Lecture 2: The Stochastic Cash-in-Advance Model

Cole, Chapters 5, 6 and 7

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Stochastic Model

- Assume productivity growth rate g_t and money growth rate τ_t follow stochastic processes
- Denote the state by

$$s_t = (Z_{t-1}, \bar{M}_{t-1}, \tau_t, g_t).$$

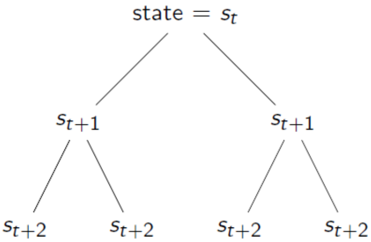
  - $Z_t = (1 + g_t)Z_{t-1}$ and $\bar{M}_t = (1 + \tau_t)\bar{M}_{t-1}$ just depends upon s_t.
  - Need current shocks to forecast tomorrow.
- Need history of events, so define the history state as

$$s^t = \{s_1, s_2, \ldots, s_t\}.$$

- History important for stocks variables like wealth and capital.
- Assume the processes are simple first-order Markov process: i.e.

$$Pr\{s_{t+1}|s^t\} = Pr\{s_{t+1}|s_t\}$$

Figure: Event Tree



- History state gives unique path through event tree.
- $s^{t-1}(s^t)$ gives unique predecessor. $S^{t+1}(s^t)$ for set of successors or $s^{t+1} \succ s^t$ successor.

Stochastic Model

- We will need to talk about probabilities and the probability of a given history state s^t is $Pr\{s^t\}$.
- The probability of a history state tomorrow given one today is

$$Pr\{s^{t+1}|s^t\} \equiv \frac{Pr\{s^{t+1}\}}{Pr\{s^t\}}$$

if $s^{t+1} \succ s^t$ and 0 otherwise.

Stochastic Model

- Our choice variables are $C(s^t)$, $L(s^t)$, $M(s^t)$, $B(s^t)$.
- And we inherit our past choices $M(s^{t-1}(s^t))$, $B(s^{t-1}(s^t))$
- Face prices $P(s^t)$, $q(s^t)$
- Our payoff is the discounted probability weighted sum of our flow payoffs

$$\begin{aligned}
&\sum_{t=1}^{T}\sum_{s^t}\beta^{t-1}\left[u(C(s^t))-v(L(s^t))\right]Pr\{s^t\} \\
&+\beta^T\sum_{s^{T+1}}V(M(s^T),\ B(s^T),\ s^{T+1})Pr\{s^{T+1}\} \\
=&E\left\{\begin{array}{c}\sum_{t=1}^{T}\beta^{t-1}\left[u(C(s^t))-v(L(s^t))\right] \\ +\beta^T V(M(s^T),\ B(s^T),\ s^{T+1})\end{array}\right\}
\end{aligned}$$

Stochastic Model

- The Lagrangian for this problem is kind of similar to what we had before:

$$
\begin{aligned}
\mathcal{L} \quad = \quad & \max_{\{L(s^t),\, C(s^t),\, M(s^t),\, B(s^t)\}_{t=1}^{T}} \ \min_{\{\mu(s^t),\, \lambda(s^t)\}_{t=1}^{T}} \\
& E\left\{ \begin{array}{c} \sum_{t=1}^{T} \beta^{t-1} \left[u(C(s^t)) - v(L(s^t)) \right] \\ +\beta^{T} V(M(s^T),\ B(s^T),\ s_{T+1}) \end{array} \right\} \\
& + \sum_{t=1}^{T} \sum_{s^t} \lambda(s^t) \left\{ M(s^{t-1}(s^t)) - P(s^t) C(s^t) \right\} \\
& + \sum_{t=1}^{T} \sum_{s^t} \mu(s^t) \left\{ \begin{array}{c} P(s^t) Z(s^t) L(s^t) + \left[M(s^{t-1}(s^t)) - P(s^t) C(s^t) \right] \\ +B(s^{t-1}(s^t)) + T(s^t) - M(s^t) - q(s^t) B(s^t) \end{array} \right\}
\end{aligned}
$$

- Have a cia and budget constraint for each history state.
- Have separate multipliers for each history state since bind differentially.

Stochastic Model

$$\beta^{t-1} u'(C_t(s^t)) \Pr(s^t) - \left[\lambda(s^t) + \mu(s^t)\right] P(s^t) = 0.$$

$$-\beta^{t-1} v'(L_t(s^t)) \Pr(s^t) + \mu(s^t) P(s^t) Z(s^t) = 0.$$

$$-\mu(s^t) + \sum_{s^{t+1} \in S^{t+1}(s^t)} \left[\lambda(s^{t+1}) + \mu(s^{t+1})\right] = 0.$$

$$-\mu(s^t) q(s^t) + \sum_{s^{t+1} \in S^{t+1}(s^t)} \mu(s^{t+1}) = 0.$$

- Because the impact of increasing money or bond holds impacts all successor nodes, we get a summation term.

Determining the Equilibrim

- The steps here are the same: (1) impose market clearing + (2) cash-in-advance constraint
  - MKt. Clearing $\implies C_t = Z_t L_t$ and $M_t = \bar{M}_t$.
  - CIA $\implies$

$$P(s^t) = \frac{\bar{M}(s^{t-1}(s^t))}{Z(s^t)L(s^t)}. \tag{1}$$

- Make our standard preference assumptions: $u(c) = \log(c)$ and $v(L) = \frac{L^{1+\gamma}}{1+\gamma}$
- So our consumption and labor conditions become:

$$\beta^{t-1} \Pr(s^t) = \left[\lambda(s^t) + \mu(s^t)\right] \bar{M}_{t-1}(s_t)$$

$$\beta^{t-1} L_t(s^t)^{\gamma} \Pr(s^t) = \mu(s^t) \frac{\bar{M}_{t-1}(s_t)}{L(s^t)}$$

Stochastic Model

- Small variation on our old change-in-variables will work:

$$
\begin{aligned}
\mu(s^t) &= \frac{\beta^{t-1}\tilde{\mu}(s^t)\,\Pr(s^t)}{\bar{M}(s^{t-1}(s^t))}, \text{ and} \\
\lambda(s^t) &= \frac{\beta^{t-1}\tilde{\lambda}(s^t)\,\Pr(s^t)}{\bar{M}(s^{t-1}(s^t))}.
\end{aligned}
$$

Stochastic Model

- Using this change, rewrite our f.o.c.'s for consumption and labor as

$$1 = \left[\tilde{\lambda}(s^t) + \tilde{\mu}(s^t)\right],$$

$$L(s_t)^{\gamma} = \tilde{\mu}(s^t)\left[\frac{1}{L(s_t)}\right].$$

$$\tilde{\mu}(s^t)\,\mathrm{Pr}(s^t) = \sum_{s^{t+1}\in S^{t+1}(s^t)}\left[\tilde{\mu}_{t+1}(s^{t+1}) + \tilde{\lambda}_{t+1}(s^{t+1})\right]\frac{\beta}{(1+\tau_t(s^t))}\,\mathrm{Pr}(s^{t+1})$$

$$\tilde{\mu}(s^t)q(s^t)\,\mathrm{Pr}(s^t) = \sum_{s^{t+1}\in S^{t+1}(s^t)}\tilde{\mu}(s^{t+1})\frac{\beta}{(1+\tau_t(s^t))}\,\mathrm{Pr}(s^{t+1}).$$

Stochastic Model

- Using this change, rewrite our f.o.c.'s for consumption and labor as

$$1 = \left[\tilde{\lambda}(s^t) + \tilde{\mu}(s^t)\right],$$

$$L(s_t)^\gamma = \tilde{\mu}(s^t)\left[\frac{1}{L(s_t)}\right].$$

$$\tilde{\mu}(s^t) = \frac{\beta}{(1+\tau_t(s^t))}$$

$$q(s^t) = \sum_{s^{t+1}\in S^{t+1}(s^t)} \frac{\beta}{(1+\tau_{t+1}(s^{t+1}))} \Pr(s_{t+1}|s_t)$$

q_t is simply the real discount rate times the expected component of inflation coming from the expected increase in the money supply.

- Nothing in these equations reflects the past, and hence everything is a function of the current state s_t only.
- Finally, labor takes a very simple form $L_t = [\beta/(1+\tau_t)]^{1/(1+\gamma)}$.

Stochastic Model

- In the bond price equation,

$$q(s_t) = \sum_{s_{t+1}} \frac{\beta}{(1+\tau_{t+1}(s_{t+1}))} \Pr(s_{t+1}|s_t)$$

  we are trying to forecast next period's growth rate from the current state.
- Really this is all about

$$\sum_{\tau_{t+1}} \frac{1}{1+\tau_{t+1}} Pr\{\tau_{t+1}|\tau_t, g_t\}.$$

- The g_t terms comes in just if there is any correlation between the two.
- Note that

$$1/E\{x\} \neq E\{1/x\}$$

- The labor equation is just analytic, so one is going to be easy

$$L_t = [\beta/(1+\tau_t)]^{1/(1+\gamma)}.$$

Stochastic Model

- From the labor equation

$$L_t = \left[\beta/(1+\tau_t)\right]^{1/(1+\gamma)}.$$

- Obtain

$$\frac{dL_t}{d\tau_t} < 0.$$

  and

$$\frac{dL_t}{dg_t} = 0$$

Stochastic Computer Model

- Our random variables are going to follow simple AR1 processes in which

$$x_t = Ax_{t-1} + B + C\varepsilon_t,$$

- So long as the expected value of $\varepsilon_t = 0$,

$$\mathbb{E}\{x_t\} = A\mathbb{E}\{x_{t-1}\} + B.$$

Since this is an unconditional expectation, $\mathbb{E}\{x_t\} = \mathbb{E}\{x_{t-1}\}$, hence

$$\mathbb{E}\{x_t\} = (I - A)^{-1} B,$$

Stochastic Computer Model

- Next, note that we can recursively substitute to get that

$$
\begin{aligned}
x_t &= B + C\varepsilon_t + A\left\{B + C\varepsilon_{t-1}\right\} + A^2\left\{B + C\varepsilon_{t-2}\right\} + \ldots \\
&= (I - A)^{-1}B + C\sum_{j=0}^{\infty} A^j \varepsilon_{t-j}.
\end{aligned}
$$

- This is because

$$I + A + A^2 + A^3 + \ldots = (I - A)^{-1}.$$

- A is governing how persistent a shock is:
  - First period $C\varepsilon_t$
  - Second period $AC\varepsilon_t$
  - Third period $A^2C\varepsilon_t$

Stochastic Computer Model

- In our set-up, x_t will be a vector

$$x_t = \begin{bmatrix} \tau_t \\ g_t \end{bmatrix}.$$

- And our AR1 equation is

$$x_t = \begin{bmatrix} \rho_\tau & 0 \\ 0 & \rho_g \end{bmatrix} x_{t-1} + \begin{bmatrix} B_\tau \\ B_g \end{bmatrix} + \begin{bmatrix} \sigma_\tau & 0 \\ 0 & \sigma_g \end{bmatrix} \begin{bmatrix} \varepsilon_{\tau,t} \\ \varepsilon_{g,t} \end{bmatrix},$$

- So, the long-run means here is

$$\begin{bmatrix} B_\tau / (1 - \rho_\tau) \\ B_g / (1 - \rho_g) \end{bmatrix}$$

Stochastic Computer Model

- In our theory we talk about $Pr\{s_{t+1}|s_t\}$. What does this mean?
- Start from our state vector $s_t = (Z_{t-1}, \bar{M}_{t-1}, \tau_t, g_t)$.
- The only state s_{t+1} that can follow must have $Z_t = Z_{t-1}(1+g_t)$ and $\bar{M}_t = (1+\tau_t)\bar{M}_{t-1}$. So:

$$s_{t+1} = (Z_{t-1}(1+g_t), (1+\tau_t)\bar{M}_{t-1}, \tau_{t+1}, g_{t+1})$$

- However, there is a wide range of possible values for τ_{t+1} and g_{t+1}.
- We need to put probabilities on these possible outcomes.
- Once we do, the probability of

$$s_{t+1} = (Z_{t-1}(1+g_t), (1+\tau_t)\bar{M}_{t-1}, \tau_{t+1}, g_{t+1})$$

is simply

$$Pr\{s_{t+1}|s_t\} = Pr\{\tau_{t+1}, g_{t+1})|\tau_t, g_t)$$

Stochastic Computer Model

- For a given τ_{t+1} to follow from τ_t, it must be the case that the realized shock satisfies

$$\tau_{t+1} = \rho_\tau \tau_t + B_\tau + \sigma_\tau \varepsilon_{\tau,t+1}.$$

- Hence, there is a unique shock associated with this realization.
- Denote the probability of that shock as

$$\Pr(\varepsilon_{\tau,t+1}) = \Pr\left(\frac{\tau_{t+1} - \rho_\tau \tau_t - B_\tau}{\sigma_\tau}\right).$$

- Similarly if g_{t+1} is the realized growth rate, then the associated shock must satisfy

$$g_{t+1} = \rho_g g_t + B_g + \sigma_g \varepsilon_{g,t+1}$$

and hence the probability of that realization is

$$\Pr\left(\varepsilon_{g,t+1}\right) = \Pr\left(\frac{g_{t+1} - \rho_g g_t - B_g}{\sigma_g}\right).$$

Stochastic Computer Model

- Thus, the conditional probability of going from $s_t = (Z_{t-1}, \bar{M}_{t-1}, \tau_t, g_t)$ to

$$s_{t+1} = (Z_{t-1}(1+g_t), (1+\tau_t)\bar{M}_{t-1}, \tau_{t+1}, g_{t+1})$$

  is given by

$$\Pr(s_{t+1}|s_t) = \Pr\left(\frac{\tau_{t+1} - \rho_\tau \tau_t - B_\tau}{\sigma_\tau}\right) \Pr\left(\frac{g_{t+1} - \rho_g g_t - B_g}{\sigma_g}\right).$$

- This last bit follows from our independence assumption for our two shocks.

Stochastic Computer Model

- Our economy can be boiled down to two key equations.
- The first is our labor supply condition

$$L(s_t)^{1+\gamma} = \frac{\beta}{(1+\tau_t(s_t))},$$

$$\Rightarrow L(s_t) = \left[\frac{\beta}{(1+\tau_t(s^t))}\right]^{\frac{1}{1+\gamma}}. \tag{2}$$

Stochastic Computer Model

- The second equation is our interest rate condition:

$$q(s_t) = \sum_{s^{t+1} \in S^{t+1}(s^t)} \left[\frac{\beta}{(1 + \tau_{t+1}(s^{t+1}))} \right] \Pr(s_{t+1}|s_t). \tag{3}$$

This equation is a bit more complicated, since it involves evaluating an integral.

- Approximating an integral can be done in clever and efficient ways, but we are not going to bother. Instead, we need only draw enough ε_{t+1} shocks using a random number generator to get a representative distribution and then use them in the approximation.

$$q(s_t) \simeq \frac{1}{N} \sum_{i=1}^{N} \left[\frac{\beta}{(1 + \rho_\tau \tau_t + B_\tau + \sigma_\tau \varepsilon_i)} \right]. \tag{4}$$

Stochastic Computer Model

- To compute the equilibrium of the model we choose the parameters $\left\{\rho_\tau, B_\tau, \sigma_\tau, \rho_g, B_g, \sigma_g, \gamma\right\}$
- Then we need to draw shocks using a random number generator and construct a realized sequence of τ_t and g_t
- Then construct the sequence L_t, and q_t.

Stochastic Computer Model

- We've focused on the stationary random variables implied by our model. We can construct the nonstationary one too.
- We can construct the money supply recursively using the fact that

$$M_t = M_{t-1}(1 + \tau_t)$$

  where we take $M_0 = 1$.
- We can do the same thing for productivity, Z_t.
- Consumption is given by

$$C_t = Z_t L_t.$$

- The price level is given by

$$P_t = \frac{M_{t-1}}{Z_t L_t}.$$

Stochastic Computer Model

- With nonstationary series it is common to view them in growth rates.
- So if we have output Y_t and it grew at rate $1 + a_t$ then

$$\frac{Y_t}{Y_{t-1}} = \frac{Y_{t-1}(1 + a_t)}{Y_{t-1}} = 1 + a_t$$

- Similarly, we can construct the inflation rate series as

$$\frac{P_t}{P_{t-1}} = 1 + \pi_t$$

- Then we can ask, is what is the mean growth rate

$$\bar{a} = \frac{1}{N} \sum_{t=1}^{N} a_t$$

what is the variance of our growth rate

$$var(a) = \frac{1}{N} \sum_{t=1}^{N} (a_t - \bar{a})^2$$

Stochastic Computer Model

- We can also ask if output and inflation are correlated by computing

$$corr(a, \pi) = \frac{\frac{1}{N}\sum_{t=1}^{N}(a_t - \bar{a})(\pi_t - \bar{\pi})}{\sqrt{var(a)var(\pi)}}$$

- We can add a lead or lag by shifting one of our variables forward or backward in time: i.e.

$$corr(a, \pi_{-1}) = \frac{\frac{1}{N}\sum_{t=1}^{N}(a_t - \bar{a})(\pi_{t-1} - \bar{\pi})}{\sqrt{var(a)var(\pi)}}$$

- Matlab has a bunch of statistical tools we can use like
    - mean
    - max
    - median
    - std
    - corrcoef

Stochastic Computer Model

- Ask if systematic inflation is correlated with systematic growth in output?
- Compute the average growth rate over some interval of time

$$
\begin{aligned}
(1+G_t) &= \left(\frac{Y_t}{Y_{t-3}}\right)^{1/3} = \left((1+g_t)(1+g_{t-1})(1+g_{t-2})\right)^{1/3} \\
(1+\Pi_t) &= \left(\frac{P_t}{P_{t-3}}\right)^{1/3} = \left((1+\pi_t)(1+\pi_{t-1})(1+\pi_{t-2})\right)^{1/3}
\end{aligned}
$$

- This will give us the (harmonic) average growth rate of that interval.
- We can compute these average growth rates at each point in time t and then see if these averages $1+G_t$ and $1+\Pi_t$ are correlated.

Stochastic Computer Model

- We regularly compute all kinds of statistics in the data and from our models, but how seriously should we take them? Will they change a lot if we just wait for some more data to come in?
- One way to examine this is to analytically derive the distribution of our statistics.
- For example: If we have a sample of size n of normal random variables with mean μ and standard deviation σ, then the sample mean has expected value

$$\mathbb{E}\bar{X} = \mathbb{E}n^{-1}\sum_{t=1}^{n} X_t = \mu$$

  and variance

$$var(n^{-1}\sum_{t=1}^{n} X_t) = \left[n^{-2}\sum_{t=1}^{n}\sigma^2\right] = \frac{\sigma^2}{n}$$

  This gives us an analytic expression for how accurate our sample mean is, given our sample size and the variance of the random variable.
- From this we can see clearly how we need to increase the sample size to maintain a desired degree of accuracy.

Stochastic Computer Model

- An easy thing to do is to examine the distribution of a statistic in our model.
- The standard way of doing this is through Monte Carlo simulation, where we draw different sequences of the shocks indexed by i, $\{\epsilon_t^i\}$ and then build up the implied outcomes for each sequence.
- Finally one computes the sample statistic for each sequence i, which we can denote by m_i, and then examines the distribution of this sample statistic.
- This distribution gives us a lot of information about the accuracy of our estimate and is suggestive as to the accuracy of the data estimate as well.
- One way to think about accuracy is the standard deviation of m_i just as in our example with the sample mean of the normally distributed random variable.
- However, a more fundamental way to examine m_i is to plot the histogram. From this plot we can determine a 95 percent confidence interval for our model-based statistic.

Some Data on M, I and Y

Variety of theories about how money growth and inflation affect output:

- Keynesian in general, and old-style Keynesians in particular, believe that fluctuations in nominal demand can affect output because prices are sticky.
    - So an increase in nominal demand coming from printing money will raise output and employment.
    - This viewpoint is encapsulated in the Phillips Curve - which is a negative relationship between inflation and unemployment, or a positive relationship between inflation, output and employment.
- Another proposed channel, is called the *Tobin effect*:
    - Both money and capital are stores of value and an increase in inflation will cause people to hold less money and more capital in their portfolios.
    - This will raise investment and hence output.
- A cash-in-advance model is well-known to predict a negative relationship in the level of output, but not much of a growth effect, in the steady state.

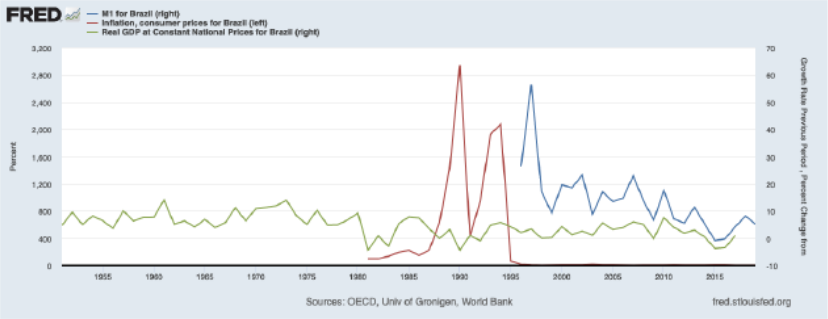
What is in the Data?

- There has been a fair amount of empirical work aimed at uncovering the relationship between money growth, inflation and output growth. In the reading section there are two examples of this work:
  - McCandless and Weber (MW), "Some Monetary Facts", *Quarterly Review of the Federal Reserve Bank of Minneapolis*, Summer 1995.
  - Haslag, "Output, Growth, Welfare, and Inflation: A Survey", *Economic Review of the Federal Reserve Bank of Dallas*, Second Quarter 1997.
- Haslag does a nice job of discussing the different theories and their implications.
- The empirical studies try and use panel data, which consists of time series data for a number of countries to try and uncover whether there is a causal relationship flowing from money growth to inflation to output growth.

What is in the Data?

What does this data look like?

Figure: Some Money, Inflation and Output Number for Brazil



What is in the Data?

What does this data look like?

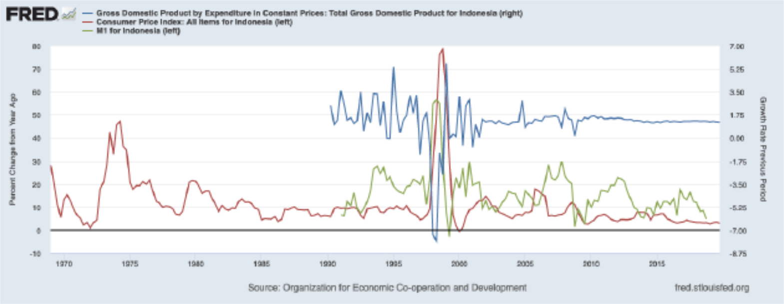


Figure: Some Money, Inflation and Output Number for Indonesia

Sadly, some of the scientifically most informative countries have the worst data.

What is in the Data?

- The key concern is that the central bank is choosing the money growth rate. Hence, variation in money growth may itself be caused by variation in, say, output growth. This would reverse the causality.
- Because we do not have laboratory experiments in Macro, we often try to find some plausible source of variation in the treatment (here money growth) to tease out the causal impact. The discussion in MW covers over this concern.
- The three main findings from this literature are:
  - There is a very tight correlation between money growth and inflation - 0.9 or higher.
  - There is very little correlation between money growth and output.
  - There is very little correlation between inflation and output.
- Are these findings consistent with our model in either the short-run or the long-run?

What is in the Data?

High correlation between money growth and inflation

Table 1

Correlation Coefficients for Money Growth and Inflation*

Based on Data From 1960 to 1990

	Coefficient for Each Definition of Money		
Sample	M0	M1	M2
All 110 Countries	.925	.958	.950
Subsamples			
21 OECD Countries	.894	.940	.958
14 Latin American Countries	.973	.992	.993

*Inflation is defined as changes in a measure of consumer prices.

Source of basic data: International Monetary Fund

What is in the Data?

Low correlation between money growth and output

Table 3

Correlation Coefficients for Money Growth and Real Output Growth*

Based on Data From 1960 to 1990

	Coefficient for Each Definition of Money		
Sample	M0	M1	M2
All 110 Countries	–.027	–.050	–.014
Subsamples			
21 OECD Countries	.707	.511	.518
14 Latin American Countries	–.171	–.239	–.243

*Real output growth is calculated by subtracting changes in a measure of consumer prices from changes in nominal gross domestic product.

Source of basic data: International Monetary Fund

Matching Model to Data

- In our set-up,

$$x_t = \begin{bmatrix} \tau_t \\ g_t \end{bmatrix}.$$

- And

$$x_t = \begin{bmatrix} \rho_\tau & 0 \\ 0 & \rho_g \end{bmatrix} x_{t-1} + \begin{bmatrix} B_\tau \\ B_g \end{bmatrix} + \begin{bmatrix} \sigma_\tau & 0 \\ 0 & \sigma_g \end{bmatrix} \begin{bmatrix} \varepsilon_{\tau,t} \\ \varepsilon_{g,t} \end{bmatrix},$$

- Many patterns can be generated by playing with our processes and varying it across countries.