



Consider execution based on the activity schedule.



(a) Using only the schedule, answer the following questions and justify your answers:

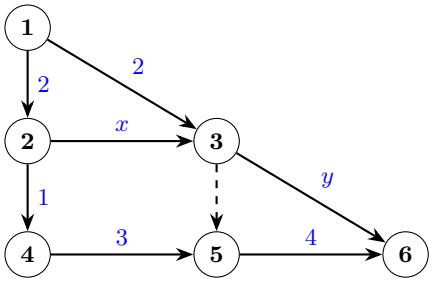
i. What is the minimum time required to complete the project?

ii. What is the critical path?
iii. Does a one-day delay in task (2,4) affect the minimum project duration?
iv. Does a two-day delay in task (3,5) affect the earliest project completion date?

(b) Now also consider the network associated with the project.
i. What is the impact of a three-day delay in task (1,3)?
ii. Identify the direct predecessors of activity (3,5).
iii. It has been established that task (3,5) must be carried out after task (2,4). How would you introduce this new precedence relationship into the network?
iv. How would you modify the network if task (3,5) had to be carried out before task (2,4)?

40. Construct an example showing that, in an activity network, a path consisting only of zero-slack nodes is not necessarily a critical path.

41. Consider the following network associated with a project.

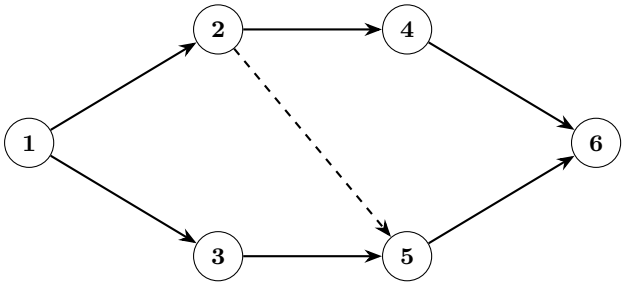


The durations of activities $(2,3)$ and $(3,6)$ are unknown and are represented by the variables x and y, respectively.

(a) Identify the activities that directly precede activity (5,6).

(b) Determine the values of x and y for which the path $((1,2),(2,3),(3,6))$ is critical.

42. Consider the following activity network for a project that must be completed within 13 weeks:

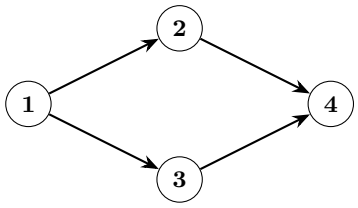


A cost analysis is required to determine the duration assigned to each activity so that the project can be completed by the deadline. The relevant information is given in the following table:

	Duration (weeks)		**Cost (monetary units)**	
Activity	normal	crashed	normal	crashed
(1,2)	6	4	42	62
(1,3)	2	1	18	33
(2,4)	4	2	54	80
(3,5)	5	3	36	58
(4,6)	7	4	59	95
(5,6)	9	6	63	87

Formulate the problem as a linear programming model.

43. Consider a project represented by the following activity network, which must be completed within 12 time units.



The normal and crashed durations of the activities, together with their respective costs, are given in the following table:

	Duration (time units)		**Cost (monetary units)**	
Activity	normal	crashed	normal	crashed
(1,2)	8	5	25	40
(1,3)	9	7	20	30
(2,4)	6	4	16	24
(3,4)	7	4	27	45

(a) Consider only the sequence of activities, $\{(1,2),(2,4)\}$.

i. Formulate a linear programming model to determine the duration assigned to each of those activities so as to minimise the cost of completing the sequence within 12 time units.

ii. Solve the resulting problem graphically.

(b) Formulate a linear programming model to determine the duration assigned to each of the four project activities so that the project can be completed within 12 time units.

44. A concert promoter has signed a contract with a famous rock band to perform in Lisbon next year. Several activities must be carried out to ensure that everything is ready by the concert date. Information about the activities is provided in the following table.

Activity	**Description**	**Precedence**	**Duration (days)**
A	Check the suitability of the selected concert venue	—	4
B	Hire sound and lighting engineers	—	3
C	Hire the opening act	A	4
D	Advertise the concert	C	3
E	Put tickets on sale	A	5
F	Prepare the venue sound and lighting	A, B	6
G	Arrange transport and accommodation for the bands	C	4
H	Sound and lighting rehearsals	F, G	5
I	Last-minute arrangements and rehearsal	H	2

(a) What is the minimum time required to complete these activities?

(b) Determine the earliest and latest start and finish times, as well as the slack of each activity.

(c) If one of the hired sound engineers is absent for three days, preparing the venue (activity F) will take two days longer than planned. What is the impact on the minimum project duration?

(d) The promoter wants to know whether the project duration can be reduced to 16 days by shortening some activities. The following table gives the total crashing costs and minimum durations of the activities.

Activity	**Minimum duration**	**Total crashing cost**
A	3	450
B	2	500
C	4	—
D	2	750
E	3	2000
F	4	900
G	3	200
H	5	—
I	1	400

Formulate the linear programming model and solve it using a *solver*.

45. A PERT analysis of a project produced the following table of activity durations (in days):

Activity	**Most likely duration (m_{ij})**	**Optimistic duration (a_{ij})**	**Pessimistic duration (b_{ij})**
(1,2)	10	5	30
(2,3)	12	10	14
(2,4)	2	1	3
(2,5)	8	4	12
(3,6)	20	10	42
(3,4)	15	12	18
(4,6)	6	4	8
(4,7)	20	16	24
(5,7)	12	10	17
(5,9)	6	4	11
(6,8)	16	10	43
(7,8)	5	2	8
(8,10)	4	3	5
(9,10)	4	3	5
(10,11)	8	7	9

(a) Calculate the expected duration of the critical path and its variance.

(b) What is the probability of completing the project within 80 days?

(c) What is the probability that path (1,2,3,4,7,8,10,11) has a duration of at most 80 days? Comment on the result by comparing it with the answer to part (b).

(d) Suppose you are the project manager. Explain what completion deadline you would commit to.

46. Bill Fredlund, president of *Lincoln Log Construction*, is considering submitting a bid for a construction project. Bill has determined that five tasks are required to complete the project. Using PERT, he obtained the following duration estimates for these tasks. The precedence relationships are also shown.

	Time required			
Task	a_{ij}	m_{ij}	b_{ij}	**Predecessors**
A	3 weeks	4 weeks	5 weeks	—
B	2 weeks	2 weeks	2 weeks	A
C	3 weeks	5 weeks	6 weeks	B
D	1 week	3 weeks	5 weeks	A
E	2 weeks	3 weeks	5 weeks	B, D

There is a penalty of $500,000 if the project is not completed within 11 weeks. Bill is therefore very interested in the probability that his company will finish the project on time.

(a) Construct the activity network for this project.

(b) Estimate the mean and variance of each activity duration.

(c) Determine the critical path.

(d) Determine the approximate probability of completing the project within 11 weeks.

(e) Bill concluded that the bid required to have a realistic chance of winning the contract would give *Lincoln Log Construction* a profit of approximately \$250,000 if the project is completed within 11 weeks. However, because of the late-completion penalty, the company would lose approximately \$250,000 if the project takes longer than 11 weeks. He therefore wants to submit the bid only if the probability of meeting the deadline is at least 50%. What would you advise?